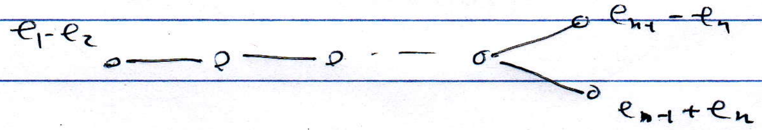
Vinberg–Conway chain terminates just after a supersingular 6-fold with 100 (−2) divisors

3/14/22 (M.)
@ Waseda Univ.

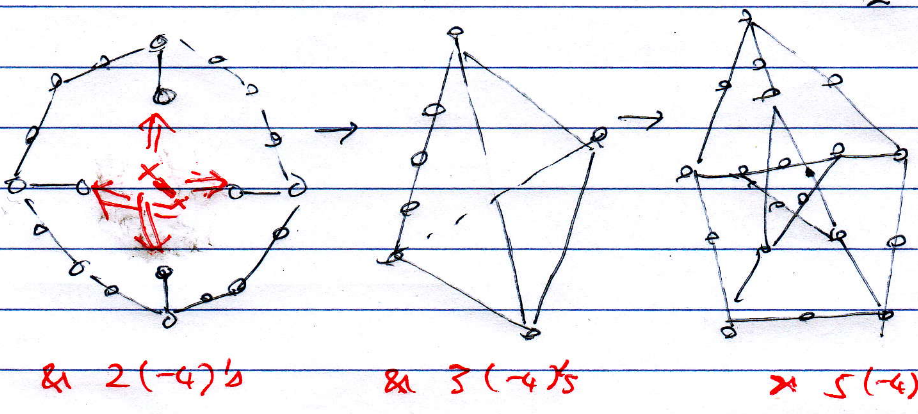
§1

$D_n \subset (\mathbb{Z}^n, \langle , \rangle)$ (index 2) $\quad \sum$ (coefficients) = even



Vinberg–Conway chain $\{(-2)$ vectors, (-4)-vectors$\} \subset U + D_{n-2}$

n	... 18	19	20	21	22	23	24
# of (−2)'s	20	22	25	30	42	100	End
# of (−4)'s	2	3	5	12	56	2	

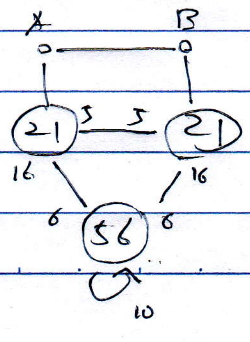


→ (15_3) → (21_5) → HS

δ-cage $\mathbb{P}^2(\mathbb{F}_4)$

Higman–Sims graph, strongly regular w. parameter (100, 22, 0, 6)

Edge $\overset{A}{\circ}$—$\overset{B}{\circ}$ decomposition



Problem Ⓡ Realize $U + D_{n-2}$ as Picard lattice of K3-like object.

Ⓐ Study (semi-) automorphism group using VC-graph.

§ 2

Progress / discovery (2019) Reid sextic

$$R : \sum_{1}^{6} x_i = \sum_{i<j} x_i x_j = \sum_{i<j<k} x_i x_j x_k = 0 \quad \subset \mathbb{P}^5_{(x)}$$

mod p is RDP K3 surface and $\mathrm{Aut}_{\mathbb{F}_p} R_{[p]}$ is computable for $p = 2, 3$ and 5, where $R_{[p]}$ is the min. resolution of R mod P. $p = 2$ is most relevant to today.

