CUBIC FOURFOLDS WITH ELEVEN CUSPS AND A RELATED MODULI SPACE

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ABSTRACT. First we construct a cubic 4-fold whose singularities are 11 cusps and which has an action of the Mathieu group M_{11}, all over the ternary field $\mathbb{F}_3$. We next consider a certain moduli space of bundles on a supersingular K3 surface of Artin invariant one in characteristic 3. We show that it has 275 (-2) Mukai vectors which form the McLaughlin graph, and ask questions on it and on its relation with our M_{11}-cubic 4-fold.

The classification of finite simple groups singles out 26 sporadic groups. We are interested in realizing some of these very large and complicated groups geometrically, as acting on *K3-like varieties*, namely, a higher dimensional analogue of K3 surfaces, in positive characteristic. In this note we investigate the case of McLaughlin group McL, relating its defining graph with the Fermat quartic surface and a certain cubic 4-fold both in characteristic 3 (cf. Remark 20).

Our model case is the Fermat cubic 4-fold $\sum_1^6 x_i^4 = 0 \subset \mathbb{P}^5_{(x)}$ in characteristic 2. Its automorphism group, that is, the finite unitary group $U_6(2)$, is important in two respects: firstly it extends to the Fisher group Fi_{22} and secondly it contains the Mathieu group M_{22}. We show an analogue of the latter for the smallest Mathieu group M_{11} in our main theorem. We note that M_{11} is one of the maximal subgroups of McL and that neither M_{11} or M_{22} has an action on a cubic 4-fold in characteristic 0 (cf. Remark 12).

Now we start to work over an algebraically closed field in characteristic 3, but varieties are mostly defined over $\mathbb{F}_3$ or $\mathbb{F}_9$. A general inseparable triple covering

$$V \to \mathbb{P}^4_{(x)}, \quad \tau^3 = G(x_0, x_1, x_2, x_3, x_4), \quad \deg G = 3. \tag{1}$$

of the projective 4-space is a cubic 4-fold in $\mathbb{P}^5_{(x\tau)}$ with 11 cusps, i.e., simple singularities of type A_2 (Lemma 6). Among such cubic 4-folds, highly symmetric one is obtained from the Segre cubic 3-fold

$$Seg^3 : \sum_{i=1}^{6} y_i = \sum_{1 \le i < j < k \le 6} y_i y_j y_k = 0 \subset \mathbb{P}^5_{(y)},$$

which has the maximal number (=10) of nodes (e.g., [8]), in the following way:

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Example 1. The inseparable triple covering

$$Seg^4 \to \mathbb{P}^4, \quad \tau^3 = \sum_{1\le i<j<k\le 6} y_iy_jy_k, \quad \text{with} \quad \mathbb{P}^4 : \sum_{i=1}^{6} y_i = 0 \subset \mathbb{P}^5_{(y)}, \tag{2}$$

with formal branch Seg^3 has 10 cusps over its nodes, and one more at $(y : \tau) = (111111 : -1)$. The automorphism group $\mathfrak{S}_6$ of Seg^4 (and also of Seg^3) acts on the 11 cusps with two orbits of length 10 and 1.

A little bit surprisingly there is a *more symmetric* cubic 4-fold with 11 cusps in the sense that the automorphism group, which is M_{11}, acts transitively on the cusps. The following is our main result of this note, and is regarded as a characteristic 3 analogue of the fact that the Fermat cubic 4-fold has an action of the Mathieu group M_{22} over $\mathbb{F}_4$ and the M_{22}-action on a set of 22 planes in it is (triply) transitive ([10], [4, p. 39]):

Theorem 2. *The cubic 4-fold*

$$V : z^3 = \sum_{i\in\mathbb{Z}/5\mathbb{Z}} (x_{i-1}x_ix_{i+1} - x_{i-2}x_ix_{i+2}) \quad \text{in} \quad \mathbb{P}^5_{(xz)} \tag{3}$$

has an action of the Mathieu group M_{11} over $\mathbb{F}_3$ Moreover, V has cusps at 11 $\mathbb{F}_3$-points, on which the M_{11} acts (quadruply) transitively. V is smooth elsewhere.

In §1 we prepare the singularity of purely inseparable covering of the projective space. In §2, we prove our main theorem by simplifying arguments in Adler[1]. Two cubic 4-folds Seg^4 and V in Theorem 2 are closely related with a supersingular K3 surface of Artin invariant one, whose standard projective model is the Fermat quartic surface. Though it does not have an action of M_{11}, there is a chance for a suitable moduli space of bundles over it to have a birational action of M_{11}. In §3, we give an 8-dimensional candidate and ask two questions.

