Degeneration of $P^n \times P^n$ and application to del Pezzo fibration

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Motivation

$\{X_t\}$ (1-parameter) family of varieties

$\{L_t, M_t\}$ pair of vector bundles on X_t

generated by n+1 global sections

$$\Phi_L, \Phi_M : X \longrightarrow \mathbb{P}^n \quad \text{morphisms}$$

$$\Phi_t := (\Phi_L, \Phi_M) : X \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$$

Confluence Problem $\quad L_t \not\cong M_t \quad$ for general t

$$L_0 \cong M_0$$

What is the reasonable limit of Φ_t as $t \longrightarrow 0$?

Answer In good cases, such a limit

$$\tilde{\Phi}_0 : X \longrightarrow \mathbb{P}^{n,n}$$

exists as a morphism to a degeneration of $\mathbb{P}^n \times \mathbb{P}^n$, which is a 2n-dimensional projective variety with only A_1-singularity along a codimension 2 subvariety.

$\tilde{\Phi}_0$ is a lift of $\Phi_0 : X \longrightarrow \mathbb{P}^n \xhookrightarrow[\text{diagonal}]{} \mathbb{P}^n \times \mathbb{P}^n$.

<u>Example</u> (n = 1)

$$\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 \qquad xt - yz = 0$$

$$\mathbb{P}^{1,1} \qquad xt - y^2 = 0$$

<u>Advanced Problem</u> Study the relation with "confluences" in other branches of mathematics, say, hypergeometric equations and (modular) representations.

§0 VIP*) varieties

① Segre variety $\qquad \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n^2+2n}$

② 2nd Veronese $\qquad \mathbb{P}^n \subset \mathbb{P}^{n(n+3)/2}$

③ Grassmannian $\qquad G(\mathbb{P}^1 \subset \mathbb{P}^n) \subset \mathbb{P}^{(n-1)(n+2)/2}$

These are projectivizations of cones of matrices of minimal rank:

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*) very important projective

① {rank 1 matrix} $\subset M_{n+1,n+1}(\mathbb{C}) = \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$

$$\rotatebox{90}{$\in$}\quad M_{p,b} = \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} (b_0 \cdots b_n) = (a_i b_j)_{0 \le i,j \le n}$$

② $\left\{ \begin{array}{l} \text{rank 1 symmetric} \\ \text{matrix} \quad M_{p,p} \end{array} \right\} \subset S^2 \mathbb{C}^{n+1}$

③ $\left\{ \begin{array}{l} \text{rank 2 skew-} \\ \text{symmetric matrix} \end{array} \right\} \subset \bigwedge^2 \mathbb{C}^{n+1}$

$$\rotatebox{90}{$\in$}\quad N_\ell = \left(\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \right)_{0 \le i,j \le n}$$

Remark $\exists$ a natural rational map

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}^n & \dashrightarrow & G(\mathbb{P}^1 \subset \mathbb{P}^n) \\ (p, q) & \longmapsto & \overline{pb} \end{array}$$

whose indeterminacy is eliminated by the blow-up with center the diagonal Δ.

Furthermore,

$$Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \big/ \text{factor change} \longrightarrow G(\mathbb{P}^1 \subset \mathbb{P}^n)$$

is a $\mathbb{P}^2$-bundle.

§1 $\mathbb{P}^{n,n}$ as projective variety

Degeneration of ① is a mixture of ② & ③.

$$\begin{array}{ccc} \mathbb{P}(\mathbb{C}^{n+1}\otimes\mathbb{C}^{n+1}) & \overset{①}{\supset} & \mathbb{P}^n\times\mathbb{P}^n \\ \| & & \\ \mathbb{P}(S^2\mathbb{C}^{n+1}\oplus\Lambda^2\mathbb{C}^{n+1}) & \supset & \mathbb{P}(S^2\mathbb{C}^{n+1}) \amalg \mathbb{P}(\Lambda^2\mathbb{C}^{n+1}) \end{array}$$

$$② \ \mathbb{P}(S^2\mathbb{C}^{n+1}) \supset \mathbb{P}^n \qquad ③ \ \mathbb{P}(\Lambda^2\mathbb{C}^{n+1}) \supset G(\mathbb{P}^1\subset\mathbb{P}^n)$$

1st Definition

$\mathbb{P}^{n,n}$ is the incidence join of Veronese $\mathbb{P}^n$ and Grassmannian $G(\mathbb{P}^1\subset\mathbb{P}^n)$. More precisely, the projectivization of

$$\bigcup_{\substack{p\in\ell \\ p\in\mathbb{P}^n,\ \ell\in G(\mathbb{P}^1,\mathbb{P}^n)}} \langle M_{p,p}, N_\ell\rangle \subset S^2\mathbb{C}^{n+1}\oplus\Lambda^2\mathbb{C}^{n+1}.$$

