JAMI talk
(Johns Hopkins U.)

Gorenstein Fano 3-fold

4/13/96 (Sat)

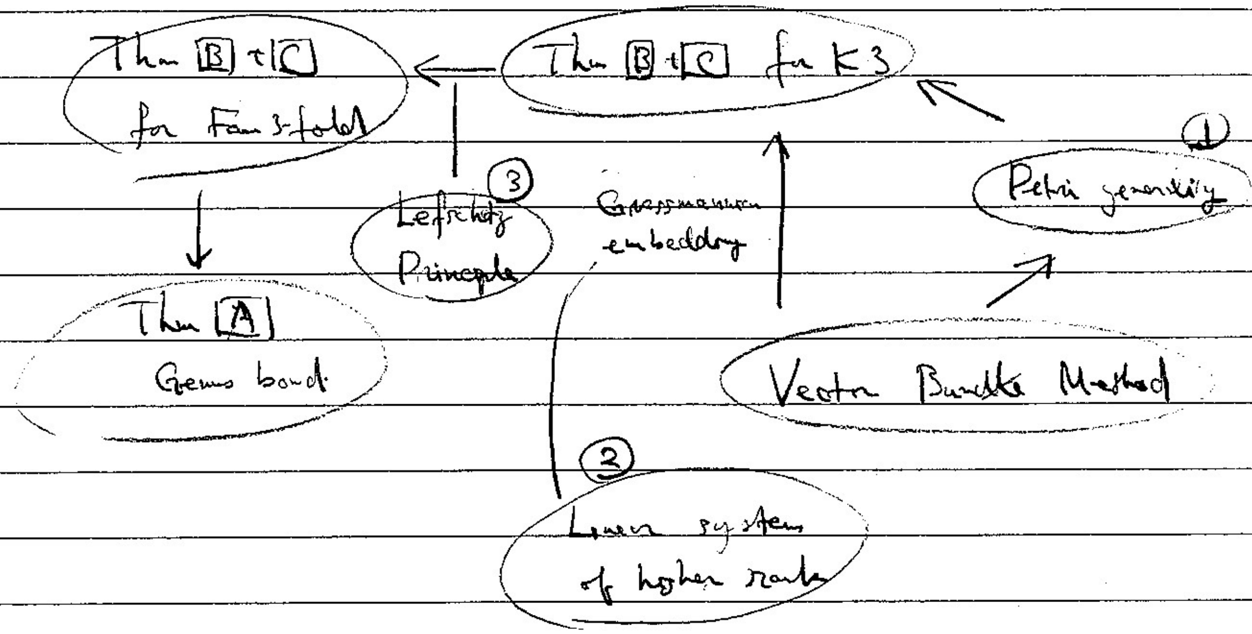
§1 Definition $\exists$ good member $S \in |K_X|$ RDP K3

Ladder $C \subset S \subset X$

$\mathbb{P}^{g-1}$ $\mathbb{P}^{g}$ $\mathbb{P}^{g+1}$

§2 Examples $Bl_C \mathbb{P}^3$, Homogeneous space of coindex 3

§3 Main Theorem $\boxed{A}$ + $\boxed{B}$ + $\boxed{C}$



§4 ① Petri generality indecomposable ⇒ Petri general ⇒ non g...

§5 ② L.S.T for K3 X: Fano S: K3 C

§6 ③ Lefschetz principle

§7 ④ Genus bound

J. Wahl 氏 by (上の③の部分に[illegible]) K3 曲面 から Fano の L.S.Th を[illegible]。

g≧8 以上 では generic の曲面の relation が[illegible]

コホモロジー群の vanishing [illegible]。 今の extension は [illegible]

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§1 X is a 3-fold with only Gorenstein canonical singularities.

X is Fano $\overset{\text{def}}{\iff}$ $-K_X$ is ample

$\Rightarrow$ $\exists\, S \in |-K_X|$ anticanonical divisor (Shokurov; Reid) with only RDP's. S is K3.

