Gorenstein Fano 3-folds

7/26/93 (A) M

Private seminar at Durham Univ.
in request of N. Shepherd-Barron

Former classification (Announced in PNAS U.S.A. (1989))

X : Fano 3-fold w. $\operatorname{Pic} X \cong \mathbb{Z}(-K)$

$\longrightarrow$ $\exists S \in |-K_X|$ $\quad \operatorname{Pic} S \cong \mathbb{Z}(-K|_S)$

Thm. of Moishezon

$\longrightarrow$ stable rigid bundle E on (S, h)

classify (S, h) $\quad S = \Sigma_{2g-2} \cap H_1 \cap \cdots \cap H_{...}$

$\longrightarrow$ Linear section thm for linear sect thm for Fano 3-fold.

Singular case

⊙ {Singular Fano 3-fold} is better category than that of smooth ones. We get simple proof of genus bound $g \leq 10$ or $g = 12$.

⊙ For singular one Pic is not good.

(We don't assume singular pt is factorial.)

Remark. In factorial case, theory of extremal ray works.
(Mori's note at lecture at Columbia and Nagoya
Univ.)

(I) Result

① {Fano 3-fold with canonical singularity}
↳ $-K$ ample

crepant blow-up → ② {weak Fano 3-fold with terminal singularity}
← anti-canonical model
↳ $-K$ nef & big

Good anticanonical member → (quasi) polarized K3 surface

① $\begin{cases} S \in |-K| & \text{K3 surface with only R.D.P's} \\ -K|_S & \text{ample} \end{cases}$

② $\begin{cases} S \in |-K| & \text{smooth K3 surface} \\ -K|_S & \text{nef \& big.} \end{cases}$

$\boxed{\text{Definition}}$ $|D|$ is movably decomposable $\iff$ $\exists A + B \sim D$ $\dim|A|, \dim|B| > 0$

$\boxed{\text{Theorem}}$ Let X be a Fano 3-fold w. only Gor. can. sing.

If $|-K_X|$ is not movably decomposable (X is called indecomposable), then $g = \frac{(-K)^3}{2}+1 \leq 10$ or $= 12$.

$\boxed{\text{Cor}}$. X is smoothable.

Classifying space Σ_{2g-2} for $g = 7, 8, 9, 10, 12$

$g \neq 12$ Homogeneous

$\Sigma^{10}_{12} \subset \mathbb{P}^{15}$ 10-dim variety of pure spinors $SO(10)$

$\Sigma^{8}_{14} = G(2,6) \subset \mathbb{P}^{14}$ 8-dim Grassmannian

$\Sigma^{6}_{16} = G(3,6,\text{symp.}) \subset \mathbb{P}^{13}$ compact dual of $\mathfrak{h}_3$

Σ^{5}_{18} G_2-variety

Genus 12 — ① $VSP(F_4, 6)$ ② Variety of twisted cubics ③ net of bivectors.
non homogeneous three descriptions.

$\boxed{\text{Def}}$. (1) a (quasi) polarized K3 surface (S, h) is numerically Petri general if $h^0(\mathcal{O}_S(a)) \cdot h^0(\mathcal{O}_S(b)) \leq g$ for every decomposition $a + b \sim h$, where $g = \frac{(h^2)}{2} + 1$.

(2) Petri general $\Leftrightarrow$ numerically Petri general and the Petri map $H^0(\mathcal{O}_S(a)) \otimes H^0(\mathcal{O}_S(b)) \longrightarrow H^0(\mathcal{O}_S(h))$ is always injective.

$\boxed{\text{Proposition}}$ If X is indecomposable and $S \in |-K_X|$ is a good member, then $(S, -K_X|_S)$ is Petri general.

$\boxed{\text{Theorem}}$ (S, h) Petri general K3 surface of genus g

$g = 7, 8, 9, 10, 12 \Rightarrow S \cong H_1 \cap \cdots \cap H_{N-2} \cap \Sigma_{2g-2}$

complete linear section. $N = \dim \Sigma$.

