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Title: Extremal Coble sextics and their mod 3 reductions

Abstract: The pair of the projective plane P2 and a rational sextic curve C is a log version of an Enriques surface. The blow-up of P2 at the 10 singular points of C is called a Coble surface. As an analogue of extremal elliptic surfaces, extremal Coble surfaces are defined, and classified in terms of root systems in characteristic zero. Defining equation of the extremal sextics are computed for the majority of them but still open for the rest. Their mod 3 reductions are studied by equations, and applied to the automorphism group of the Fermat quartic surface in characteristic 3 (joint work with H. Ohashi).

References

Allcock: , D.: The period of Enriques surfaces, Math. Ann., 2000.

Mukai, S.: Lecture notes on Enriques surfaces, in "Contribution to Algebraic Geometry", 2011.

Edge, W.: A pencil of four-nodal plane sextics, Math, Proc. Camb. Phil. Soc., 1981.

Winger, R.M.: Self-projective rational sextics, Amer. J. Math. 1916.

Kondo, S. and Shimada, I.: The automorphism group of a supersingular K3 surface with Artin invariant 1 in characteristic 3, IMRN, 2014.

Borcherds, R.: Automorphism groups of Lorentzian lattices, J. Algebra, 1987.

Extremal Coble sextics and their mod 3 reductions

於 復旦大学

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9:30〜10:30

§1 Enriques/Coble surface

$$S = X/\varepsilon \qquad K3/\text{involution of EC type}$$

that is,

$$\begin{tikzcd} \mathrm{Fix}(\varepsilon) = \coprod_m \mathbb{P}^1\text{'s} \arrow[d, "2"] & \subset X \arrow[d, "2:1"] \\ \coprod_{i=1}^m B_i & \subset S \end{tikzcd} \qquad (B_i^2) = -4$$

$B := \sum_{i=1}^m B_i$ a.b.c bdry $\in |-2K_S|$

$m = 0$ $\quad (S, \emptyset)$ Enriques surface

$m \geq 1$, $\quad (S, B)$ pair of a rational surface &

$\leftarrow (S, \frac{1}{2}B)$ の方が[illegible]

(Coble, log-Enr.)

$m = 1$ Classical Coble

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] \arrow[d, equal] & \mathbb{P}^2 \\ B \arrow[r, hook] & Bl_{10}\mathbb{P}^2 =: S \arrow[u] \end{tikzcd}$$

birat. embedding of deg 6

π : resolution of $f(\mathbb{P}^1)$

(S, B) Coble w. red. bdry

$\#$ of moduli $= 10 - m$

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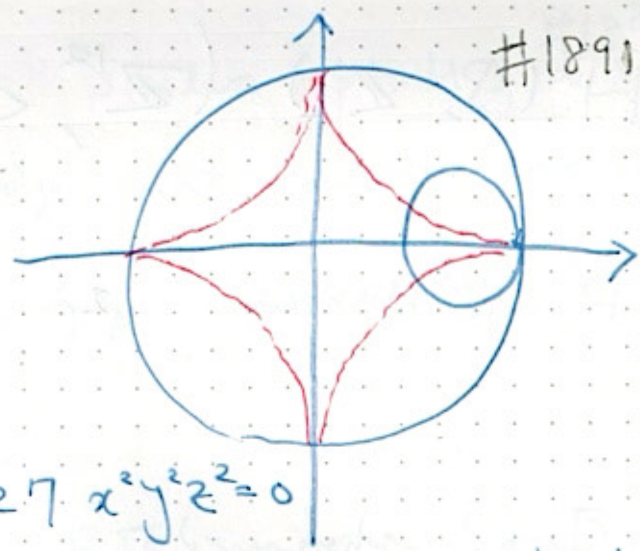
Example 1 (astroid)



$Ast_0 : x^{2/3} + y^{2/3} = 1$

alg. curve $/\mathbb{R}$ with

4 $\mathbb{R}$-cusps.

$/\mathbb{C}$ $\quad (x^2+y^2+z^2)^3 - 27\,x^2y^2z^2 = 0$

sextic with $\mathfrak{S}_4$-symmetry, 6 cusps at $(0:1:\pm i)$ etc.

and 4 nodes at $(\pm 1 : \pm 1 : 1)$. genus $= p_a - 6 - 4 = 0$.

Claim: This astroid Coble sextic is extremal.