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1. Preliminary

Let V be an n-dimensional smooth hypersurface of degree d over the complex number. Then the primitive Betti number of its middle cohomology $H^n(V)$ is equal to

$$b_n(V)_{pr} = (d-1)J_n^{(d)}$$

by [9, Corollary 1.12], where we put

$$J_n^{(d)} := \frac{1}{d}((d-1)^{n+1} + (-1)^n).$$

The case $d = 3$, that is,

$$J_n = J_n^{(3)} = 1, 1, 3, 5, 11, 21, 43, 85, \dots \quad \text{for} \quad n = 0, 1, 2, 3, 4, 5, 6, 7, \dots$$

is known as the Jacobsthal sequence (OEIS A001045).

$J_n^{(d)}$ is also equal to the top Chern number $c_n(\Omega_{\mathbb{P}}(d))$ of the twisted sheaf of differentials of the n-dimensional projective space $\mathbb{P}^n$ by the exact sequence

$$0 \to \Omega_{\mathbb{P}}(d) \to \mathcal{O}_{\mathbb{P}}(d-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}}(d) \to 0. \tag{4}$$

This number $c_n(\Omega_{\mathbb{P}}(d))$ has the following meaning in algebraic geometry of positive characteristic. Assume that d is a power of a prime p and consider a hypersurface of the *special* form

$$V : \tau^d - G(x) = 0 \subset \mathbb{P}^{n+1}_{(x_0:\ldots:x_n:\tau)}$$

for a homogeneous polynomial G of degree d over an algebraically closed field of characteristic p. This hypersurface is special in the sense that its polar at $(0 : \ldots : 0 : 1)$ is identically zero but not a cone in general. The projection from $(0 : \ldots : 0 : 1)$ induces a purely inseparable covering $\pi : V \to \mathbb{P}^n$ of degree d. So we can say that a hypersurface $V \subset \mathbb{P}^{n+1}$ has $(0 : \ldots : 0 : 1)$ as *its inseparable point*.

The following is obvious.

Lemma 3. *The singular locus of V is bijected by π onto the critical locus of $G(x)$, that is, the common zero locus of all partials $\partial_i G$ of $G(x)$.*

Since $\sum_0^{n+1} x_i \partial_i G = 0$, we have the well-defined differential map

$$d : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d)) \to H^0(\mathbb{P}^n, \Omega_{\mathbb{P}}(d)), \quad G \mapsto dG \tag{5}$$

by (4). The critical locus of G is the zero locus of dG.

Lemma 4. *The unique singular point of V over a critical point $p \in \mathbb{P}^n$ is a simple singularity of type A_{d-1} if and only if $dG \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}}(d))$ has a reduced isolated zero at p.*

Proof. Since the assertion is local and since we can add the d-th power of linear forms freely to G, we may assume that $G(x)$ is of the form $q(x) + (\text{higher order terms})$ in a neighborhood of p. ($\tau^d = G(x)$ and $\tau^d = G(x) + L(x)^d$ define coverings which are isomorphic to each other.) Then dG has a reduced isolated zero at p if and only if the quadratic term $q(x)$ is non-degenerate. Hence we have our lemma. □

The author does not know the general answer of the following:

Question 5. Is the zero locus of dG reduced and of dimension 0 for a general homogeneous polynomial G?

If this holds true, then V has only simple singularity of type A_{d-1} and the number of singular pints is the generalized Jacobsthal number $J_n^{(d)}$. Since the reducedness and 0-dimensionality is an open condition and since we have an example of a good G, say, the Segre cubic or Klein's, we have the following:

Lemma 6. *The inseparable triple covering* (1) *of $\mathbb{P}^4$ has 11 cusps for general cubic $G(x_0, \ldots, x_4)$.*

2. CUBIC 4-FOLDS WITH ACTION OF M_{11} OVER $\mathbb{F}_3$

2.1. **From Klein's to the pentagon-pentagram cubic.** We start our study with Klein's cubic 3-fold

$$U : \sum_{i\in\mathbb{Z}/5\mathbb{Z}} y_i^2 y_{i+1} = 0 \subset \mathbb{P}^4, \tag{6}$$

which is invariant under the transformation $A' = \mathrm{diag}[\zeta, \zeta^9, \zeta^4, \zeta^3, \zeta^5]$ of order 11 and cyclic permutation $B' = y_i \mapsto y_{i+1}$ of order 5, where ζ is a primitive 11-th root of unity, assuming that the base field is of characteristic 3. The minimal polynomial of ζ over $\mathbb{F}_3$ is of degree 5 and there are two possibilities, among which choose $X^5+X^4-X^3+X^2-1$. In studying the automorphism of U the following is crucial:

Lemma 7. *The critical locus of Klein's cubic 3-fold* (6) *consists of the 11 points*

$$P_i\,(\zeta^i : \zeta^{9i} : \zeta^{4i} : \zeta^{3i} : \zeta^{5i}), \quad i \in \mathbb{Z}/11\mathbb{Z}$$

and the inseparable triple covering $V' : \tau^3 = \sum_{i\in\mathbb{Z}/5\mathbb{Z}} y_i^2 y_{i+1} \subset \mathbb{P}^4$ has 11 simple singularities of type A_2 over them.

Proof. By Lemma 3, the singular locus of V' is bijected onto the common zero locus of partials $-y_{i-1}^2 + y_i y_{i+1} = 0$ for $i \in \mathbb{Z}/5\mathbb{Z}$ and hence consists of the 11 points above. Since $J_4 = 11$, our claim follows from and from Lemma 4. □

We make the following change of coordinates of $\mathbb{P}^4$

$$\begin{aligned}
y_0 &= \zeta x_0 + \zeta^9 x_1 + \zeta^4 x_2 + \zeta^3 x_3 + \zeta^5 x_4\\
y_1 &= \zeta^9 x_0 + \zeta^4 x_1 + \zeta^3 x_2 + \zeta^5 x_3 + \zeta x_4\\
y_2 &= \zeta^4 x_0 + \zeta^3 x_1 + \zeta^5 x_2 + \zeta x_3 + \zeta^9 x_4\\
y_3 &= \zeta^3 x_0 + \zeta^5 x_1 + \zeta x_2 + \zeta^9 x_3 + \zeta^4 x_4\\
y_4 &= \zeta^5 x_0 + \zeta x_1 + \zeta^9 x_2 + \zeta^4 x_3 + \zeta^3 x_4
\end{aligned} \tag{7}$$

so that the six critical points P_1, P_9, P_4, P_3, P_5 and P_0 become the five coordinate points and $(-1 - 1 - 1 - 1 - 1)$, respectively. In this new coordinate system (x), Klein's cubic (6) is defined by

$$U : \sum_{i\in\mathbb{Z}/5\mathbb{Z}} (-x_i^3 + x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2}) = 0. \tag{8}$$

The cyclic group $\langle B' \rangle$ of order 5 in (y)-coordinate is generated by $B : x_i \mapsto x_{i+1}$ in our new (y)-coordinate. The transformation A' of order 11 becomes

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 & 1\\ -1 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 0 & -1\\ 1 & -1 & 1 & 0 & -1 \end{pmatrix},$$

in (y)-coordinate by computation. In particular, it is defined over $\mathbb{F}_3$. (This is not surprisng because A' induces a permutation of critical points all of which are defined over $\mathbb{F}_3$.)

Now we are ready to explain that U and V' have extra automorphisms. Firstly the permutation

$$x_1 \leftrightarrow x_4, \quad x_2 \leftrightarrow x_3$$

of type $(2)^2$ is an automorphism of U since it preserves the pentagon supporting $\sum_{i\in\mathbb{Z}/5\mathbb{Z}} x_{i-1}x_ix_{i+1}$ and also the pentagram supporting $\sum_{i\in\mathbb{Z}/5\mathbb{Z}} x_{i-2}x_ix_{i+2}$ in (8). Together with the cyclic group $\langle B\rangle$, this involution generates a dihedral group D_{10} of order 10.

Secondly what is more crucial in characteristic 3 is to consider the cyclic permutation

$$x_1 \mapsto x_2 \mapsto x_4 \mapsto x_3 (\mapsto x_1)$$

of type (4) whose square is the involution above. Since this permutation interchanges the pentagon and pentagram above, we consider the signed permutation

$$C : x_1 \mapsto -x_2 \mapsto x_4 \mapsto -x_3 (\mapsto x_1)$$

instead and observe the following:

Lemma 8. *The pentagon-pentagram cubic form* (8) *is preserved by the linear transformations A and B. It is not preserved by C but transformed under C to*

$$\sum_{i\in\mathbb{Z}/5\mathbb{Z}} (x_i^3 + x_{i-1}x_ix_{i+1} - x_{i-2}x_ix_{i+2}). \tag{9}$$

In particular, it is invariant under the action of $\langle A, B, C\rangle$ modulo cubes of linear forms.