$\boxed{\text{Theorem}}$ + $\boxed{\text{Lefschetz Principle}}$ $\longrightarrow$ Same classification for Fano 3-folds

$\boxed{2}$ Linear system of higher rank

Pair $\left\{\begin{array}{l} \text{vector bundle } E \text{ (on } S) \\ V \subset H^0(E) \end{array}\right.$ linear system of rank $r = r(E)$

(E, V) is free $\Leftrightarrow$ $V \otimes \mathcal{O} \longrightarrow E$ surjective

$\Rightarrow$ morphism $\Phi_{(E,V)} : S \longrightarrow G(V, r)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ||

Grassmannian of r-dim. quotient of V.

When $V = H^0(E)$, $(E, H^0(E))$ is denoted by $|E|$, called complete linear system. We always assume that

i) $|E|$ is almost free, i.e. $\mathrm{Coke}\,[H^0(E) \otimes \mathcal{O} \to E]$ has supp of codim ≥ 2.

ii) $\mathrm{Hom}\,(E, \mathcal{O}_X) = 0$.

<u>Definition</u> (1) $|E|$ is <u>irreducible</u> $\Leftrightarrow$ $\forall\, U^r \subset H^0(E)$

$$U^r \otimes \mathcal{O}_S \longrightarrow E \quad \text{is injective.}$$

(2) $|E|$ is <u>semi-irreducible</u> $\Leftrightarrow$ $\forall\, U^r \subset H^0(E)$ either

i) $U^r \otimes \mathcal{O}_S \longrightarrow E$ injective, or

ii) $\mathrm{Im}\,[U^r \otimes \mathcal{O}_S \longrightarrow E]$ is a locally free sub-bundle of rank $r-1$. ($\Rightarrow$ Coke is a line bundle.)

<u>Rmk.</u> semi-irred. & $\dim U < r \Rightarrow U^r \otimes \mathcal{O}_S \to E$ is injective.

(S, h) : ((quasi-) polarized) K3 surface of genus g

E : vector bundle of rank r with $\overset{r}{\wedge} E \simeq \mathcal{O}_S(h)$

$s := \chi(E) - r$ "corank"

<u>Proposition</u> $|E|$ irreducible $\Rightarrow$ $r(h^0(E) - r) \leq g$.

<u>Proof.</u> (Castelnuovo argument)

$G(r, H^0(E))$ Grassmannian $\dim = rs$

$$\text{Plücker} \quad G(r,H^0(E)) \subset \mathbb{P}_*(\overset{r}{\wedge} H^0(E)) \dashrightarrow H^0(\overset{r}{\wedge} E) = H^0(\mathcal{O}_S(h)) \quad \dim g+1.$$

$g+1$ linear forms have no common zero on $G(r, H^0(E))$.

$\dim G(r, H^0(E)) < g+1$. General case Omitted. q.e.d.

<u>Proposition</u> E semi-irred $\Rightarrow$ simple

<u>Proof.</u> $f : E \longrightarrow E$ nonzero homomorphism

$H^0(f) : H^0(E) \longrightarrow H^0(E)$ also nonzero

rk $H^0(f) < r$ Im f is trivial. contradicts our assumption $H^0(E^\vee) = 0$.

rk $H^0(f) = r$ $E \cong \mathrm{Coke}\,[L^{-1} \to \mathcal{O}^{\oplus n}] \oplus L$ contradiction

rk $H^0(f) \geq r$ f is generically surjective. Hence surjective and isom.

q.e.d.

<u>Cor.</u> E semi-rigid $\Rightarrow r\,s \leq g$

$\because$ $2 \geq \chi(E^\vee \otimes E) = 2rs - (2g-2)$

<u>Def.</u> Semi-irreducible E is <u>maximal</u> if $rs = g$.

<u>Prop.</u> E, F maximal semi-rigid, of the same rank $\Rightarrow E \cong F$.

$\because$ $\dim \mathrm{Hom}(E, F) + \dim \mathrm{Hom}(F, E)$
$\geq \chi(E^\vee \otimes F) = \chi(E^\vee \otimes E) = 2.$

q.e.d.

<u>Proposition</u> E maximal semi-rigid. 1) $H^1(E) = 0$

2) $H^1(\overset{r-1}{\wedge} E) = 0$ $\left(\Leftrightarrow H^0(E) \xrightarrow{\sim} H^0(E|_c) \quad c \in \mathbb{P}^1 \right)$

<u>Proof.</u> Omit. Use the fact that $E|_c$ is also semi-irred and bound $h^0(E|_c)$.

