Finite and infinite generation of the Nagata invariant rings

Dedicated to Professor Masaki Maruyama on his 60th Birthday

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An m-dimensional linear representation of a group induces an action on the polynomial ring $\mathbf{C}[z_1, \dots, z_m]$ of m variables. This is called a *linear action* on the polynomial ring. In 1890, Hilbert[H] showed that the invariant ring was finitely generated for classical representations of the special linear groups. The following is known as his original fourteenth problem (see [N2]):

Problem 1 Is the invariant ring $\mathbf{C}[z_1, \dots, z_m]^G$ of a linear action of an algebraic group G finitely generated?

The answer is affirmative for the additive algebraic group $\mathbf{G}_a$ (Theorem of Weitzenböck [Se]). In 1958, Nagata[N1] considered the standard unipotent linear action

$$(t_1, \dots, t_n) \in \mathbf{C}^n \curvearrowright \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_n] =: S, \quad \begin{cases} x_i \mapsto x_i \\ y_i \mapsto y_i + t_i x_i \end{cases}, \quad 1 \le i \le n, \tag{1}$$

of $\mathbf{C}^n$ on the polynomial ring S of $2n$ variables and showed that the invariant ring S^G with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension 3 was not finitely generated for $n = 16$. Since then the problem has been studied in the following form:

Metaproblem Search a *good* condition on a linear representation $G \curvearrowright V$ for the invariant ring $\mathbf{C}[V]^G$ to be (in)finitely generated.

In this article, we shall answer this problem for the Nagata action:

Theorem *The invariant ring S^G of* (1) *with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension r is infinitely generated if and only if*

$$\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} \le 1. \tag{2}$$

In particular, S^G is infinitely generated if $\dim G = s \ge 3$ and if $n \ge s^2/(s-2)$. So the answer to Problem 1 is negative for $\mathbf{G}_a^3$. But the following part is still open:

Problem 2 Is the invariant ring $\mathbf{C}[z_1, \dots, z_m]^G$ of a linear action of the 2-dimensional additive group $G = \mathbf{G}_a \times \mathbf{G}_a$ finitely generated?

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For non-linear actions, there is an example of $\mathbf{G}_a$-action, due to Roberts [R], whose invariant ring is infinitely generated.

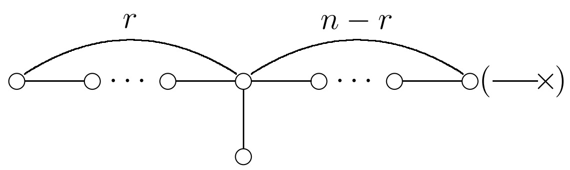
Our proof of the theorem is based on the fact that the invariant ring S^G is a certain Rees algebra (§1). In geometric term, the Rees algebra is isomorphic to the *total coordinate ring*, or the Cox ring, $\mathcal{TC}(X_G)$ of the blow-up X_G of the projective space $\mathbf{P}^{r-1}$ at n points (§2). More precisely, the projective space is $\mathbf{P}_*(\mathbf{C}^n/G)$ and the n points, denoted by $p_1, \ldots, p_n$, are the images of the standard basis of $\mathbf{C}^n$. This ring $\mathcal{TC}(X_G)$ is graded by the Picard group $\operatorname{Pic} X_G \simeq \mathbf{Z}^{n+1}$ and its support is $\operatorname{Eff} X_G$, the semi-group of effective classes on X_G. Hence $\mathcal{TC}(X_G)$ is not finitely generated if $\operatorname{Eff} X_G$ is not so as semi-group (Lemma 2).

The simplest case is

$$G = \left\{ (t_1, \ldots, t_9) \middle| \sum_{i=1}^{9} t_i = \sum_{i=1}^{9} \wp(c_i) t_i = \sum_{i=1}^{9} \wp'(c_i) t_i = 0 \right\} \subset \mathbf{C}^9, \tag{3}$$

where $\wp(z)$ is Weierstrass's $\wp$-function of an elliptic curve $C = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ and $c_1, \ldots, c_9$ are distinct points of C. In this case, X_G is the blow-up of $\mathbf{P}^2$ at the nine points $(1 : \wp(c_i) : \wp'(c_i))$, $1 \le i \le 9$. Assume that the sum $\sum_{i=1}^{9} c_i \in C$ is zero, for simplicity. Then the nine points are the intersection of two cubics, X_G has an elliptic fibration $f : X_G \to \mathbf{P}^1$ and the nine exceptional curves are sections of f. If the difference $c_i - c_{i+1}$ is of infinite order for some $1 \le i \le 8$, then there are infinitely many exceptional curves of the first kind (cf. [N3]). So S^G is not finitely generated. (Cf. Remark 1 at the end of §4.)