$\Rightarrow$ $|-K_X|$ is very ample with few exceptions

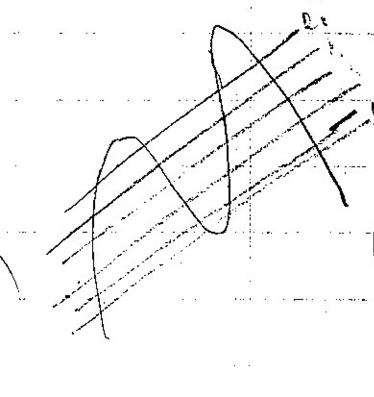
$\Rightarrow$ The image $\Phi_{-K_X} : X_{2g-2} \hookrightarrow \mathbb{P}^{g+1}$, called the anti-canonical model, is a projective variety with canonical curve section

$$\begin{cases} g = \dfrac{(-K_X)^3}{2} + 1 \\ \dim |-K_X| = g+1 \\ \text{(R-R + Vanishing)} \end{cases}$$

$$\begin{array}{llcl}
\text{a.c. model} & & X_{2g-2} & \subset \mathbb{P}^{g+1} \\
 & & | & \quad | \\
\text{proj. model of pol. RDP K3} & X \cap H = & S_{2g-2} & \subset \mathbb{P}^{g} \\
 & & | & \quad | \\
\text{canonical model of smooth curve} & X \cap H \cap H' = & C_{2g-2} & \subset \mathbb{P}^{g-1}
\end{array}$$

§2 Example 1. Anticanonical model of $Bl_C\, \mathbb{P}^3$

a) $C_6 \subset \mathbb{P}^3$ general (smooth) rational curve of degree 6



$$\begin{array}{ccc} C_6 & \subset & \mathbb{P}^3 \\ \uparrow & & \uparrow \pi \\ E & \subset & Bl_C \mathbb{P}^3 = Y \end{array} \qquad \begin{array}{l} -K_Y = 4\pi^* H - E \\ (-K_Y)^3 = 14 \end{array}$$

$|-K_Y|$ is free and gives morphism $\Phi_{-K_Y} : Y \longrightarrow \mathbb{P}^9$.

There exists 6 4-secant lines $l_1, —, l_6$.

Let $Z_1, \dots, Z_6$ be their strict transforms on Y.

Φ_{-K_Y} is an isomorphism outside $Z_1, \dots, Z_6$.

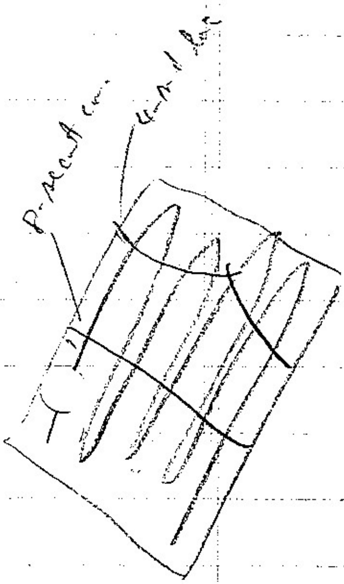
On the other hand, $(-K_Y . Z_i) = 0$. Z_i's are contracted to ODP's. The image X_{14} of Φ_{-K_Y} is Fano 3-fold of genus 8 with 8 ODP's.

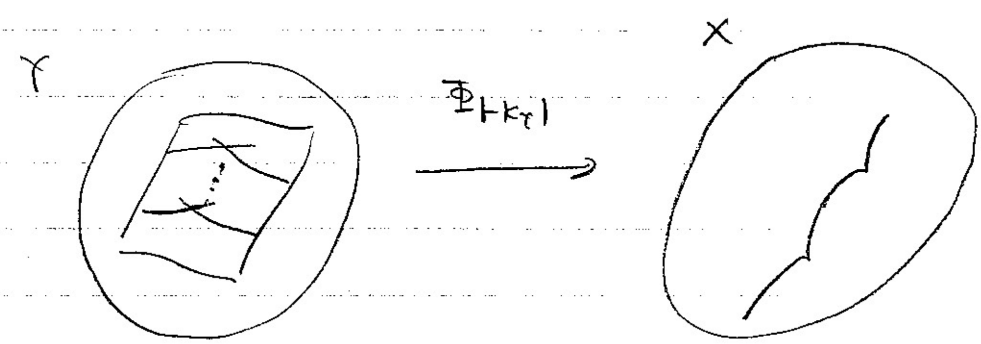
b) $C_{10} \subset \mathbb{P}^3$ degree 10, genus 12

$-K_Y = 4\pi^*H - E$, $(-K_Y)^3 = 6$, $|-K_Y|$ free

(# of 4-secant lines) $= 10$.

C_{10} has 1-dim'l family of 8-secant conics.





These strict transforms are contracted by $\Phi_{|-K_Y|}$. The anticanonical model X has canonical singularities along a curve.

c) $C_{12} = S_3 \cap S_4 \subset \mathbb{P}^3$ The anticanonical model X of Y is a quartic with a triple point. The strict transform of S_3 is contracted to the isolated can. singularity.