<u>Theorem</u> (S, h) Petri general, $g = rs$
$\Longrightarrow$ $\exists !$ mixed semi-rigid linear system $|E|$ of rk r.

<u>Theorem</u>

g	8	9	10	12
r	2	3	5	3
s	4	3	2	4

$\Phi_{|E|}$ embeds S into Σ and its image is complete linear section.

$g=8$ $\quad \Phi_{|E|} : S \longrightarrow G(2,6)$

$g=9$ $\quad \overset{2}{\wedge} H^0(E) \longrightarrow H^0(\overset{2}{\wedge} E)$ $\quad \mathrm{Ker} = \mathbb{C}\cdot\sigma$ $\leftarrow$ non-deg. symplectic form.

$\Phi_{|E|} : S \longrightarrow G(3,6,\sigma)$

$g=10$ $\quad \overset{4}{\wedge} H^0(E) \longrightarrow H^0(\overset{4}{\wedge} E)$ $\quad \mathrm{Ker} = \mathbb{C}\cdot\tau$ $\leftarrow$ non-deg. 4-vector

$\Phi_{|E|} : S \longrightarrow G(5,7,\tau)$ $\leftarrow$ G_2-variety

$g=12$ $\quad N = \mathrm{Ker}\left[\overset{2}{\wedge} H^0(E) \longrightarrow H^0(\overset{2}{\wedge} E)\right]$ $\quad$ 3-dim. family of bivectors

$\Phi_{|E|} : S \longrightarrow G(3,7,N)$ $\leftarrow$ smooth Fano 3-fold of genus 12.

$\boxed{3}$ Lefschetz principle

$$\begin{array}{lccc} \text{K3 section.} & S & \hookrightarrow & \Sigma_{2g-2} \\ & \cap & & \cap \\ \text{Fano 3-fold} & X & & \mathbb{P}^N \end{array}$$

(Image) = complete linear section

Vanishing $\quad H^1(T_\Sigma|_S \otimes \mathcal{O}_S(-n)) = 0 \quad \forall n \geq 1$

$\Uparrow$

Bott Vanishing

$\Downarrow$

$S \hookrightarrow \Sigma_{2g-2}$ extends to the formal nbd $\hat{S}_X$ of $S \subset X$.

$$\begin{tikzcd}
\mathbb{P}^g \supset S \arrow[rr, hook] & & \Sigma_{2g-2} \arrow[d, phantom, "\cap"] \\
\mathbb{P}^{g+1} \supset \hat{S}_X \arrow[rr, "\hat{f}"] & & \mathbb{P}^N \\
\mathbb{P}^{g+1} \supset X & &
\end{tikzcd}$$

Consider

$$\begin{tikzcd}
 & H^0(\mathbb{P}^N, \mathcal{O}(1)) \arrow[d, "\hat{f}^*"] \arrow[ddl, "\text{surjective}"'] & \\
 & H^0(\hat{S}_X, \mathcal{O}_{\hat{S}}(-K)) \arrow[d, phantom, "\cong"] & \\
0 \leftarrow H^0(S, \mathcal{O}_S(-K)) & H^0(X, \mathcal{O}_X(-K)) \arrow[l] & \mathbb{C} \leftarrow 0 \arrow[l]
\end{tikzcd}$$

Two possibilities ① $\hat{f}^*$ is surjective

② Image of $\hat{f}^*$ is $H^0(S, \mathcal{O}_S(-K))$

Case ① $\hat{f}$ extends to $\mathbb{P}^{g+1} \xrightarrow{\Phi} \Sigma$

Φ maps X onto Σ since $\Phi(\hat{S}_X) \subset \Sigma$.

Case ② Image of $\Phi_{|-K|}$ is a cone of $S_{2g-2} \subset \mathbb{P}^{g+1}$

Contradicts to "X only can. singularity".

$\boxed{\text{Conclusion}}$ $g = 8, 9, 10, 12$ Same Lemma shows

$$X = \Sigma_{2g-2} \cap H_1 \cap \cdots \cap H_{n-3}.$$

[4] Indecomposable Fano 3-folds.