Remark 9. Similar claims, especially the last one, in Lemma 8 were first found in Adler[1, Lemma 3.1] for Klein's cubic.

2.2. **Proof of Theorem 2.** In order to eliminate the modulo cubes ambiguity, we introduce a new independent variable τ and consider the cubic 4-fold

$$V : \tau^3 + \sum_{i\in\mathbb{Z}/5\mathbb{Z}} (-x_i^3 + x_{i-1}x_ix_{i+1} - x_{i-2}x_ix_{i+2}) = 0 \tag{10}$$

in $\mathbb{P}^5$, or equivalently, the cubic 4-fold in Theorem 2 by change of variables $\tau = -z + \sum_{i\in\mathbb{Z}/5\mathbb{Z}} x_i$. We extend the action of A, B, C to $\mathbb{P}^5$ by

$$\tilde{A} : \tau \mapsto \tau, \quad \tilde{B} : \tau \mapsto \tau \quad \text{and} \quad \tilde{C} : z \mapsto z - \sum_{i\in\mathbb{Z}/5\mathbb{Z}} x_i.$$

By Lemma 8, this action of $\tilde{A}, \tilde{B}$ and $\tilde{C}$ preserves V.

In our (xz)-coordinate system, the singularity of V locates at 11 points

$$(x_0 : \ldots : x_4 : z) =$$

$$(10000; 0), (01000; 0), (00100; 0), (00010; 0), (00001; 0), (-1-1-1-1-1; 0), \tag{11}$$
$$(01-1-11; 1), (101-1-1; 1), (-1101-1; 1), (-1-1101; 1), (1-1-110; 1),$$

which are the 11 points $1, 2, 3, 4, 5, 6$ and a, b, c, d, e in the notation of [15, p. 406]. By Coxeter-Todd ([6], [15]), the automorphism group of the 12-pointed projective space

$$(\mathbb{P}^5; 1, 2, \dots, 6, a, b, \dots, f)$$

is known to be the Mathieu group M_{12}, where we put $f(00000 : 1)$. Furthermore the permutation action of M_{12} on the 12 points is quintuply transitive.

Lemma 10. *An automorphism of V preserves the point $f(00000 : 1)$.*

Proof. As we saw in §1, the point f is an inseparable point of $V \subset \mathbb{P}^5$. It suffices to show there are no other inseparable points. This is obvious since the five partials $-y_{i-1}^2 + y_i y_{i+1}$ of Klein's cubic are linearly independent. □

An automorphism of V induces a permutation of its singular locus. Hence, by the lemma, the automorphism group of $V \subset \mathbb{P}^5$ is contained in M_{11}, the stabilizer of M_{12} at f. The following completes our proof of Theorem 2.

Lemma 11. *The three linear transformations $\tilde{A}, \tilde{B}$ and $\tilde{C}$ generate M_{11} in $PGL(6, \mathbb{F}_3)$.*

Proof. Le $G \subset M_{11}$ be the subgroup generated by $\tilde{A}, \tilde{B}$ and $\tilde{C}$. Since $\tilde{A}, \tilde{B}, \tilde{C}$ are of order 11, 5, 4, the order of G is divisible by 220. By the classification of maximal subgroups of $\mathfrak{S}_{11}$ (e.g., [4]), G is isomorphic to either M_{11} or $L_2(11)$. The latter is impossible since the subgroup $\langle \tilde{B}, \tilde{C} \rangle \subset G$ is the semi-direct product $5 : 4$ or $\mathrm{Hol}(C_5)$ by our construction but $L_2(11)$ does not contain such a semi-direct product. □

Remark 12. (1) The cubic 4-fold $\tau^3 - \sum_{i \in \mathbb{Z}/5\mathbb{Z}} y_i^2 y_{i+1} = 0 \subset \mathbb{P}^5_{(\tau y)}$ is also interesting over the complex number field $\mathbb{C}$ in the sense that its automorphism group $L_2(11)$ is maximal among all finite groups with a symplectic action on a smooth cubic 4-fold ([12]). Similar holds for for Klein's cubic 3-fold $\sum_{i \in \mathbb{Z}/5\mathbb{Z}} y_i^2 y_{i+1} = 0 \subset \mathbb{P}^4_{(y)}$ ([16]).