~~Obstacle~~ ~~Fano 3-fold~~

Basic Example 2 Homogeneous variety of index $n-2$

$$\Sigma^n = G/P \subset \mathbb{P}(V) \quad \left\{ \begin{array}{l} G: \text{semi-simple alg. gp} \\ V: \text{fundamental representation} \\ X: \text{orbit of h.w. vector} \end{array} \right.$$

$\exists$ 4 homogeneous projective variety with $-K_\Sigma = (n-2)H$

$g = \frac{1}{2}\deg \Sigma + 1$	$n = \dim \Sigma$	G	V	
7	10	$SO(10)$	16 dim	half-spin representation
8	8	$SL(6)$	$\bigwedge^2 \mathbb{C}^6$	$G(2,6) \subset \mathbb{P}^{14}$ 8-dim. Grassmannian
9	6	$Sp(3)$	14-dim.	Compact dual of $\mathfrak{h}_{g_3}$
10	5	Exep. type G_2	14-dim	adjoint rep.

$X = \Sigma^n \cap H_1 \cap \cdots \cap H_{n-3}$ is Gorenstein Fano 3-fold of genus g if $\dim X = 3$.

We call these Fano 3-fold of linear section type.

§3 Main Theorem

Def. $-K_X \sim A_1 + A_2$ is a movable decomposition if both A_i are movable Weil divisors, i.e., $\dim |A_i| > 0$.

A Fano 3-fold X is indecomposable if $-K_X$ has no movable decompositions.

<u>Remark</u> If X is smooth, then indecomposable $\Leftrightarrow$ $\mathrm{Pic}\, X$ is generated by $-K_X$ (Implication "$\Rightarrow$" is not easy).

<u>Theorem</u> Let X be an indecomposable Fano 3-fold of genus g.

$\boxed{A}$ Genus bound: $g \leq 10$ or $g = 12$.

$\boxed{B}$ Description for $g \leq 10$: X is isomorphic to one of the following:

$g=2$ $\quad X \xrightarrow{2:1} \mathbb{P}^3 \quad \deg(\mathrm{Branch}) = 8$

$g=3$ $\quad X_4 \subset \mathbb{P}^4$ or $X \xrightarrow{2:1} Q_3 \subset \mathbb{P}^4$

$g=4$ $\quad X_6 = (2) \cap (3) \subset \mathbb{P}^5$

$g=5$ $\quad X_8 = (2) \cap (2) \cap (2) \subset \mathbb{P}^6$

$g=6$ $\quad X_{10} = (2) \cap V_5^4 \subset \mathbb{P}^7 \quad V_5^4 \subset \mathbb{P}^7$ smooth or cone over smooth $V_5^3 \subset \mathbb{P}^6$.

$g=7,8,9,10$ $\quad X \simeq \Sigma^n_{2g-2} \cap H_1 \cap \cdots \cap H_{n-3}$ linear section type

$\boxed{C}$ If $g=12$, then X is smooth and $\mathrm{Pic}\, X$ is generated by $-K_X$. Moreover, $X \cong G(3, 7, N)$

for 3-dim. subspace $N \subset (\overset{2}{\wedge} \mathbb{C}^7)^*$.

$N = \langle \sigma_1, \sigma_2, \sigma_3 \rangle_{\mathbb{C}}$ $\quad \sigma_i : \underset{\text{skew-symmetric}}{\mathbb{C}^7 \times \mathbb{C}^7 \longrightarrow \mathbb{C}}$

$$G(3,7,N) = \left\{ U \;\middle|\; \begin{array}{l} \text{3-dim. subspace of } \mathbb{C}^7 \text{ which are} \\ \text{totally isotropic w.r.t. } \sigma_i,\ i=1,2,3 \end{array} \right\}$$

$\cap$

$G(3,7)$ 12-dim. Grassmannian

<u>Remark</u> Fano 3-fold of genus 12 has two other descriptions. One is

$$VSP(F_4, 6) = \left\{ ([f_1], \cdots, [f_6]) \;\middle|\; \sum_{i=1}^{6} f_i^4 = F_4 \right\}$$

where $F_4 = F_4(x, y, z)$ and f_i's are linear forms.