Proof of Thm 1. i.e. $g \leq 10$ or $g = 12$.

Double projection from a point

$p \in X$ general point

$$\begin{tikzcd} Bl_p(X) \arrow[r] \arrow[dr, "|-K|"'] & X_{2g-2} \subset \mathbb{P}^{g+1} \arrow[d, dashed] \\ & \overline{X} \subset \mathbb{P}^{g-3} \end{tikzcd}$$

genus decreases by 4.

contains Veronese surface.

Show this is impossible by classification of X of $g \leq 10$ or $g = 12$.

X indecomposable $\Rightarrow$ $\overline{X}$ indecomposable

since $|-K_{\overline{X}}| = |-K_X - 2p| \subset |-K_X|$

Gorenstein Fano 3-folds (2nd lecture)

7/28/93 (W.)

X Fano 3-fold

Vector bundle classification (w. comparison)

Smooth	Gorenstein canonical singularity
$\mathrm{Pic}\, X = \mathbb{Z}(-K_X)$	$\lvert -K_X \rvert$ is not mixedly decomposable i.e. $-K_X \sim A + B \Rightarrow \dim \lvert A \rvert = 0$ or $\dim \lvert B \rvert = 0$

General member $S \in \lvert -K_X \rvert$

Polarization $h = (-K_X)|_S$

$\mathrm{Pic}\, S = \mathbb{Z} h$	(S, h) is numerically Petri general i.e., $h \sim a + b \Rightarrow h^0(\mathcal{O}_S(a))\, h^0(\mathcal{O}_S(b)) \leq g$

$\exists !\ E \in M_S(r, h, s)$ rigid stable bundle

$r(E) = r \quad c_1(E) = h \quad \chi(E) = r + s \qquad \rho := g - rs = 0$

$\lvert E \rvert$ is irreducible, i.e., $\forall\, U^r \subset H^0(E)$ $U^r \otimes \mathcal{O}_S \xrightarrow{ev} E$ is injective.	$\lvert E \rvert$ is semi-irreducible

g	8	9	10	12	
r	2	3	5	3	2
s	4	3	2	4	6

Grassmannian embedding of S by $\lvert E \rvert$

$$\Phi_{|E|} : S \longrightarrow G(H^0(E), r) = G(r, V)$$

$$V = H^0(E)^\vee$$

Image is fermion complete intersection.

$$\lambda : \bigoplus_{i=0}^{r} \bigwedge^{i} H^0(E) \longrightarrow H^0(\bigwedge^{i} E)$$

$\bigcap$

$\left\{ \begin{array}{l} \text{skew-symmetric multi-} \\ \text{linear form on } V \end{array} \right\}$

$$\operatorname{Ker}\left[\bigwedge^{n-1} H^0(E) \longrightarrow H^0\left(\bigwedge^{n-1} E\right)\right] = \begin{cases} 0 \\ \mathbb{C}\sigma \\ \mathbb{C}\tau \\ N \end{cases}$$

$$\Phi_{|E|} : S \xrightarrow{\ \Phi\ } \Sigma_{2g-2} \subset G(n, V)$$

(LST) Image is complete linear section

$$H^1(S, \Phi^* T_\Sigma \otimes \mathcal{O}_S(-n)) = 0 \quad \forall n \geq 2$$

E extends to a vector bundle $\tilde{E}$ on X (formal nbd $\to$ Zariski open $\to X$)	$\Phi : S \longrightarrow \Sigma$ extends to a morphism $\tilde{\Phi} : X \longrightarrow \Sigma$ (formal nbd $\to$ ambient projective space $\longrightarrow X$)

(L. S. T) for Fano 3 folds of $g \leq 10$, $g = 12$

Genus bound by independent method, e.g., generically injectivity of $\mathcal{P}_g \longrightarrow \mathcal{M}_g$. $\mathcal{P}_g$ is $\mathbb{P}^g$-bundle over $\mathcal{F}_g = \{K3 \text{ of genus } g\}$ parametrizing curves.	Genus bound by (L. S. T) and double projection from a point. $$\begin{array}{ccc} p & \dashrightarrow & P \text{ Veronese} \\ \cap & & \cap \\ X_{2g-2} & \dashrightarrow & X_{2g-10} \\ \cap & & \cap \\ \mathbb{P}^{g+1} & \dashrightarrow & \mathbb{P}^{g-3} \end{array}$$

