Enriques surfaces and root lattices

— Enriques surfaces of type E_7 —

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Enriques surface $S = X/\varepsilon =$ (K3 surface)/ (free involution)

$\mathbf{Z}^\omega := (\mathbf{Z} \times X)/(-1, \varepsilon) \to S$ nontrivial local system on S

$H := H_S := H^2(S, \mathbf{Z}^\omega) \simeq \mathbf{Z}^{12}$: Hodge structure of weight 2, that is, $H_S \otimes_{\mathbf{Z}} \mathbf{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$. Hodge # $= (1, 10, 1)$

H carries an integral symmetric bilinear form induced by $\mathbf{Z}^\omega \times \mathbf{Z}^\omega \to \mathbf{Z}$. As a lattice, $H \simeq I_{2,10}$, odd unimodular lattice with signature $(2, 10)$. H_S is a polarized Hodge structure.

Twisted Picard group $\mathrm{Pic}^\omega S := H^2(S, \mathbf{Z}^\omega) \cap H^{1,1}$

is a negative definite lattice which does not contain (-1) elements.

Relation with traditional formulation

$H^2(S, \mathbf{Z}^\omega) = \mathrm{Ker}[H^2(X, \mathbf{Z}) \longrightarrow H^2(S, \mathbf{Z})(\simeq \mathbf{Z}^{12} \oplus \mathbf{Z}/2)]$

$\mathrm{Pic}^\omega S = \mathrm{Ker}[\mathrm{Pic}\, X \longrightarrow \mathrm{Pic}\, S](1/2)$, twisted Picard # $= \rho(X) - 10$.

Torelli type theorem (in new formulation)

S, S': two Enriques surfaces.

$H^2(S, \mathbf{Z}^\omega) \simeq H^2(S', \mathbf{Z}^\omega)$ as polarized Hodge structures $\Rightarrow S \simeq S'$.

In other words,

$\{$Enriques surface$\}$ / isom. $\longrightarrow D^{10}/O(I_{2,10}), \quad S \mapsto H_S,$

is injective (and almost surjective).

Inverse Problem: (Re)construct S from its period H_S, or KS(H_S), the Kuga-Satake abelian variety of dimension 2^{10}.

Two parts: a) Construct the K3-cover $X = \tilde{S}$.

b) Construct the free involution ε.

Today I answer in the case of type E_7. In this case KS(H_S) is isogeneous to the self product A^{2^8} for an abelian surface A. Still both a) and b) are non-trivial.

Definition Let L be a negative definite lattice which does not contain a (-1) element. An Enriques surface S is of (lattice) type L if there exists a primitive embedding $L \longrightarrow \mathrm{Pic}^{\omega} S$.

Assume

(*) the primitive embeddings of L into $I_{2,10}$ is unique

and let $M := L^{\perp}$ be the orthogonal complement.

By Torelli, the period map

$\{$Enriques surface of type $L\}/L$-isom. $\longrightarrow D^{m-2}/O(M)'$,

$S \mapsto$ Hodge structure on M,

is injective, where m is the rank of M,

$D^{m-2} = \{z \in M \otimes \mathbf{C} \,|\, (z,z) = 0,\ (z,\overline{z}) > 0\}$

is the $(m-2)$-dimensional bounded symmetric domain of type IV, on which the orthogonal group $O(M)$ acts, and $O(M)'$ is the image of $O(I_{2,10}, L) \to O(M)$.

I consider the case where $L = E_7$. E_7 satisfies (*) and the orthogonal complement is $< 1, 1, -1, -1, -2 >$, that is, $\mathbf{Z}^5$ with inner product diag$[1, 1, -1, -1, -2]$. Moreover, $O(I_{2,10}, E_7) \longrightarrow O(< 1, 1, -1, -1, -2 >)$ is surjective. Hence the period map

$$\{\text{Enriques of type } E_7\}/\text{isom.} \longrightarrow D^3/O(< 1, 1, -1, -1, -2 >)$$

is injective.

Lemma (1) D^3 is the Siegel upper half space H_2 of degree 2, and $D^3/O(< 1, 1, -1, -1, -2 >)$ is the quotient of $H_2/\Gamma_0(2)$ by the Fricke (or Atkin-Lehner) involution.

(2) $H_2/\Gamma_0(2)$ is the moduli space of pairs (A, G) of a principally polarized abelian surface A and a Göpel subgroup $G \subset A_{(2)}$.

(3) The Fricke involution maps (A, G) to $(A', G') := (A/G, A_{(2)}/G)$.

Inverse Problem for E_7

Construct an Enriques surface S of type E_7 from (A, G) corresponding to the period of S.

Enriques surfaces of type E_7, E_8 and $(E_7 + A_1)^+$

Consider the quartic surfaces in $\mathbf{P}^3_{x:y:z:t}$

$$\{a(xt + yz) + b(yt + xz) + c(zt + xy)\}^2 - xyzt = 0.$$

for nonzero constants $a, b, c \in \mathbf{C}$. This is a K3 surface with 4 rational double points of type D_4. Let X be the minimal resolution. Standard Cremona transformation

$$(x : y : z : t) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{t})$$

induces an involution ε of X, which is free if $\pm a \pm b \pm c \neq 1/2$.