§4 Consequence of indecomposability

<u>Def.</u> A ~~generic~~ polarized (Smith) K3 surface (S, h) is <u>Petri general</u> if, for every $h \sim a+b$, (sum of Weil div.)

i) $h^0(\mathcal{O}_S(a))\, h^0(\mathcal{O}_S(b)) \leq g$, and

ii) $\mu : H^0(\mathcal{O}_S(a)) \otimes H^0(\mathcal{O}_S(b)) \longrightarrow H^0(\mathcal{O}_S(h))$

is injective.

min. res. RDP K3 Fano

$$S \xrightarrow{\pi} \overline{S} \subset X$$

$\overline{S} \in |-K_X|$ good member

$h = \pi^*(-K_X)$

(S, h) is a quasi-pol. K3 surface

E

$\ker f \to E \to \operatorname{Coker} f \to$

$\Downarrow$ movable decomp

$\det E = \mathcal{O}(-K_X)$

<u>Proposition</u> If X is indecomposable, then (S, h) is Petri general.

~~Proof~~ Rough sketch X を crepant res. $0 \to E^\vee \to \mathcal{O}_X^{\oplus r} \to \pi_* \mathcal{O}_S(a) \to 0$

$M_{S,h}$ not inj. $\Rightarrow \operatorname{End}(E) \geq 2 \Rightarrow$ $f: E \to E$

E free in codim 1.

<u>Proof</u> Sは非特異, Xは... Assume (S, h) is not Petri general. $\exists\, a + b \sim h$

either i) $h^0(\mathcal{O}_S(a))\, h^0(\mathcal{O}_S(b)) > g$, or

ii) $\mu_{a,b}$: $H^0(\mathcal{O}_S(a)) \otimes H^0(\mathcal{O}_S(b)) \longrightarrow H^0(\mathcal{O}_S(h))$ not injective

We show $\exists$ of movable decomposition of $-K_X$ in the case ii).

May assume $|a|$ is free. (Over K3, fixed component free $\Rightarrow$ free.)

Put $r = h^0(\mathcal{O}_S(a)) \geq 2$. Consider $\mathcal{O}_X$-module $\pi_* \mathcal{O}_S(a)$, sheaf supported on $\overline{S}$. Define torsion free sheaf F by

$$0 \longrightarrow F \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow \pi_* \mathcal{O}_S(a) \longrightarrow 0 \quad \text{(exact)}$$

Take dual

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow F^\vee \longrightarrow \pi_* \mathcal{O}_S(b) \longrightarrow 0$$

$$F^\vee = E$$

Then $\bigwedge^r E \cong \mathcal{O}_X(\overline{S}) \cong \mathcal{O}_X(-K_X)$. E is generated by global sections in codim 1. $\operatorname{Hom}(E, \mathcal{O}_X) = 0$ by construction.

Lemma $\mu_{a,b}$ not injective $\Rightarrow$ $\dim \operatorname{End}(E) \geq 2$

Take nonzero $f: E \longrightarrow E$ which is not an isomorphism.

$\operatorname{Im} f$ and $\operatorname{Coker} f$ are quotients of E. Hence they are generated by global sections in codim one. They are not trivial since $\operatorname{Hom}(E, \mathcal{O}) = 0$. Hence

$$\underbrace{h^0(\det \operatorname{Im} f) \geq 2 \text{ and } h^0(\det \operatorname{Coker} f)}_{\text{defined as Weil div.}} \geq 2.$$

$$-K_X \sim \det E \sim \det \operatorname{Im} f + \det \operatorname{Coker} f \quad \times$$

movable decomposition $\square$

Remark By similar argument, one can show

$C = X \cap H \cap H'$ is numerically general,

that is $h^0(\xi)\, h^0(K_C \xi^{-1}) \leq g$ for every $\xi \in \operatorname{Pic} C$.

§5 Linear section theorem for K3 surfaces

(S, h) polarized RDP K3 surface $\quad g = \dfrac{(h^2)}{2} + 1$

Assume (S, h) (or its resolution) is Petri general.

The following is easy to show.

Proposition $g=2$ $S \xrightarrow{2:1} \mathbb{P}^2$

$g=3$ $S_4 \subset \mathbb{P}^3$ $g=4$ $S_6 = (2) \cap (3)$ $g=5$ $S_8 = (2) \cap (2) \cap (2)$

($\Rightarrow$ Thm Ⓑ $g=2,3,4,5$.)

Theorem Ⓓ (1) $g=7,8,9,10$ $\begin{cases} S \cong \Sigma^n \cap H_1 \cap \cdots \cap H_{n-2} \\ h = (\text{hyperplane section}) \end{cases}$

(S,h) is of linear section type, i.e.,

(2) $g=12$, $\exists N \in (\wedge^2 \mathbb{C}^7)^*$ $S \cong G(V,3,N) \cap H$

Proved by Grassmannian embedding

$S \overset{i}{\supset} C \in |h|$ smooth member.