Problem: Classify extremal EC surfaces as explicit as possible

Remark extremal $\Rightarrow$ rigid

Theorem 1 # of extremal EC surfaces is 68

m	0	1	2	3	4	5	6	7	8	9	10
#	12	14	13	9	7	4	4	1	2	1	1

Today $\ni$ Ex. 1

6 lines conf. w. 4 3-ple pts.

§2 Period map (& Hodge part)

$$\begin{array}{ccc} & 1 & \\ 0 & 10 & 0 \\ & 1 & \end{array} \xrightarrow[\text{rotation}]{\pi/2} \begin{array}{ccc} & 0 & \\ 1 & 10 & 1 \\ & 0 & \end{array}$$

(描書忘れ)

S Enriques surface

$\pi_1 = \mathbb{Z}/2$

$\mathbb{Z}^\omega$ non-trivial local system

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$H^2(S, \mathbb{Z}^w) = (\mathbb{Z}^{12}, \langle\ ,\ \rangle) \cong I_{2,10}$

$\langle\ ,\ \rangle$: cup prod. $\qquad I_{2,10}$: odd unimodular sgn $(2,10)$

Hodge decomp $\quad H^2(\ \ ) \underset{\mathbb{Z}}{\otimes} \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$

$(1\ ,\ 10\ ,\ 1)$

- Torelli thm $\quad \left\{ \begin{array}{c} \text{Enriques} \\ \text{surface} \end{array} \right\} \Big/ \cong \hookrightarrow \mathcal{D}^{10} \Big/ O_{\mathbb{Z}}(I_{2,10})$

$\mathcal{D}^{10} = SO_{\mathbb{R}}(2,10) \Big/ \text{max. cpt} \qquad$ type IV bdd sym. domain

$\underset{\text{open}}{\subset} Q^{10}_{\mathbb{C}} \subset \mathbb{P}^{11} \qquad$ cpt dual

- "Surjectivity" Image is the complement of a divisor, which parametrizes all Coble surfaces (w. $m \geq 1$). Enriques であるにEC曲面と すると完全にwt2 pol. HSが対応.

Hodge part $\quad H^{1,1}(S, \mathbb{Z}^w) = H^{1,1} \cap H^2(S, \mathbb{Z}^w)$

negative definite (primitive) sublattice of $I_{2,10}$

Definition Enriques surface S $(m=0)$ is extremal ($\Leftrightarrow$ $H^2(S, \mathbb{Z}^w)$ is extremal as pol. HS)

$\Leftrightarrow$ Hodge part is of maximal rank $(=10)$ and virtually generated by (-2)-elements.

General case: Similar pol. H.S. is associated and its Hodge part contains orthonormal (-1)-elements $b_1, \dots, b_m$ corresponding to bdry components $B_1, \dots, B_m$.

Definition (cont'd) EC surface (S, B) is extremal $\iff$ complement H.S. $\langle b_1, \dots, b_m \rangle^\perp \subset H^2(S, \mathbb{Z}^\omega)$ $\cong I_{2,10-m}$ as lattice, is extremal.

§ 3 Twisted fundamental class (& main theorem)

X K3-cover, $\pi: X \to S$. $C \subset S$ interior curve, i.e., $C \cap B = \emptyset$ w. reducible pull-back i.e.

$$\pi^{-1}(C) = C^+ \amalg C^- \quad (\text{e.g. } C \cong \mathbb{P}^1)$$

$$[C]_\omega := [C^+] - [C^-] \in H^2(S, \mathbb{Z}^\omega) \quad \text{twisted f. class}$$

perpendicular to $b_1, \dots, b_m$

Hodge, defined up to sign

$C \cong \mathbb{P}^1 \Rightarrow (C^2) = -2$ (well-known). Furthermore $([C]_\omega^2) = -2$ in $H^2(S, \mathbb{Z}^\omega)$

$\boxed{\text{Proposition: } S \text{ contains ADE configuration of rank } 10-m \Rightarrow S \text{ (Hodge theoretically) extremal}}$

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Remark (1) The hypothesis is never satisfied in Enriques case ($m=0$), since $H^2(S,\mathbb{Z})$ has sgn $(1,9)$.

(2) " $\Leftarrow$ " does not hold.