Projecting from (0001), we get a double plane expression

$$X : \tau^2 = p(\frac{x}{z} + \frac{z}{x}) + q(\frac{y}{z} + \frac{z}{y}) + r(\frac{x}{y} + \frac{y}{x}) + s$$

for constants $p, q, r, s \in \mathbf{C}$ with $pqr \neq 0$.

Theorem (1) The quotient of a quartic $\{a(xt + yz) + b(yt + xz) + c(zt + xy)\}^2 - xyzt = 0$ by Cremona is an Enriques of type E_7 (if $\pm a \pm b \pm c \neq 1/2$).

(2) Every Enriques S of type E_7 is obtained in this way.

(3) The coefficient $(p : q : r : s)$ is explicitly determined from the period H_S explicitly.

Remark on $\{a(xt+yz)+b(yt+xz)+c(zt+xy)\}^2 - xyzt = 0$

(1) In characteristic 2, this is the equation of the Jacobian Kummer surface Km(Jac (C)) (Laszlo-Pauly).

(2) This K3 is not a Jacobian Kummer but *isogeneous* to it.

(3) This deforms to the double covering of the quadric $Q : a(xt+yz)+b(yt+xz)+c(zt+xy) = 0$ with branch $Q \cap \{xyzt = 0\}$. This is a product Kummer surface Km $(E_1 \times E_2)$ and its quotient by Cremona is an Enriques surface of type $(E_7 + A_1)^+$.

(4) An Enriques surfrace becomes of type E_8 if one of p, q, r becomes 0 in the double plane expression

$$\tau^2 = p(\frac{x}{z} + \frac{z}{x}) + q(\frac{y}{z} + \frac{z}{y}) + r(\frac{x}{y} + \frac{y}{x}) + s.$$

How to recover S from H_S

(A, G) the pair corresponding to H_S

A is a p.p.a.s. and $G = \{0, a, b, c\} \subset A_{(2)}$ is a Göpel.

I consider the case where A is the Jacobian of a curve C of genus 2.
(When A is product, then S is E_8-type.)

C is a double cover of $\mathbf{P}^1$ with 6 points $P_1, \ldots, P_6$. A 2-torsion point $a \neq 0 \in A_{(2)}$ is $\tilde{P}_{i(a)} + \tilde{P}_{j(a)} - K_C$ for different $i(a), j(a)$.
Regard $\mathbf{P}^1$ as a conic Q in $\mathbf{P}^2_{u:v:w}$. Let $l_{i(a)} + l_{j(a)} : q_a(u, v, w) = 0$ be the sum of two tangent lines of Q at $P_{i(a)}, P_{j(a)}$.