The proof of the 'if' part of the theorem (§4) is similar but we replace the elliptic fibration by the symmetry of $\operatorname{Pic} X_G$ with respect to the Weyl group of the Dynkin diagram $T_{2,r,n-r}$ with n vertices (§3):



(4)

which was introduced in Dolgachev[D]. As is well known the inequality (2) is equivalent to the infiniteness of the Weyl group of this diagram (Lemma 4). If $G \subset \mathbf{C}^n$ is general and if (2) is satisfied, then there exist infinitely many exceptional divisors on X_G. Therefore, $\operatorname{Eff} X_G$ and hence $\mathcal{TC}(X_G)$ are not finitely generated (Lemma 3).

The 'only if' part is proved case by case. [1] There are four infinite series [1]–[4] and five exceptional cases [5]–[9] for which $1/2 + 1/r + 1/(n-r) > 1$ holds:

	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
Cartan's symbol			BDII	DIII	EIII	EVII	EVI	EIX	EVIII
r	1		2		3	3	4	3	5
$n-r$		1		2	3	4	3	5	3
diagram	A_n	A_n	D_n	D_n	E_6	E_7	E_7	E_8	E_8

[1]But it is worth to mention that the condition $1/2 + 1/r + 1/(n-r) > 1$ is equivalent to that X_G is isomorphic to a Fano variety in codimension one.

In the cases [1] and [3], the invariant ring is very explicit and the proof is immediate (Examples 1 and 2 in §1). The case [2] is classical and the invariant ring S^G is the homogeneous coordinate ring of the Grassmannian variety $G(2, n+1)$. In the case $r = 3$, X is a del Pezzo surface and the theorem follows from [BP].

In the remaining cases, we make use of the fact that X_G is the moduli spaces of certain vector bundles. Note that $G \subset \mathbf{C}^n$ and the standard basis determine the n points $q_1, \dots, q_n$ on the projective space $\mathbf{P}_* G \simeq \mathbf{P}^{s-1}$ also, where we put $s := \dim G$. We reduce the finite generation of $\mathcal{TC}(X_G)$ to a geometry of the n-pointed projective space $(\mathbf{P}^{s-1}; q_1, \dots, q_s)$, which is the *Gale transform* of $(\mathbf{P}^{r-1}; p_1, \dots, p_s)$ ([DO, III], [EP]). Let $I_{q_1,\dots,q_n} \subset \mathcal{O}_\mathbf{P}$ be the ideal sheaf of the set of n points $\{q_1, \cdots, q_n\} \subset \mathbf{P}^{s-1}$. Then we obtain a family of exact sequences of coherent sheaves of $\mathcal{O}_\mathbf{P}$-modules

$$\mathbf{E}_x : 0 \longrightarrow \mathcal{O}_\mathbf{P}(1) \otimes I_{q_1,\dots,q_n} \longrightarrow E_x \xrightarrow{\pi} \mathcal{O}_\mathbf{P} \longrightarrow 0 \tag{5}$$

on $\mathbf{P}^{s-1}$ parameterized by $x \in \mathbf{P}_* H^1(\mathcal{O}_\mathbf{P}(1) \otimes I_{q_1,\dots,q_n}) = \mathbf{P}^{r-1}$. By the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_\mathbf{P}(1)) \longrightarrow H^0(\bigoplus_{i=1}^n \mathbf{C}(p_i)) = \mathbf{C}^n \longrightarrow H^1(\mathcal{O}_\mathbf{P}(1) \otimes I_{q_1,\dots,q_n}) \longrightarrow 0,$$

$H^1(\mathbf{P}^{s-1}, \mathcal{O}_\mathbf{P}(1) \otimes I_{q_1,\dots,q_n})$ is isomorphic to the vector space $\mathbf{C}^n/G$ including the assignment of bases. The exact sequence $\mathbf{E}_{p_i}$ splits outside q_i for every $1 \le i \le n$, that is, E_{p_i} contains a subsheaf $\simeq I_{q_i}$ on which π is nonzero.