Take $\xi \in \mathrm{Pic}\, C$ with $h^0(\xi) = r$, $h^1(\xi) = s$ and $rs = g$

g	8	9	10	12
(r,s)	$(2,4)$	$(3,3)$	$(2,5)$	$(3,4)$
			(or $(5,2)$)	(or $(2,6)$)

Define vector bundle E_r on S by the exact seq.

$$0 \longrightarrow E_r^\vee \longrightarrow \mathcal{O}_S^{\oplus r} \longrightarrow i_* \xi \longrightarrow 0$$

sheaf supported on C

$\det E_r \cong \mathcal{O}_S(h)$

$h^0(E_r) = r+s$

$$\Phi_{E_r} : S \longrightarrow G(H^0(E_r), r)$$

g-dim. Grassmannian.

Ex. $g=8$ $\quad E_2$ rank 2 $\quad$ commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, phantom, "\cap"] & G(2,6) \arrow[d, phantom, "\cap", "\text{Plücker}"'] \\ \mathbb{P}^8 \arrow[r, hook] & \mathbb{P}^{14} \end{tikzcd}$$

Hl

is Cartesian ($\Longrightarrow$ Theorem.)

Remark.
$g=7$ We use twisted conormal bundle $N^\vee_{S/\mathbb{P}^7}(2)$ of $S \subset \mathbb{P}^7$. to embed S into Grassmann
$rk=5$ $h^0=10$

§6 Proof of Theorem Ⓑ & Ⓒ

$$S \subset X$$
K3 $\quad$ Fano

By Th. Ⓓ, S is a linear section of Σ_{2g-2}.

$$\begin{tikzcd} i : S \arrow[r, hook] \arrow[d, phantom, "\cap"] & \Sigma_{2g-2} \\ X & \end{tikzcd}$$

$$\mathcal{O}_X(S) = \mathcal{O}_X(-K_X) =: \mathcal{O}(1)$$

Step 1. i extends to (a morphism $j : \hat{S}_X \longrightarrow \Sigma$ of) the formal nbd $\hat{S}_X$ of $S \subset X$.

☺ By Bott vanishing, $H^1(S, (i^* T_\Sigma)(-n)) = 0 \quad \forall n \geq 1$.

Step 2. j induces

$$\begin{tikzcd} H^0(\mathcal{O}_\Sigma(1)) \arrow[r, "j^*"] & H^0(\hat{S}_X, \mathcal{O}(1)) \\ & H^0(X, \mathcal{O}(-K_X)) \arrow[u, "\wr"] \end{tikzcd}$$

Let β be the (contravariant ~~projective tangent~~

$$\mathbb{P}^* H^0(\mathcal{O}_\Sigma(1)) \dashleftarrow \mathbb{P}^{g+1}$$
subset of Σ $\qquad$ subset of X

$$\begin{tikzcd}
\hat{S}_X \arrow[rrr, "\hat{j}"] \arrow[d] & & & \Sigma \arrow[dd] \\
X \arrow[d] & & S \arrow[ur, "i"] \arrow[d] \arrow[r, phantom, "\square"] & \\
\mathbb{P}^{s+1} \arrow[rrr, dashed, bend left, "\beta"] & & \mathbb{P}^s \arrow[r, "\alpha"] & \mathbb{P}
\end{tikzcd}$$

β maps $\hat{S}_X$ into Σ. Hence X into Σ.

It suffices to show

<u>Claim</u> β is embedding. ($\Leftrightarrow j^*$ is surjective)

$\because$ otherwise β is the composite of the projection off a point p and α. This implies X is the cone ~~of~~ over S with vertex p. contradict. //

By the claim, $X = \Sigma \cap \operatorname{Im} \beta$ complete intersection since S is so.

$$
\begin{array}{c}
|-K_Y| \nearrow \overline{Y} \subset \mathbb{P}^{g-3} \quad \text{projection from tangent space at } p \\
\uparrow \\
Y_p \to \underset{\substack{\cup \\ p}}{X_{2g-2}} \subset \mathbb{P}^{g+1} \\
|-K_X|
\end{array}
$$

§7 Genus bound

Inductive structure of {Gor. Fano w. can. sing.}.