E vector bundle, or more generally, coherent sheaf on a K3 surface S.

$$v(E) = (r,\ c_1(E),\ s = \chi(E) - r) \in \mathbb{Z} \oplus H^2(S,\mathbb{Z}) \oplus \mathbb{Z}$$

"action" of $O\left(\begin{pmatrix}0&1\\1&0\end{pmatrix} \oplus H^2(S,\mathbb{Z})\right)$

Moduli theoretically, one essential invariant,

$$v(E)^2 := c_1(E)^2 - 2rs \quad \left(= -\chi(E^\vee \otimes E) \text{ if } E \text{ l.f.}\right)$$

Theorem (1) Semi-stable sheaf E with $v(E) = v$

$\Leftrightarrow (v^2) \geq -2$

(2) $\dim M_S(v) = (v^2) + 2$

(3) $(v^2) = -2 \Rightarrow \overline{M}_S(v) \ni$ unique member

Proof (1) $\Rightarrow$ & (2) If E is simple, then $\dim \mathrm{Ext}^1(E,E) = (v^2) + 2$.

(3) $E, E' \in \overline{M}_S(v)$

$\dim \mathrm{Hom}(E, E') + \dim \mathrm{Hom}(E', E)$

$\geq \chi(E^\vee \otimes E') = \chi(E^\vee \otimes E) = 2$. q.e.d.

Special divisors on $C \subset S$ is closely related to ~~$M(r, [C], s>0)$~~.

$$\begin{tikzcd} \xi \arrow[d] & \in & W_C^{r,s} = \left\{ \xi \,\middle|\, \begin{matrix} \chi(\xi) = r - s \\ h^0(\xi) \geq r \end{matrix} \right\} \subset \mathrm{Pic}^{r-s+g-1} C \arrow[d, dashed] \\ \mathrm{Ker}[\mathcal{O}_S^{\oplus r} \to \xi]^\vee & \in & VB(r, [C], s) \end{tikzcd}$$

$$0 \longrightarrow E^\vee \longrightarrow \mathcal{O}_S^{\oplus r} \longrightarrow \xi \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_S^{\oplus r} \longrightarrow E \to \eta \longrightarrow 0 \qquad \eta = \omega_C \xi^{-1}$$

$$\chi(E) = \chi(E^\vee) = 2r - \chi(\xi) = r + s$$
$$c_1(E) = h$$

$$(\# \text{ of moduli of } E) = v(E)^2 + 2 = 2g - 2 - 2rs + 2$$
$$= 2(g - rs) = 2\ (\text{Brill-Noether } \#)$$

Assume $s > 0$. $h^0(\eta) \neq 0$. $|E|$ is almost free.
By construction $\mathrm{Hom}(E, \mathcal{O}_S) = 0$. e.g. if $\mathrm{Pic}\, S = \mathbb{Z} \cdot h$

Proposition if $|h|$ is not movably decomposable

then $|E|$ is irreducible, i.e., $\forall U^r \subset H^0(E)$

$$U^r \otimes \mathcal{O}_S \longrightarrow E$$

is injective.

Proof. Assume $U^r \otimes \mathcal{O}_S \longrightarrow E$ is not injective. Then the image F is of rank $< r$. Let $\tilde{F}$ be saturation.
Get exact sequence torsion free

$$0 \longrightarrow \tilde{F} \longrightarrow E \to G \to 0$$
$$\cap$$
$$\tilde{G} \quad \text{double dual}$$

$h^0(\tilde{F}) \geq r > r(\tilde{F})$
$|\tilde{G}|$ is almost free since so is E.
$\mathrm{Hom}(\tilde{G}, \mathcal{O}_S) = 0$ since $\mathrm{Hom}(E, \mathcal{O}_S) = 0$

$$\det E \simeq \det \tilde{F} \otimes \det \tilde{G}$$
$$h \sim c_1(\tilde{F}) + c_1(\tilde{G}) \quad \text{movable decomposition}$$

q.e.d.