In the case $s = 2$, $\mathbf{E}_x$ is regarded as a quasi-parabolic rank 2 vector bundle on the n-pointed projective line $(\mathbf{P}^1; q_1, \dots, q_n)$. By the correspondence $x \mapsto \mathbf{E}_x$, the moduli space $\mathcal{U}(\alpha)$ of parabolic 2-bundles with a certain weight α is isomorphic to $\mathbf{P}^{r-1}$ (5). The moduli space $\mathcal{U}(\alpha')$ is isomorphic to the blow up X_G for another weight α'. We apply the result of Bauer[B] on the variation of the moduli spaces $\mathcal{U}(\alpha)$ to determine the movable cone of them. Then the finite generation follows from the GIT construction of such moduli spaces by Mehta-Seshadri[MS] and a result of Zariski.

In the case $s \ge 3$, the sheaf E_x is not locally free at $q_1, \dots, q_n$ but determines uniquely a vector bundle $\tilde{E}_x$ on the blow-up $S = Bl_{q_1,\dots,q_n}\mathbf{P}^{s-1}$. Especially, In the cases [9] and [7], the correspondence $x \mapsto \tilde{E}_x \otimes \mathcal{O}_S(1)$ gives rise to an isomorphism

$$\mathbf{P}^{r-1} \xrightarrow{\sim} M_{S,L}(2, -K_S, c_2 = 2) \tag{6}$$

of the $(r-1)$-dimensional projective space to the moduli space of 2-bundles with the above described invariants on a del Pezzo surface S (of degree 1 and 2) which are stable with respect to a certain ample divisor L. The blow-up X_G is isomorphic to $M_{S,L'}(2, -K_S, c_2 = 2)$ for another ample divisor L'. The finite generation essentially follows from the ampleness of $-K_S$ (6).

The first half (§§1–4) of this article, except for Remark 2 in §4, is essentially [M]. The author is grateful to Professor Akihiko Tsuchiya for his interest and useful comments to [M] and to Professor Tetsuji Shioda for his characteristic two example in Remark 2. In the preparation of the latter half the author received a preprint 'Hilbert's 14-th problem and Cox ring' from Professors Ana-Maria Castravat and Jenia Tevelev, to whom he is also grateful. A stronger theorem than ours is proved there by a different technique.

1 Invariant ring is Rees algebra

Let $G \subset \mathbf{C}^n$ be a linear subspace of codimension r and

$$\sum_{i=1}^n a_i^{(1)} t_i = \sum_{i=1}^n a_i^{(2)} t_i = \cdots = \sum_{i=1}^n a_i^{(r)} t_i = 0 \tag{7}$$

a system of defining equations. Since $x_1, \ldots, x_n$ are G-invariant, we obtain the induced action of G on the localization

$$S[x_1^{-1}, \ldots, x_n^{-1}] = \mathbf{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n] = \mathbf{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \frac{y_1}{x_1}, \ldots, \frac{y_n}{x_n}].$$

Since $(t_1, \ldots, t_n) \in G$ acts by the translation $y_i/x_i \mapsto y_i/x_i + t_i$, the invariant ring $S[x_1^{-1}, \ldots, x_n^{-1}]^G$ is generated by

$$\sum_{i=1}^n a_i^{(1)} \frac{y_i}{x_i}, \quad \sum_{i=1}^n a_i^{(2)} \frac{y_i}{x_i}, \quad \ldots, \quad \sum_{i=1}^n a_i^{(r)} \frac{y_i}{x_i} \tag{8}$$

over the Laurent polynomial ring $\mathbf{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let

$$J^{(1)}(x,y), \quad J^{(2)}(x,y), \quad \ldots, \quad J^{(r)}(x,y) \in S^G \tag{9}$$

be the products of (8) and the monomial $\prod_{i=1}^n x_i$. Let V be the subspace and R the subring of S^G generated by them. R is a polynomial ring and V is its degree one part. The invariant ring S^G contains $R[x_1, \ldots, x_n]$ and $S[x_1^{-1}, \ldots, x_n^{-1}]^G$ coincides with $R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Obviously we have

$$S^G = S[x_1^{-1}, \ldots, x_n^{-1}]^G \cap S = R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \cap S. \tag{10}$$