Answer to Ⓡ

Picard # n	≤ 20	21	22	23	24
object	$K3/\mathbb{C}$	$K3/\mathbb{F}_2$	$K3/\mathbb{F}_4$	symplectic 6-fold	? OG10 ?
theme	$\mathrm{Aut}_{\mathbb{C}} S$	$\mathrm{Aut}_{\mathbb{F}_2} S$	$S\mathrm{Aut}_{\mathbb{F}_4} S'$	Bir-Aut	

↑ (under 21)

$R_{[2]}$

§3 Picard number $n \leq 20$

$n=20$ $\qquad S_4 / \mathbb{C} \qquad T_S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\mathrm{Pic}\, S_4 = U + D_{18}$

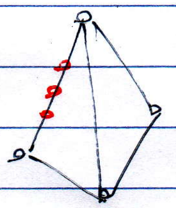
Theorem (Vinberg 1983)

$$1 \longrightarrow C_2 \longrightarrow \mathrm{Aut}\, S_4 \longrightarrow (C_2^{*5} \rtimes \mathfrak{S}_5) \longrightarrow 1$$

$n=19$ $\qquad$ Mirror quartic

$$\overline{M}_\lambda : xyz(x+y+z+\lambda t) + t^4 = 0 \subset \mathbb{P}^3$$

with 6 A_3-singularity and 4 lines. Min. resolution M_λ has $18+4=22$ (-2) $\mathbb{P}^1$'s with



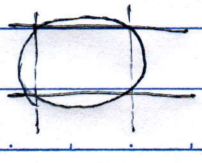
configuration.

Theorem 1 (Answer to Ⓐ) (λ: very general) Aut M_λ is generated by $\mathfrak{S}_4$, 3 Enriques involutions $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and 12 Shioda-Inose involutions. More precisely $\mathrm{Aut}\, M_\lambda = C_2^{*(3+12)} \rtimes \mathfrak{S}_4$.

$n \leq 18$ $\quad$ Easier. Aut S is either finite or Aut S $\sim \mathbb{Z}$ modulo finite groups.

Remark $(n=18)$ S is K3-cover of Barth-Peters Enr.

$(n=19)$ S/ε_i $(i=1,2,3)$ are Enr. of Kondo type I



$\mathbb{P}^1 \times \mathbb{P}^1$

B-P

§4 $n=21$ ($\nexists$ K3 of this Picard number over $k = \bar{k}$)

$$R : \sum_1^6 x_i = \sum' x_i x_j = \sum' x_i x_j x_k = 0$$

R mod 2 $\supset$ 15 lines $\begin{cases} x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 0 \\ x_1 + x_3 + x_5 = 0 \end{cases}$ $(2)^3$

$\Rightarrow$ 15 ODP's (110000) etc. (2)

Min. res. $R_{[2]}$ has 30 (−2) $\mathbb{P}^1$'s with $(15_3 - 15_3)$

configuration. $\mathrm{Pic}_{\mathbb{F}_2} R_{[2]} \simeq U + D_{17}$ (generated by these 30 $\mathbb{P}^1$'s)

(Borcherds 1987) $O^+(U + D_{17}) / \langle (-2)\text{-reflection} \rangle$

$$\cong (\mathfrak{S}_6 . 2) *_{3^2 C_4} 3^2 D_8$$

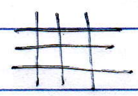
Theorem 2 (Answer to (A))

$$\mathrm{Aut}_{\mathbb{F}_2} R_{[2]} \cong \text{Borcherds' amalgam}$$

Two ingredients of pf. ① Extension of $\mathfrak{S}_6 \curvearrowright R_{[2]}$ to $\mathfrak{S}_6 . 2$

② $R_{[2]}$ is mod 2 reduction of 2nd Vinberg surface

$\mathcal{S}_3$ w. $T_S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, that is, $\exists\, R_{[3]} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, $\tau^3 = x(x-1)y(y-1)$.