Theorem 2 ($m=1$, extremal classical Coble)

Hodge lattice $I_{2,9} \cap H^{1,1}$ of 13 extremal Coble surface w. $m=1$ is as follows:

$\lvert$disc$\rvert$	root lattice
2	E_8+A_1
4	D_9
6	E_7+A_2
8	A_7+2A_1
10	A_9 Ex. 3.
12	E_6+A_3, $A_5+A_3+A_1$ (Ex. 2, Ex. 1)
18	A_8+A_1, $E_6+A_2+A_1$
20	D_5+A_4 (*2)
30	A_5+A_4
42	$A_6+A_2+A_1$
90	$A_4+2A_2+A_1$

Astroid

$\exists$ $15+6+1=32$ conics passing thru 6 of 10 sing pts Ast_8

$\Updownarrow$ twisted fund. class

These $\mathbb{P}^1$'s generate A_5, A_3, and A_1, respectively. ($5+3+1=9$, maximal rank)

$A_5+A_3+A_1 \hookrightarrow H^{1,1}$ is of index 2.

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§4 Two more examples

Example 2. $\overline{X} : \begin{cases} \sum_1^4 x_i + 4x_5 = 0 & \text{in } \mathbb{P}^4 \\ \sum_1^4 \frac{1}{x_i} + \frac{4}{x_5} = 0 \end{cases}$

EC surface の説明 ~ 最適で係数を一般化して、もっと早く出してもよかった。

Blow-up of 10 nodes, $X \longrightarrow \overline{X}$, is K3 surface

$S = X/\varepsilon$, $\varepsilon(x) = \left(\frac{1}{x}\right)$ EC type

In this case $(1111-1)$ is fixed, and $m = 1$, that is $\mathrm{Fix}(\varepsilon) \cong \mathbb{P}^1$. S is the blow-up of 10 nodes of (Desargues type) sextic $\boxed{Bl_{10}\mathbb{P}^2}$

$\overline{B} : x^6 + y^6 + z^6 - (x^4y^2 + \cdots) + 3x^2y^2z^2 = 0$

(S, B) is extremal with $E_6 + A_3 \subset \langle b \rangle^{\perp} = I_{2,9}$

10 nodes 上への10以外 四子分 古来

6個のEckard 点が自然に出てくる 33 conic の例

(言忘れた 当時は未だ)

Example 3 Edge pencil

$E_\lambda : x^6 + y^6 + z^6 + (x^2+y^2+z^2)(x^4+y^4+z^4) - 12x^2y^2z^2$
$+ \lambda (y^2 - z^2)(z^2 - x^2)(x^2 - y^2) = 0$

of 4-nodal plane sextics with icosahedral ($\cong \mathfrak{A}_5$) symmetry. (Genus 6 when parameter λ is general.) Special values $\lambda = 0, \pm 5\sqrt{5}, \infty, \cdots$

Wiman 曲線

5次 del

Period 10次

5 conic

$\lambda = 5\sqrt{5} \Rightarrow$ irred., acquire 6 more nodes, hence rational.

$(S, \tilde{E}_{5\sqrt{5}})$ extremal with $A_9 \subset I_{2,9}$.

10 node 中の 3つを通る 直線 15本、それらが 6つを通る conic か 合計 15本 (未確認)

§5 mod 3 reductions

Observation $\begin{pmatrix}\text{K3-cover}\\ \text{of Ex. 2}\end{pmatrix} \bmod 3 \cong \begin{pmatrix}\text{K3-cover}\\ \text{of Ex. 3}\end{pmatrix} \bmod 3$

$$\cong X^{ss1} \qquad /\overline{\mathbb{F}}_3$$

Fermat quartic $\sum_{i=1}^{4} x_i^4 = 0 \subset \mathbb{P}^3$

contains 48 lines and has $C_4^3 \rtimes \mathfrak{S}_4$ symmetry $/\mathbb{C}$

$\rightsquigarrow$ mod 3 reduction: 112 lines and $PGU(4, \mathbb{F}_9)$ symmetry over $\overline{\mathbb{F}}_3$.

- supersingular (maximal Picard number $\rho = 22$)
- Artin invariant 1 (minimal discriminant $\operatorname{disc} \operatorname{Pic} = -3^2$)

Denoted by X^{ss1}.

Theorem 3 (M.-Ohashi)

$$\operatorname{Aut}(X^{ss1}) = \langle PGU(4, \mathbb{F}_9), \sigma, \tau \rangle$$

covering involution of Example 1 & 2.

$$X^{ss1} \to S_{E_6+A_3}, \quad X^{ss1} \to S_{A_9}$$

Remark This group structure of $\operatorname{Aut} X^{ss1}$ was proved by Kondo-Shimada (2014). Their method to find σ, τ is computational rather than conceptual.

Keum 氏の質問 $\to$ expected from Borcherds theory and