Let V_1 be the linear subspace of V consisting of $J(x,y)$ which do not contain the monomial $y_1 \prod_{i=2}^n x_i$. Then $V_1 \subset V$ is of codimension ≤ 1. A polynomial $J(x,y) \in V$ is divisible by x_1 if and only if it belongs to V_1. Let $I_1 \subset R$ be the ideal generated by V_1. Define $V_i \subset V$ and $I_i \subset R$ for $2 \leq i \leq n$ similarly. If $F(x,y) \in R$ belongs to the b_i-th power $I_i^{b_i}$, then $F(x,y)$ is divisible by $x_i^{b_i}$ and the quotient $F(x,y)/x_i^{b_i}$ belongs to S^G. Hence S^G contains

$$\sum_{b_1, \ldots, b_n \in \mathbf{Z}} (I_1^{b_1} \cap \cdots \cap I_n^{b_n}) x_1^{-b_1} \cdots x_n^{-b_n} \subset R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \tag{11}$$

as its subring. Here we understand that every negative power I^b, $b < 0$, of an ideal is R. The following was proved in [N1] in the case of codimension 3.

Proposition *The invariant ring S^G of the action* (1) *with respect to a subspace $G \subset \mathbf{C}^n$ coincides with the extended multi-Rees algebra* (11) *of $(R : I_1, \ldots, I_n)$.*

Proof. It suffices to show the following

claim : $f(J^{(1)}(x,y), \ldots, J^{(r)}(x,y)) \in R$ is divisible by $x_i^{b_i}$ if and only if $f(J^{(1)}, \ldots, J^{(r)})$ belongs to $I_i^{b_i}$.

If $a_i^{(1)}, \ldots, a_i^{(r)}$ are all zero, then $J^{(1)}(x,y), \ldots, J^{(r)}(x,y)$ are all divisible by x_i. The claim is obvious, since none is divisible by x_i^2 and since $V_i = V$. So assume the contrary. By reordering (9), we may assume that $a_i^{(1)} \neq 0$. Put

$$z_1 = J^{(1)}/a_i^{(1)}, z_2 = J^{(2)} - a_i^{(2)} z_1, \ldots, z_r = J^{(r)} - a_i^{(r)} z_1.$$

Then

$$f(J^{(1)}, \ldots, J^{(r)}) = f(a^{(1)} z_1, a^{(2)} z_1 + z_2, \ldots, a^{(r)} z_1 + z_r)$$

and this belongs to the ideal $(z_2, \ldots, z_r)^{b_i}$ if and only if $f(J^{(1)}, \ldots, J^{(r)})$ belongs to $I_i^{b_i}$ by the lemma below. When regarded as polynomials of $x_1, \ldots, x_n, y_1, \ldots, y_n$, the $r-1$ polynomials $z_2, \ldots, z_r$ are divisible by x_i and only z_1 is not. Therefore, f belongs to $(z_2, \ldots, z_r)^{b_i}$ if and only if $f(J^{(1)}(x,y), \ldots, J^{(r)}(x,y))$ is divisible by $x_i^{b_i}$. □

Lemma 1 *Let I be the ideal of $\mathbf{C}[z_1, \ldots, z_r]$ generated by linear forms vanishing at*

$$(a^{(1)}, a^{(2)}, \ldots, a^{(r)}) \in \mathbf{C}^r.$$

Assume that $a^{(1)} \neq 0$. Then a polynomial $f(z_1, \ldots, z_r)$ belongs to the b-th power I^b if and only if

$$f(a^{(1)} z_1, a^{(2)} z_1 + z_2, \ldots, a^{(r)} z_1 + z_r)$$

belongs to the b-th power of the homogeneous ideal $(z_2, \ldots, z_r)$.

For small values of r, the invariant ring is very explicit.

Example 1 ($r = 1$) Assume that $G \subset \mathbf{C}^n$ is defined by $\sum_{i=1}^m t_i = 0$ for $1 \leq m \leq n$. Then S^G is generated by $x_1, \ldots, x_n$ and

$$\left(\frac{y_1}{x_1} + \cdots + \frac{y_m}{x_m}\right) \prod_{i=1}^m x_i.$$

Example 2 ($r = 2$) Assume that $G \subset \mathbf{C}^n$ is defined by $\sum_{i=1}^n t_i = \sum_{i=1}^n c_i t_i = 0$. Then $c_i J_1(x,y) - J_2(x,y)$ is divisible by x_i and the quotient $(c_i J_1(x,y) - J_2(x,y))/x_i$ belongs to S^G for every $1 \leq i \leq n$. S^G is generated by these invariants over $\mathbf{C}[x_1, \ldots, x_n]$ if $c_1, \ldots, c_n$ are distinct.