(A) By definition, $\mathbb{P}^{n,n}$ is contained in the weighted projective space:

$$\begin{tikzcd}
\mathbb{P}^{n,n} \arrow[r, hook] & \mathbb{P}(\underbrace{\overset{x_0,\cdots,x_n}{1,\cdots,1}}_{n+1}, \underbrace{\overset{y_{ij}}{2,\cdots,2}}_{\binom{n+1}{2}}) \subset \mathbb{P}^{n^2+2n} \\
G(\mathbb{P}^1 \subset \mathbb{P}^n) \arrow[u] \arrow[r, hook] & \mathbb{P}(2,\cdots,2) \arrow[u]
\end{tikzcd}$$

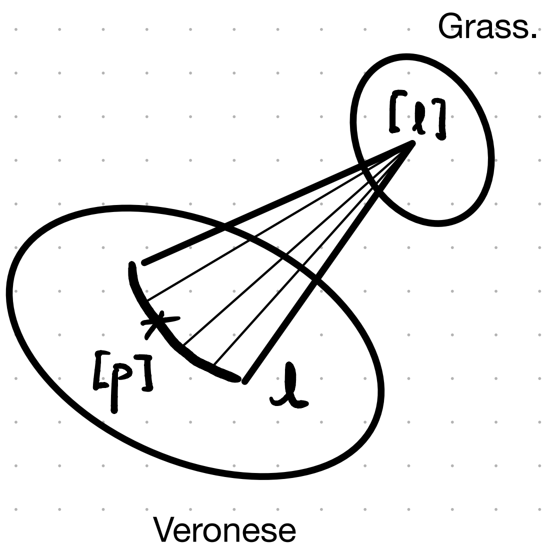
Defining equation of

	degree
• Pliucker relations on y_{ij}	4
• Incidence relations among x and y	3

(B) Fix a line ℓ.

Then $\bigcup_{p \in \ell} \langle M_{p,p}, N_\ell \rangle$ is a quadric cone.

$\mathbb{P}^{n,n}$ has A_1-singularity along $G(\mathbb{P}^1 \subset \mathbb{P}^n)$.



(c) Minimal resolution is a $\mathbb{P}^n$-bundle over $\mathbb{P}^n$:

$$\begin{array}{c} \widetilde{\mathbb{P}}^{n,n} \\ \downarrow \\ \mathbb{P}^{n,n} \end{array}$$

In fact,

$$\widetilde{\mathbb{P}}^{n,n} \cong \mathbb{P}(\mathcal{O}(2) \oplus \Omega(2))$$

& $-K_{\widetilde{\mathbb{P}}}$ = (n+1) H, H: tautological line bundle.

$\mathbb{P}^{n,n}$ is the image of

$$\Phi_H : \widetilde{\mathbb{P}}^{n,n} \longrightarrow \mathbb{P}(S^2\mathbb{C}^{n+1} \oplus \wedge^2 \mathbb{C}^{n+1}).$$

<u>Example</u> (n = 2) $\mathbb{P}^{2,2} \subset \mathbb{P}(\overbrace{111}^{x}\overbrace{222}^{y})$

is a cubic hypersuraface $\sum_{i=0}^{2} x_i y_i = 0$.

§2 Degeneration of $\mathbb{P}^n \times \mathbb{P}^n$ to $\mathbb{P}^{n,n}$

1. Bundle method

2. Elementary transformation

§3 $\mathbb{P}^{n,n}$ in Grassmanian

$$\left.\begin{array}{l} \mathcal{O}_X^{n+1} \twoheadrightarrow L \\ \mathcal{O}_X^{n+1} \twoheadrightarrow M \end{array}\right\} \Rightarrow \mathcal{O}_X^{2n+2} \twoheadrightarrow E = L \oplus M$$

pair of line bundles — rank 2 bundle

$$\begin{tikzcd} \Phi_E : X \arrow[rr] \arrow[dr] & & G(2n+2, 2) \\ & \mathbb{P}^n \times \mathbb{P}^n \arrow[ur] & \end{tikzcd}$$

First we describe this map.

$\mathbb{P}^{2n+1} \supset \mathbb{P}^n \amalg \mathbb{P}^n$ (When n=1, just a pair of skew lines in $\mathbb{P}^3$.)