§5 $n=22$

(Dolgachev–Kondo 2003) $\begin{matrix} \bar{S} \\ \downarrow \\ \mathbb{P}^2 \end{matrix}$ $\tau^2 = xyz(x^3 + y^3 + z^3)$

Sing. $\bar{S} = \mathbb{P}^2(\mathbb{F}_4)$

21 ODP's.

Min. resolution $S \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$ $\begin{cases} x^2X + y^2Y + z^2Z = 0 \\ xX^2 + yY^2 + zZ^2 = 0 \end{cases}$

S has 42 (-2) $\mathbb{P}^1$'s w. $(21_5 - 21_5)$ configuration.

$$\mathrm{Pic}_{\mathbb{F}_4} S = U + D_{20} \quad \left(\begin{array}{c}\text{generated by} \\ 42\ (-2)\ \mathbb{P}^1\text{'s}\end{array}\right)$$

$$S\mathrm{Aut}(S) = \mathrm{Aut}_{\mathbb{F}_4} S \amalg \left\{ \begin{array}{ccc} S & \xrightarrow{\sim} & S \\ \downarrow & & \downarrow \\ \mathbb{F}_4 & \xrightarrow[\text{Frob.}]{} & \mathbb{F}_4 \end{array} \right\}$$

Theorem 3 (Answer to Ⓐ) $S\mathrm{Aut}(S)$ is generated by $PGL(3, \mathbb{F}_2).2$ and 196 semi-automorphisms associated with hyperovals.

6-set $C = \{p_1, \dots, p_6\} \subset \mathbb{P}^2(\mathbb{F}_4)$ is hyperoval if no 3 is colinear.

Action $PGL(3, \mathbb{F}_4) \curvearrowright \{168 \text{ h.o's}\}$ is transitive.
Decomposes 3 orbits (of length 56) when restricted to $PSL(3, \mathbb{F}_4)$.

§6 $n = 23$ ($\nexists$ K3 with this Picard number)

Find K3-like object with $\mathrm{Pic} \cong U + D_{21}$.

1st approximation. $\mathrm{Pic}\, K3^{[3]} = (\mathrm{Pic}\, K3) \perp \mathbb{Z}\delta$, $\delta^2 = -4$

Hence $\mathrm{Pic}\, (S^{[3]} \text{ in } §5) = U + D_{20} + \langle -4 \rangle \subset U + D_{21}$ (index 2)

Step 1. ∃ 44 (−2) divisors on $S^{[3]}$ with sub-HS-configuration.

☺ $S = (1,2) \cap (2,1) \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$ $h = h_1 + h_2$ $\begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}$

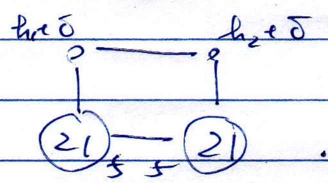
invariant polarization $(h^2) = 14$

Consider $h_i - \delta$ $(i = 1, 2)$.

Then their intersection matrix is $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

So we have an edge $\overset{h_1-\delta}{\circ}$ — $\overset{h_2-\delta}{\circ}$ in $\mathrm{Pic}\, S^{[3]}$.

Moreover, the edge has correct adjacency with $(21_5 - 21_5)$ in $\mathrm{Pic}\, S \subset \mathrm{Pic}\, S^{[3]}$. Namely we have sub-HS configuration



Step 2. ∃ 56 (−2) ⊕-divisors which completes the above to HS graph.

☺ For h.o. $C = \{p_1, \dots, p_6\} \subset \mathbb{P}^2(\mathbb{F}_4)$, put

$$\alpha_C := 2h_1 - \sum_1^6 e_i$$

$\{e_i\}$ are (-2) $\mathbb{P}^1$ over C.

(-4)-element

Choose a $PSL(3, \mathbb{F}_2)$-orbit G of h.o.'s. $|G| = 56$. $\left\{ \frac{\alpha_C + \delta}{2} \right\}_{C \in G}$

satisfies our requirement.

Theorem 4 (Answer to Ⓑ) Fix $C_0 \in G$ and put $\alpha_0 :=$ α_{C_0}. Replace $S^{[3]}$ with moduli of 2-bundles

$$X := M_S(2, \alpha_0, -2)$$

on S. Then X has 100 (-2) divisors w. HS configuration.

Problem Does X has a birational action of the HS simple group?