2 Total coordinate ring

For our purpose, it is more convenient to state the proposition in geometric term. Let $\mathbf{P}^{r-1} = \operatorname{Proj} R$ be the $(r-1)$-dimensional projective space whose homogeneous coordinates are (9). In the sequel we assume that

($\diamondsuit$) $r \geq 3$ and any two of n vectors $(a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(r)}) \in \mathbf{C}^r, 1 \leq i \leq n$, are linearly independent.

(The study of S^G for the action (1) is easily reduced to this case.) Then n points

$$p_i := (a_i^{(1)} : a_i^{(2)} : \ldots : a_i^{(r)}) \in \mathbf{P}^{r-1}, \quad 1 \leq i \leq n, \tag{12}$$

are well-defined and distinct. The ideal $I_i \subset R$ is generated by the linear forms vanishing at p_i. Let

$$\pi : X = X_G \longrightarrow \mathbf{P}^{r-1}$$

be the blow-up at these n points. The isomorphism class of X_G does not depend on the choice of the defining equation (7). The Picard group is a free abelian group of rank $n+1$. The pull-back h of the hyperplane class H and the classes e_i, $1 \le i \le n$, of the exceptional divisors form a basis, which is called *the standard basis* of $\operatorname{Pic} X_G$ (with respect to π). The direct sum of the spaces of global sections of all line bundles (up to isomorphism)

$$\bigoplus_{a,b_1,\dots,b_n \in \mathbf{Z}} H^0(X, \mathcal{O}_X(ah - b_1e_1 - \cdots - b_ne_n)) \simeq \bigoplus_{L \in \operatorname{Pic} X} H^0(X, L) \tag{13}$$

is a graded ring, which is called the *total coordinate ring* of X and denoted by $\mathcal{TC}(X)$. In our case, $\mathcal{TC}(X_G)$ is the Rees algebra (11), or more precisely, it is the $\mathbf{Z}^n$-graded ring (11) plus the extra grading of the polynomial ring R. By the proposition, we have

Corollary *Under the condition of* $(\diamondsuit)$*, the invariant ring* S^G *of the action* (1) *with respect to* $G \subset \mathbf{C}^n$ *is the total coordinate ring* $\mathcal{TC}(X_G)$ *of the blow-up* X_G*.*

Let $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ be an integral domain graded by a free abelian group Λ. The subset $\{\lambda \mid A_\lambda \neq 0\}$ of Λ is a semi-group. This is called the *support* of A and denoted by $\operatorname{Supp} A$.

Lemma 2 *If* $\operatorname{Supp} A$ *is not finitely generated as semi-group, neither is* A *as a ring over* A_0*.*

Proof. Assume that A is finitely generated. Then finite nonzero homogeneous elements $a_i \in A_{\lambda_i}$, $1 \le i \le N$, generate A and $\lambda_1, \dots, \lambda_N$ generate $\operatorname{Supp} A$. $\square$

For example, the support of $\mathcal{TC}(X)$ as $\mathbf{Z}^{n+1}$-graded ring is the semi-group

$$\operatorname{Eff} X := \{L \in \operatorname{Pic} X \mid H^0(X, L) \neq 0\},$$

of linear equivalence classes of effective divisors on X. If $\operatorname{Eff} X$ is not finitely generated as semi-group, neither is $\mathcal{TC}(X)$. The following is basic for our analysis of $\operatorname{Eff} X$.

Lemma 3 *Let* $\pi : X \longrightarrow Y$ *be the blowing up of a projective variety* Y *at a point. Then the linear equivalence class of the exceptional divisor* E *of* π *belongs to any system of generators of the effective semi-group* $\operatorname{Eff} X$*.*

Proof. Assume that E is linearly equivalent to the sum $D_1 + D_2$ of two effective divisors. Let H be the pull-back of an ample divisor on Y. Then the intersection number $(E.H^{m-1})$, $m = \dim X$, is zero. Hence so are $(D_1.H^{m-1})$ and $(D_2.H^{m-1})$. Therefore, both $\operatorname{Supp} D_1$ and $\operatorname{Supp} D_2$ are contained in E and either D_1 or D_2 is zero. $\square$