$\mathbb{C}^{2n+2} = V_1 \oplus V_2 \qquad \dim V_1 = \dim V_2 = n+1$

$\mathbb{P}^n \times \mathbb{P}^n$ is a subvariety of $G(\mathbb{P}^1 \subset \mathbb{P}^n)$ defined by two Schubert conditions:

$$\{ [U] \in G(2, \mathbb{C}^{2n+2}) \mid U \cap V_1 \neq 0,\ U \cap V_2 \neq 0 \}$$
$$= \{ \text{line } l \text{ intersecting both } \mathbb{P}(V_1) \ \& \ \mathbb{P}(V_2) \}$$
$$= \mathbb{P}(V_1) \times \mathbb{P}(V_2)$$

Thus $\mathbb{P}^n \times \mathbb{P}^n \subset G(2, \mathbb{C}^{2n+2})$ is defined by a pair of points $[V_1], [V_2] \in G(n+1, 2n+2)$.

Our $\mathbb{P}^{n,n}$ is the limit case where $[V_2]$ becomes an infinitely near point, that is, a tangent direction at $[V_1]$.

Fact : Tangent space of $G(n+1, \mathbb{C}^{2n+2})$ at $[V]$ is $\mathrm{Hom}(V, \mathbb{C}^{2n+2}/V)$. (linear map from sub to quotient)

A tangent vector $\varphi : V \to \mathbb{C}^{2n+2}/V$ is non-degenerate if φ is an isomorphism.

2nd Definition φ : non-degenerate tangent vector

$$\mathbb{P}^{n,n} := \left\{ [U] \in G(2, \mathbb{C}^{2n+2}) \;\middle|\; \begin{array}{l} \mathrm{Im}\, U \text{ in } \mathbb{C}^{2n+2}/V \\ \subset \varphi(U \cap V). \end{array} \right\}$$

2 cases 1) $\dim U \cap V = 1$

$\varphi : U \cap V \xrightarrow{\sim} \mathrm{Im}\, U$

2) $U \subset V$ $(\dashrightarrow \mathbb{P}^{n,n} \supset G(2, V))$.

The 2nd agrees with the 1st definition in §1.

§4 Functorial property

① L : free line bundle

exact sequence $0 \to L \to E \to L \to 0$

with surjective $H^0(E) \twoheadrightarrow H^0(L)$.

$$\Longrightarrow \quad \Phi_E : X \longrightarrow G(H^0(E), 2)$$

factors through $(\mathbb{P}^{n,n})_{reg}$.

② More generally L : free line bundle

D : effective divisor with $H^0(L) \xrightarrow{\sim} H^0(L(D))$.

exact sequence $0 \to L(D) \to E \to L \to 0$ with

$H^0(E) \twoheadrightarrow H^0(L)$.

$\Longrightarrow \quad \Phi_E$ factors through $\mathbb{P}^{n,n}$.

§5 Del Pezzo surface of degree 6

S : RDP del Pezzo surface of degree 6

minimal resolution of S is the blow-up of $\mathbb{P}^2$ at 3 points p_1, p_2 and p_3, which may be infinitely near.

<u>case a</u> p_1, p_2, p_3 are not colinear.

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed, "2h-p_1-p_2-p_3"'] & & \mathbb{P}^2 \end{tikzcd}$$

$S \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$

$S = (\mathbb{P}^2 \times \mathbb{P}^2) \cap H_1 \cap H_2$

(graph of quadratic Cremona transformation)

<u>case b</u> p_1, p_2, p_3 are colinear.

$\exists\ D \in |h - p_1 - p_2 - p_3| \quad (-2)\ \mathbb{P}^1$

$\exists$ exact sequence $0 \to L(D) \to E \to L \to 0$

as in §4, ②. $\qquad S = \mathbb{P}^{2,2} \cap H_1 \cap H_2$.

<u>Proposition</u> $\mathcal{S} \longrightarrow C$ RDP dP_6-fibration over a curve with only central monodromy*) (contained in the first factor C_2 of $C_2 \times \mathfrak{S}_3 = D_{12}$)

$\Longrightarrow \exists\ \mathcal{P}$ in $\mathbb{P}^8$-bundle such that

$$\begin{array}{c} \mathcal{P} \\ \downarrow \\ C \end{array}$$

$\mathcal{S} \cong \mathcal{P} \cap \mathcal{H}_1 \cap \mathcal{H}_2$ and every fiber of $\mathcal{P}/C$ is either

$\mathbb{P}^2 \times \mathbb{P}^2$ or $\mathbb{P}^{2,2}$.

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*) In the talk this monodromy condition was erroneously omitted.