Proposition If $\mathrm{Pic}\, S = \mathbb{Z}\cdot h$, then E is μ-stable.

Proof. $E \longrightarrow F \longrightarrow 0$ proper quotient bundle.

$|F|$ is almost free. $\det F \cong \mathcal{O}_S(nh)$ $n \geq 1$.

$\mu(F) > \mu(E)$. q.e.d.

Existence of special divisors (Kempf, Kleiman-Laksov)

$W_C^{n,s} \neq \emptyset$ if $\rho = g - ns \geq 0$

$\longrightarrow$ Irreducible linear system $|E|$ $\begin{cases} |h| \text{ not simply dec.} \\ s > 0 \end{cases}$

$\longrightarrow$ $M(n, h, s) \neq \emptyset$ if $\mathrm{Pic}\, S = \mathbb{Z}h$ $(v(E)^2) \geq -2$

deformation argument $\Big\downarrow$ $\begin{cases} \text{smoothness of } \bigcup_S M_S(v) \\ \text{properness of } \bigcup_S \overline{M_S(v)} \end{cases}$

(1) of Theorem.

Linear section theorem (Weaker version)

$|E|$ irreducible linear system on (S, h)

g:	7	8	9	10	12
$v(E)$	$(5,2h,5)$	$(2,h,4)$	$(3,h,3)$	$(5,h,2)$	$(3,h,4)$
a		0	1	1	3

$(2,h,6)$ another description of V_{22}.

$$\Rightarrow \quad \Phi_{|E|} : S \longrightarrow G(H^0(E), r) = G(r, V)$$

$$\Phi'_{|E|} : S \longrightarrow \Sigma_{2g-2} \quad \text{cpt intersection w.r.t. } (\textstyle\bigwedge^{r-1} E)^{\oplus a}$$

Image of $\Phi'_{|E|}$ is a complete linear section of Σ_{2g-2}.

$g=9$. $\quad \bigwedge^2 H^0(E) \xrightarrow{\lambda_2} H^0(\bigwedge^2 E)$

$\binom{6}{2} = 15 \qquad \dim = 14$ by R.R. + Vanishing $H^1(\bigwedge^2 E) = 0$

$0 \neq \sigma \in \operatorname{Ker} \lambda_2 \qquad \operatorname{rk} \sigma = 2, 4, 6.$

Claim $\operatorname{rk} \sigma = 6$

☺ $\operatorname{rk} \sigma < 6 \Rightarrow \sigma = v_1 \wedge v_2 + v_3 \wedge v_4$ or $v_1 \wedge v_2$ for suitable choice of bases.

$\sigma \wedge v_3 \in \sigma \wedge H^0(E) \subset \operatorname{Ker}\left[\bigwedge^3 H^0(E) \to H^0(\bigwedge^3 E)\right]$

$\parallel$

$v_1 \wedge v_2 \wedge v_3$

v_1, v_2, v_3 are lin. dep. at gen. pt.

$|E|$ is reducible

q.e.d.

Claim $\dim \operatorname{Ker} \lambda_2 = 1$.

☺ $a\sigma + b\sigma' \in \operatorname{Ker} \lambda_2$

$\operatorname{Pfaff}(a\sigma + b\sigma')$ cubic homog. polynomial

$\operatorname{rk}(a\sigma + b\sigma') < 6$ for some $(a:b) \neq (0,0)$

q.e.d.

4章: $|\bar{E}|$ irred. $\Rightarrow \begin{cases} n \le 8 \\ \bigwedge^2 H^0(E) \to H^0(\bigwedge^2 E) \\ \text{surjective} \end{cases}$

$\operatorname{Ker} \lambda_2 = \mathbb{C}\sigma$

σ is symplectic form on $V := H^0(E)^\vee$

$$\Phi_{|E|} : S \longrightarrow G(H^0(E), 3) = G(3, V)$$

$$\cup$$

$$\Sigma_{16} = G(3, \underbrace{V}_{\substack{6\text{-dim} \\ \text{symplectic} \\ \text{space}}}, \sigma)$$

$$\begin{array}{ccc} S & \longrightarrow & \Sigma_{16} \\ \cap & & \cap \\ \mathbb{P}^{13} & \longrightarrow & \mathbb{P}^{13} = \mathbb{P}^*(\bigwedge^2 H^0(E) / \sigma \wedge H^0(E)) \end{array}$$