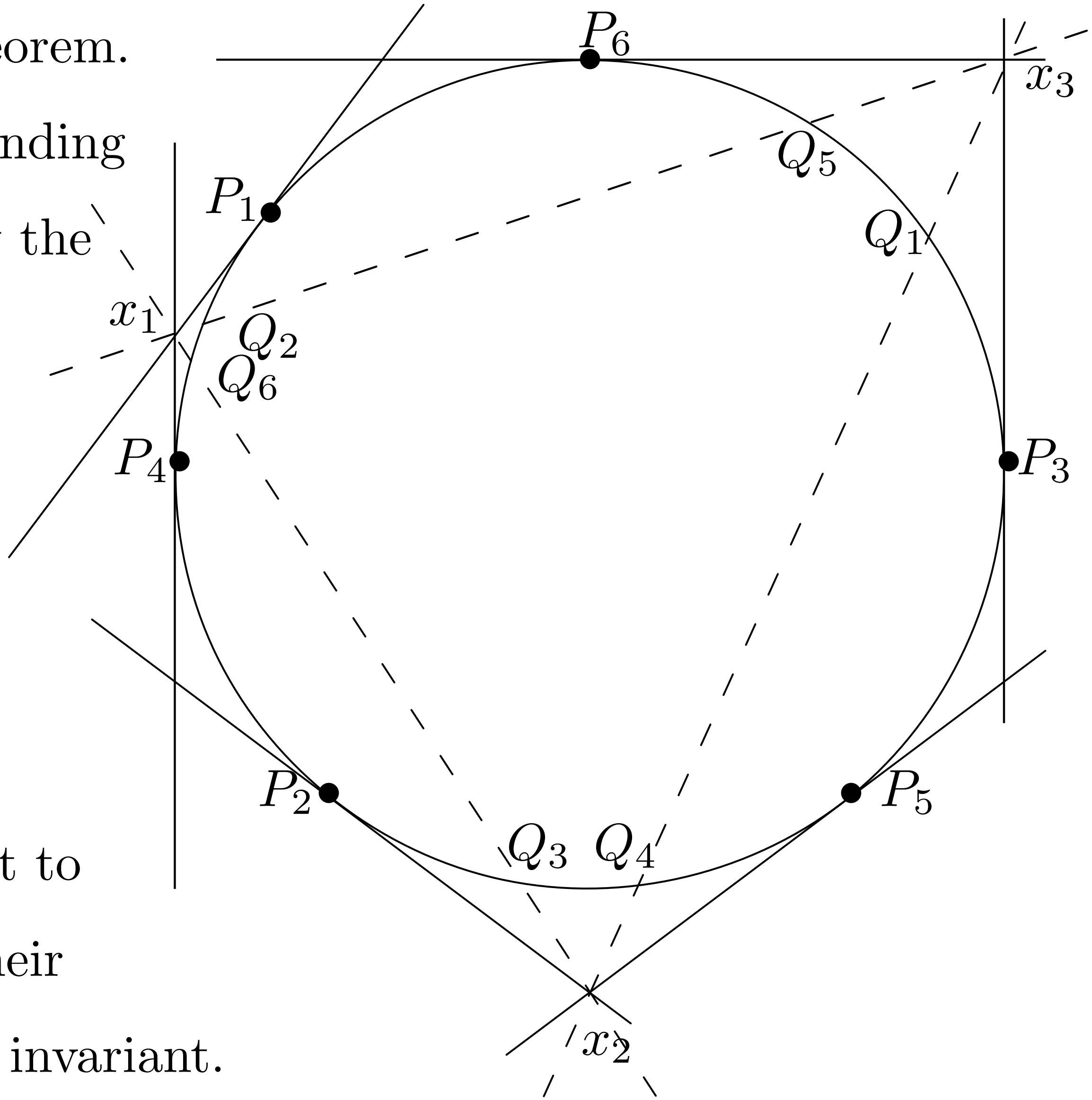
Then $\tilde{S}$ is the double plane

$$\tau^2 = \det(xq_a + yq_b + zq_c)/xyz$$

and S is its quotient by $(x, y, \tau) \mapsto (1/x, 1/y, -\tau)$.

Remark (1) G is Göpel $\Leftrightarrow \{i(a), j(a), \ldots, j(c)\} = \{1, \ldots, 6\}$.

Remark (2) (A, G) and $(A/G, A_{(2)}/G)$ give the same S by Richelot's theorem. 6 points $Q_1, \ldots, Q_6$ corresponding to $(A/G, A_{(2)}/G)$ is given by the diagram.



$Q_1, \ldots, Q_6$ are not equivalent to $P_1, \ldots, P_6$ in general. But their nets of conics have the same invariant.

Proof of Theorem

$\mathrm{Km}(A/G)$ is the double $\mathbf{P}^2_{u:v:w}$ with equation $\tau^2 = q_a q_b q_c$.

$\mathrm{Km}(A)$ is the $(2,2,2)$-covering of $\mathbf{P}^2_{u:v:w}$ with equation $\tau_1^2 = q_a,\ \tau_2^2 = q_b,\ \tau_3^2 = q_c$. This is a complete intersection of three quadrics in $\mathbf{P}^5$.

Double plane

$$\tau^2 = \det(xq_a + yq_b + zq_c)/xyz$$

is the moduli space of 2-bundles on $\mathrm{Km}(A)$ with Mukai vector $(2,h,2)$. Its period is $v^\perp/\mathbf{Z}\cdot v$. By computation this is the same as the period of $\tilde{S}$. By Torelli, the double plane is isomorphic to $\tilde{S}$.

Remark Fixed point condition $\pm a \pm b \pm c \neq 1/2 \Leftrightarrow A$ has a real $\sqrt{2}$-multiplication $\varphi \in \mathrm{Aut}\ A$, $\varphi^2 = 2$, and $G = \mathrm{Ker}\ \varphi$.

Enriques surfaces of type $(D_6 + A_1)^+$

$D_6 + A_1$ has two type of primitive embeddings into $I_{2,10}$. One has odd orthogonal complement and the other even one. The latter embedding is denoted by $(D_6 + A_1)^+$.

Let $q(u, v, w) = 0$ be a smooth plane conic and consider the quartic surface $q(xt + yz, yt + xz, zt + xy) + xyzt = 0$.

This is Kummer's quartic surface Km (Jac C) with 16 nodes. 4 nodes at the coordinate points form a Göpel subgroup G of the Jacobian. Standard Cremona transformation induces a free involution and we obtain an Enriques surface (Km Jac C)$/\varepsilon$.

Theorem (1) (Km Jac C)$/\varepsilon$ is an Enriques surface of type $(D_6 + A_1)^+$.

(2) Every Enriques surface S of type $(D_6 + A_1)^+$ is obtained in this way or of type $(E_7 + A_1)^+$ or $(D_8)^+$. (In the latter two cases, the K3-cover $\tilde{S}$ is Km $(E_1 \times E_2)$.)