(2) The stabilizer group M_{10} of the standard permutation action of M_{11} is also maximal among all finite groups with a symplectic action on a smooth cubic 4-fold ([12]).

3. Conjectural symplectic 8-fold as moduli of bundles on Fermat quartic

3.1. **Two questions.** The Fermat quartic surface $Fer_4 : \sum_1^4 x_i^4 = 0 \subset \mathbb{P}^3_{(x)}$ has an action of the finite unitary group $PGU_4(3)$. The action of a subgroup of index 4, namely, of $U_4(3) := PSU_4(3)$ is symplectic. Though $U_4(3)$ does not contain M_{11} as a subgroup, the moduli space $\overline{M}_{Fer}(v)$ of (semi-)stable sheaves on the Fermat quartic Fer_4 might have a birational action of M_{11}, or even a much larger finite simple group, for suitable Mukai vector $v = (r, *, s) \in \mathbb{Z} \oplus \mathrm{Pic} \oplus \mathbb{Z}$. A hopeful candidate, in view of symmetry of the Leech lattice, is 8-dimensional, i.e., $\langle v^2 \rangle = 6$, and the group containing M_{11} should be the McLaughlin group McL.

Question 13. Does the moduli space $\overline{M}_{Fer}(3, \alpha, -3)$ have a birational action of McL, where α is a (-12)-divisor class attached to Segre's hemisystem (see §3.2)?

Remark 14. As is explained e.g. in [4, p. 100], the McL is the pointwise stabilizer of a triangle ABC of type 322 in the Leech lattice Λ. This means that the orthogonal complement L of a negative root lattice $\simeq A_1 + A_2$ in $U + \Lambda(-1)$ has an action of McL. Since the Picard lattice of Fer_4 has a primitive embedding into L and its orthogonal complement is generated by an element of norm -6, it is natural to seek the possibility above, namely $\langle v^2 \rangle = 6$ and 8-dimensioonal.

McL contains the simple groups $U_4(3)$ and M_{11} as maximal subgroups, and hence it is generated by these two subgroups. The action of the former on the moduli space is not surprising since its $\mathbb{Q}$-twisted expression is $\overline{M}_{Fer}(3, 0, -1)$ (Proposition 18). Seeking after an action of the latter, we pose the following

Question 15. Is $\overline{M}_{Fer}(3, \alpha, -3)$ birational to the conjectural LLSvS 8-fold associated with the M_{11}-cubic 4-fold V?

Remark 16. The LLSvS 8-fold in the question is conjectural since it is constructed in [13] only for smooth cubic 4-folds over $\mathbb{C}$ which does not contain a plane. Our M_{11}-cubic 4-fold has 11 cusps, defined in characteristic 3 and the author does not know whether it contains a plane or not.

3.2. **Segre's hemisystem and the McLaughlin graph in a Picard lattice.** The Fermat quartic surface Fer_4 has 280 $\mathbb{F}_9$-(rational) points, with weight distribution 2: 24, 3: 64 and 4: 192. For each $\mathbb{F}_9$-point p, the tangent plane T_p cuts out the union of 4 lines passing through p from Fer_4. Since every line has ten $\mathbb{F}_9$-points, the number of lines in Fer_4 is $280 \times 4/10 = 112$. The Picard lattice is generated by these line classes. Its discriminant group $\mathrm{Disc}(Fer_4)$ is isomorphic to $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ (see e.g. [11]).

Segre's hemisystem is a set H of 56 lines, among the 112, which covers $Fer_4(\mathbb{F}_9)$ doubly, that is, every $\mathbb{F}_9$-point is contained in exactly two members of H. There are 648 hemisystems and they are divided into 4 orbits of length 162 by the action of $U_4(3)$. These 4 orbits corresponds to the four elements of norm $2/3$ modulo $2\mathbb{Z}$ in the discriminant group $\mathrm{Disc}(Fer_4)$ as we will see below. We choose one of them. Then the intersection size $|H \cap H'|$ of two among our 162 hemisystems are either 20 or 32 ([3, §10.34]).