If X and X' are isomorphic in codimension one, then the Picard groups are the same and $\operatorname{Eff} X = \operatorname{Eff} X'$. So we call $D \subset X$ a (-1)*-divisor* if there is a birational map $f : X \cdots \to X'$ and a morphism $\pi : X' \to Y$ such that f is an isomorphism in codimension one, π is the blowing up of a projective variety Y at a smooth point and D is the strict transform of the exceptional divisor of π. By the lemma, the class of a (-1)-divisor is contained in any system of generators of $\operatorname{Eff} X$. Hence $\operatorname{Eff} X$ is not finitely generated if X has infinitely many classes of (-1)-divisors.

3 Root systems and elliptic curves

Let Λ be the lattice of rank $n+1$ with orthogonal basis $h, e_1, \dots, e_n$. In view of the standard Cremona transformation (see the next section especially the formula (18)), we set $(h^2) = r-2$ and $(e_i^2) = -1$ for $1 \le i \le n$. For $\lambda = ah - \sum_{i=1}^n b_i e_i \in \Lambda$, we denote its coefficient a in h by $\deg \lambda$. We put $\kappa = rh - \sum (r-2) \sum_{i=1}^n e_i$, which corresponds to the anti-canonial class of the blow-up of $\mathbf{P}^{r-1}$ at points. The orthogonal complement of κ together with its basis

$$e_1 - e_2, \quad e_2 - e_3, \quad \dots, \quad e_{n-1} - e_n \quad \text{and} \quad h - \sum_{i=1}^r e_i \tag{14}$$

becomes a root system. The Dynkin Diagram is (4), that is, $T_{2,r,n-r}$ with three-legs of length $2, r$ and $n-r$. For a subset $I \subset [n] := \{1, 2, \dots, n\}$ of cardinality r, $\alpha_I = h - \sum_{i \in I} e_i$ is a root. The reflection R_I with respect to α_I is as follows:

$$\begin{cases} h & \mapsto & h + (r-2)\alpha_I & = (r-1)h - (r-2)\sum_{i \in I} e_i \\ e_i & \mapsto & e_i + \alpha_I & \text{for } i \in I \\ e_j & \mapsto & e_j & \text{for } j \notin I \end{cases} \tag{15}$$

Let W be the Weyl group of (14). By definition, W leaves κ invariant, that is, $rw(h) - (r-2)\sum_{i=1}^n w(e_i) = \kappa$ for every $w \in W$. In particular, we have

$$r \deg w(h) - (r-2) \sum_{i=1}^n \deg w(e_i) = r. \tag{16}$$

Lemma 4 *If the inequality* (2) *holds, then the W-orbit of e_n is infinite.*

Proof. The assumption implies $r \ge 3$. Let w be an element of the Weyl group. There exists a subset $I \subset [n]$ of cardinality r such that

$$\sum_{i \in I} \deg w(e_i) \le \frac{r}{n} \sum_{i=1}^n \deg w(e_i).$$

By (16) we have

$$\deg w(\alpha_I) = \deg w(h) - \sum_{i \in I} \deg w(e_i) \ge \deg w(h) - \frac{r^2}{n(r-2)}(\deg w(h) - 1),$$

which is positive by (2). Therefore, $\deg w(R_I(h)) - \deg w(h) = (r-2)\deg w(\alpha_I)$ is also positive. It follows that the degree is increased by a suitable reflection R_I. Hence, the orbit $W \cdot h$ is infinite. So is $W \cdot e_n$ by the equality (16). □

The Weyl group of $T_{p,q,r}$ is infinite if and only if $1/p + 1/q + 1/r \le 1$ ([K] Chap. 4). The lemma also follows from this.