$\uparrow$ 14-dim irred. rep. of $Sp(6)$

$g = 12$ $\qquad G(3, V, N)$ smooth

$\Uparrow$

$|\bar{E}|$ smooth

$N = \operatorname{Ker}[\bigwedge^2 H^0(E) \to H^0(\bigwedge^2 E)]$ $\qquad \forall \sigma \in N$, $\operatorname{rk}(\sigma) = 6$

この部分の証明

2章 Clean $\mathbb{P}^1 \to \mathbb{P}'$ approx.

Ker $\bigwedge^2$ ... is symplectic ...

~~Smooth~~

Gorenstein Fano 3-folds

For Second Lecture

7/28/93 (SK)

~~Traditional~~

Vector Bundle ~~Method~~ Classification

~~[illegible]~~ smooth X ~~[illegible]~~	~~[illegible]~~ Gorenstein can. singularities
$\operatorname{Pic} X = \mathbb{Z}(-K_X)$	$\lvert -K_X \rvert$ is not [illegible] decomps.
~~[illegible]~~ $S \in \lvert -K_X \rvert$, $h = -K_X\lvert_S$	general member (quasi) polarization
$\operatorname{Pic} S = \mathbb{Z}h$ (Moishezon)	(S, h) is Petri
$\exists!\ E \in M_S(r, h, s)$, $rs = g$	stable rigid bundle
$\lvert E \rvert$ is ~~[illegible]~~ irreducible	$\lvert E \rvert$ is semi-irred.

~~Existence of lines~~

Uniqueness follows from Riemann-Roch
$\chi(E^\vee \otimes E') = \underset{=2}{\chi(E^\vee \otimes E)}$

$\Phi_{|E|} : S \longrightarrow G(H^0(E), r)$

~~Image is complete intersection with respect to~~

Image is Fermionically complete intersection.

~~i.e. complete intersection with respect to~~

more precisely, complete intersection with respect to $(\bigwedge^{r-1} E)^{\oplus a} \oplus (\bigwedge^{r} E)^{\oplus b}$

for some a, b if $g = 8, 9, 10, 12$

$\boxed{\text{Remark}}$

By the uniqueness of E,

~~E descend to E_k on S_k of S is $S_k \otimes_k \mathbb{C}$.~~

~~makes the classi~~ [illegible] classification ~~[illegible]~~ possible extends over $\forall k \subset \mathbb{C}$.

$g \neq 8$ ~~[illegible]~~

E descend to a vector bundle E_k on S_k, i.e. $E = E_k \otimes_k \mathbb{C}$ if $S = S_k \otimes \mathbb{C}$.

Same description of S_k.

$g = 8$ E does not nec. descend. but $\mathbb{P}(E)$ descends.

~~[illegible]~~

E extends to a vector bundle $\tilde{E}$ on a Fano 3-fold (Fujita).

$H^2(S, \Phi^* T_\Sigma \otimes \mathcal{O}_S(-n)) = 0$ $n \geq 1$.

$\Phi : S \longrightarrow \Sigma$

extends to a map between ~~[illegible]~~ spaces. and extend to $X \to \Sigma$.

Genus bound by using double projection from a (general) point

Genus bound by using generically injectivity of

$$\begin{array}{ccc} P_g & \longrightarrow & M_g \\ \downarrow & & \\ \mathcal{F}_g & & \end{array}$$

Dynkin 図形との関連 [illegible]

$$\left\{\begin{array}{ll} E_8[\ ] & g = 7 \\ E_7[\ ] & g = 8 \\ F_4 & g = 9 \\ G_2 & g = 10 \end{array}\right.$$

[illegible]

[illegible] $\left\langle \begin{array}{l} g = 9 \text{ の [illegible]} \\ g = 10 \ \text{〃} \\ g = 12 \ \text{〃} \end{array}\right.$ $\begin{array}{l} g = 7+4n \text{ の [illegible]} \\ g = 10+6n \ \text{〃} \\ g = 12+ [illegible] \ \text{〃} \end{array}$