Assume $(-K_X)^3 \geq 10$

$X \ni p$ general point

$Y_p = Bl_p X \qquad -K_Y = \pi^*(-K_X) - 2E \quad \left(\begin{array}{l} E \cong \mathbb{P}^2 \\ \mathcal{O}_E(E) = \mathcal{O}(-1) \end{array} \right)$

Y_p is not Fano in general, since there ~~exists~~ may exist (usually exists) a conic passing through p. But weak Fano, $-K_Y$ is nef and big. The a.c. model $\overline{Y}_p = \mathrm{Proj} \bigoplus_{n \geq 0} H^0(\mathcal{O}(-nK_Y))$

is a Gorenstein Fano 3-fold. Genus drops by 4.

$\mathbb{P}^{g+1} \supset X \rightsquigarrow \overline{Y}_p \subset \mathbb{P}^{g-3}$ ($\overline{Y}_p$ contains a Veronese surface as the image of E)

genus g $\qquad$ genus $g-4$.

If X is indecomposable, then so is $\overline{Y}_p$.

Proof of Thm $\boxed{A}$. It suffices to show the non-existence

for $g = 11, 13, 14$ and 16.

X genus $16 \Longrightarrow \overline{Y}_p$ genus 12. By Thm $\boxed{C}$, $\overline{Y}_p$ is smooth.

$\mathrm{Pic}\, \overline{Y}_p = \mathbb{Z}(-K)$. But $\overline{Y}_p$ contains $E \cong \mathbb{P}^2$.

contradiction

X genus $14 \Rightarrow \overline{Y}_p$ genus 10 $\quad \overline{Y}_p \subset \Sigma^5_{16}$ by Thm $\boxed{B}$. But Σ^5_{16} does not contain a Veronese surface.

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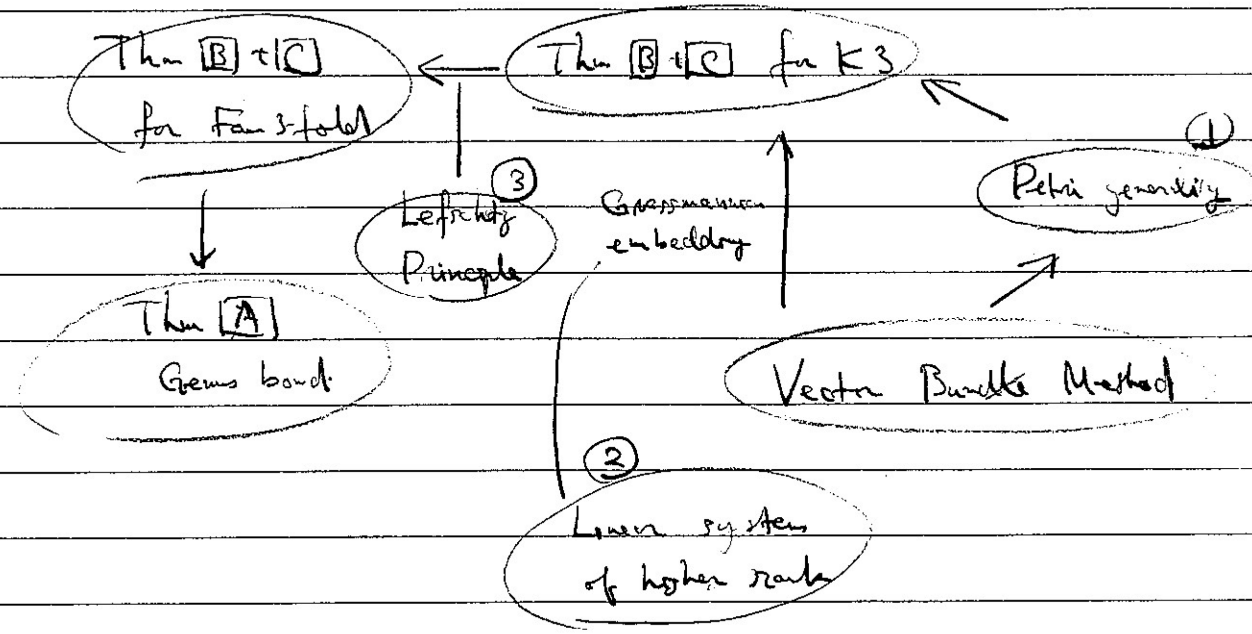

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(上の③の部分に関して)

J. Wahl 氏 が K3 曲面 の S. Fano の L.S.T. [illegible]

g≧[illegible] generic の [illegible] relation [illegible]

[illegible] vanishing [illegible] extension [illegible]