Let C be an elliptic curve and Λ_C the $(n+1)$-dimensional variety $\mathrm{Pic}^r C \times C^n$. This is canonically isomorphic to $\mathrm{Pic}^r C \times (\mathrm{Pic}^1 C)^n$. So the factor permutation of C^n and the automorphism

$$(D; c_1, \dots, c_n) \mapsto (D'; c'_1, \dots, c'_n),$$

$$\begin{cases} D' &= (r-1)D - (r-2)\sum_{i=1}^r c_i \\ c'_i &= D - c_1 - \cdots - \check{c}_i - \cdots - c_r \quad \text{for } 1 \le i \le r \\ c'_j &= c_j \quad \text{for } r+1 \le j \le n \end{cases}$$

define the action of the Weyl group W on the variety Λ_C. For a real root $\alpha = ah - \sum_{i=1}^n b_i e_i \in \Delta^{re}$ ([K] Chap. 5), the reflection R_α interchanges

$$f_\alpha : \Lambda_C \longrightarrow \mathrm{Pic}^0 C, \quad (D; c_1, \dots, c_n) \mapsto aD - \sum_{i=1}^n b_i c_i.$$

with $-f_\alpha$. We denote the fiber $f_\alpha^{-1}(0)$ by $\mathcal{D}(\alpha)$.

Example 3 $\mathcal{D}(e_i - e_j)$, $i \neq j$, is the diagonal $\{c_i = c_j\}$. $\mathcal{D}(h - \sum_{i=1}^r e_i)$ consists of $(D; c_1, \dots, c_n)$ such that $\sum_{i=1}^r c_i \in |D|$.

The Weyl group W acts on the complement of all these fibers:

$$\Lambda_C - \bigcup_{\alpha \in \Delta^{re}} \mathcal{D}(\alpha). \tag{17}$$

4 Standard Cremona transformation

The map

$$\Psi : \mathbf{P}^{r-1} \cdots \to \mathbf{P}^{r-1}, \quad (x_1 : x_2 : \cdots : x_r) \mapsto (\frac{1}{x_1} : \frac{1}{x_2} : \cdots : \frac{1}{x_r}), \quad r \ge 3,$$

is a birational transformation of the projective space $\mathbf{P}^{r-1}$. It contracts the r coordinate hyperplanes to the r coordinate points and its square is the identity. A birational map which is projectively equivalent to Ψ is called *a standard Cremona transformation.* Let $P = \{p_1, \dots, p_r\}$ and $Q = \{q_1, \dots, q_r\}$ be a pair of sets of r points of $\mathbf{P}^{r-1}$. If both P and Q span $\mathbf{P}^{r-1}$, then there exists the unique standard Cremona transformation which contracts the hyperplane H_i passing through the $r-1$ points $p_1, \dots, \check{p}_i, \dots, p_r$ to the point q_i for every $1 \le i \le r$. We denote this by $\Psi_{P,Q}$. P and Q are called its *center* and *cocenter*, respectively. $\Psi_{P,Q}$ is the rational map associated with $|(r-1)H - (r-2)\sum_{i=1}^n p_i|$, the linear system of hypersurfaces of degree $(r-1)$ passing through P with multiplicity $\ge r-2$. (The sum of $r-1$ of $H_1, \dots, H_r$ form a basis of the linear system.) The indeterminacy locus of $\Psi_{P,Q}$ is the union $I_P := \cup_{1 \le i < j \le r} H_i \cap H_j$ of the intersection of all pairs of the hyperplanes H_i's.

Let X_P and X_Q be the blow-up of $\mathbf{P}^{r-1}$ with center P and Q, respectively. $\Psi_{P,Q}$ induces the birational map $\tilde{\Psi}_{P,Q}$ from X_P to X_Q. The diagram

$$\begin{array}{ccc} & \tilde{\Psi}_{P,Q} & \\ X_P & \cdots \longrightarrow & X_Q \\ \downarrow & & \downarrow \\ \mathbf{P}^{r-1} & \cdots\cdots \to & \mathbf{P}^{r-1} \\ & \Psi_{P,Q} & \end{array}$$

is commutative and $\tilde{\Psi}_{P,Q}$ induces an isomorphism between the complement of the strict transform of I_P and that of I_Q. Hence $\tilde{\Psi}_{P,Q}$ is an isomorphism in codimension one. (More precisely, $\tilde{\Psi}_{P,Q} : X_P \cdots \to X_Q$ is the composite of certain flops.) In particular it induces an isomorphism $\operatorname{Pic} X_P \xrightarrow{\sim} \operatorname{Pic} X_Q$ between the Picard groups and that between the semi-groups of effective classes. Let $\{h, e_1, \dots e_r\}$ be the standard basis of $\operatorname{Pic} X_P$. Then the standard basis of $\operatorname{Pic} X_Q$ consists of

$$(r-1)h - (r-2)\sum_{i=1}^{r} e_i, \quad \text{and} \quad h - e_1 - \cdots - \check{e}_i - \cdots - p_r, \quad 1 \le i \le r. \tag{18}$$