Proposition 17. ([3, §10.61]) *The graph with the following three types of vertices and a suitable adjacency is a strongly regular graph $srg(275, 112, 30, 56)$, isomorphic to the McLaughlin graph: (i) ∞, (ii) the 112 lines in Fer_4 and (iii) the 162 hemisystems. (See* [7, §7] *for the adjacency.)*

We realize this graph inside the extended Picard lattice $U(-1) \oplus \mathrm{Pic}\, Fer_4$ of the Fermat quartic surface, or more precisely, in the sublattice $(3, \alpha, -3)^{\perp}$, which is expected to be the Picard lattice of the conjectural moduli symplectic 8-fold ([14], [17], [18] but only over $\mathbb{C}$). Here U denotes the standard hyperbolic lattice of rank 2. The intersection pairing $(D.D')$ on the Picard lattice extends to the orthogonal sum $\mathbb{Z} \oplus \mathrm{Pic} \oplus \mathbb{Z}$ obviously but with changing the sign of U, namely,

$$\langle (r, D, s), (r', D', s') \rangle = -rs' + (D.\, D') - sr', \quad (r, s), (r', s') \in U(-1). \tag{12}$$

Now we define a divisor class for a hemisystem H. Consider the sum $\sum_{m \in H} m$ of its all members in the Picard grup $\mathrm{Pic}\, Fer_4$. Then we have

$$(\sum_{m \in H} m.\, l) = \begin{cases} 8 & \text{if } l \in H, \\ 20 & \text{otherwise.} \end{cases} \tag{13}$$

In particular, $\sum_{m\in H} m$ is divisible by 4 in the Picard group. So we define

$$\alpha_H := 2h - \frac{1}{4}\sum_{m\in H} m \in \operatorname{Pic} Fer_4,$$

where h is the hyperplane section class of Fer_4. Since $(\alpha_H.l)$ is divisible by 3 for all lines l, $\alpha_H/3$ defines an element in the discriminant group, whose norm is $2/3$ since $(\alpha_H^2) = -12$.

Proposition 18. *The graph on the following three types of (-2)-vectors in $(3,0,-1)^\perp \otimes \mathbb{Q}$, adjacent when non-orthogonal, is isomorphic to the McLaughlin graph:*

- $(3,h,1)$,
- $(0,l,0)$ *for the 112 lines l in Fer_4 and*
- $(1,-\frac{\alpha_H}{3},\frac{1}{3})$ *for the 162 hemisystems H chosen as above.*

Proof. We just check adjacencies here and that in [7, §7] are the same. For example, $(3,h,1)$ has inner product 1 with $(0,l,0)$ and hence they are adjacent in $(3,0,-1)^\perp \otimes \mathbb{Q}$. The corresponding ∞ and all 112 lines are adjacent in [7, §7] by definition. Other cases are similar but tedious and we omit it. (The MOG computation in [19, §5.5.2] may be useful for a better proof.) □

Geometrically, these are the Mukai vectors of the rank 3 bundle $T_{\mathbb{P}^3}(-1)$ restricted to Fer_4, torsion sheaves supported on lines and $\mathbb{Q}$-line bundles on Fer_4, respectively.

Now we fix a hemisystem F among our 162, put $\alpha = \alpha_F$ and take twist by tensor product of the $\mathbb{Q}$-line bundle $\mathcal{O}_{Fer}(\frac{\alpha}{3})$. Then all the vertices in the proposition become integral. Since the the tensor of a line bundle preserves the inner product (12), we have

Corollary 19. *The graph on the following three types of (-2) Mukai vectors in $(3,\alpha,-3)^\perp$, adjacent when non-orthogonal, is isomorphic to the McLaughlin graph:*

- $(3,h+\alpha,-3)$,
- $(0,l,*)$ *for the 112 lines l in Fer_4 and*
- $(1,\frac{\alpha-\alpha_H}{3},**)$ *for the 162 hemisystems H,*

where $*$ is equal to 0 if $l \in F$ and 1 otherwise, and $**$ is equal to $1,-1,-2$ according as $H = F, |H\cap F| = 20$ and $|H \cap F| = 32$.

Proof. $\alpha-\alpha_H$ is divisible by 3 since both $\alpha/3$ and $\alpha_H/3$ defines the same element in $\operatorname{Disc}(Fer_4)$. Hence the vertices are Mukai vectors of a rank 3 bundles, torsion sheaves and the 162 line bundles $\mathcal{O}_{Fer}(\frac{\alpha-\alpha_H}{3})$. □

Remark 20. Two more strongly regular graphs are similarly realized by taking (-2) Mukai vectors as their vertices in characteristic 2 and 5, which will be discussed elsewhere.

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