Proof of 'if'part of Theorem. Let C be an elliptic curve and take an $(n+1)$-tuple $(D; c_1, \dots, c_n)$ from the W-invariant open subset (17) of Λ_C. The complete linear system $|D|$ embeds C into the $(r-1)$-dimensional projective space $\mathbf{P}_D := \mathbf{P}^* H^0(C, \mathcal{O}_C(D))$. Let $p_1, \dots, p_n \in \mathbf{P}_D$ be the image of $c_1, \dots, c_n$ by the embedding Φ_D. Since $(D; c_1, \dots, c_n)$ does not belong to the divisor $\mathcal{D}(e_i - e_j) \subset \Lambda_C$ for any $1 \le i < j \le n$, the n points $p_1, \dots, p_n$ are distinct. Moreover, since it does not belongs to $\mathcal{D}(\alpha_I)$ for any $I \subset [n]$ with $|I| = r$, any r of $p_1, \dots, p_n$ spans the projective space $\mathbf{P}_D$ (Example 3). Hence we can perform the standard Cremona transformation of $\mathbf{P}_D$ with any r of $p_1, \dots, p_n$ as center. Put $(D'; c'_1, \dots, c'_n) = R_I(D; c_1, \dots, c_n)$ and $p'_i = \Phi_{D'}(c'_i)$ for $1 \le i \le n$. Then we have the commutative diagram:

$$\begin{array}{ccccc} & C & = & C & \\ \Phi_D & \downarrow & & \downarrow & \Phi_{D'} \\ & \mathbf{P}_D & \cdots \to & \mathbf{P}_{D'} & \\ & & \Psi_I & & \end{array}$$

where Ψ_I is the standard Cremona transformation whose center is $\{p_i \mid i \in I\}$ and cocenter is $\{p'_i \mid i \in I\}$. Any point of C other than $\{p_i \mid i \in I\}$ does not lie in the indeterminacy locus of Ψ_I. Let $\pi : X \longrightarrow \mathbf{P}_D$ be the blowing up at the n points $p_1, \dots, p_n$ and $\pi' : X \longrightarrow \mathbf{P}_{D'}$ at $p'_1, \dots, p'_n$. Then Ψ_I induces $\tilde{\Psi}_I$ between X and X' and we have the commutative diagram:

$$\begin{array}{ccccc} & C & = & C & \\ & | & & | & \\ & \downarrow & \tilde{\Psi}_I & \downarrow & \\ & X & \cdots \longrightarrow & X' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & \mathbf{P}_D & \cdots\cdots \to & \mathbf{P}_{D'} & \\ & & \Psi_I & & \end{array}$$

By our choice of $(D; c_1, \dots, c_n)$, the images $p'_1, \dots, p'_n$ of $c_1, \dots, c_n$ are distinct and any subset of cardinality r spans $\mathbf{P}_{D'}$. Hence we can perform the standard Cremona transformation with any r of $p'_1, \dots, p'_n$ as center. We can continue this as many times as we like. Hence we have the following by (15) and (18):

Lemma 5 *If an $(n+1)$-tuple $(D; c_1, \dots, c_n)$ belongs to the open subset (17) of Λ_C and if α is in the orbit $W \cdot e_n$, then there exists a (-1)-divisor D whose linear equivalence class is α.*

It is obvious that the same holds for the blow-up $\tilde{X}$ at $\tilde{p}_1, \dots, \tilde{p}_n$ if the n-tuple $(\tilde{p}_1, \dots, \tilde{p}_n) \in \mathbf{P}^{r-1} \times \cdots \times \mathbf{P}^{r-1}$ belongs to a neighborhood of $(p_1, \dots, p_n)$ in the classical topology. Hence, by virtue of Lemma 4, $\tilde{X}$ contains infinitely many classes of (-1)-divisors if (2) holds. Therefore, S^G for a general $G \subset \mathbf{C}^n$ is not finitely generated by Corollary and two lemmas in §2. □

Remark 1 Following [N1], Steinberg [St] and independently the author [M2] consider the diagonal subring

$$S^{T \cdot G} := R[x] + \sum_{b \geq 0} (I_1^b \cap \cdots \cap I_n^b) x^{-b} \subset R[x^{\pm 1}], \quad x = \prod_{i=1}^{n} x_i,$$

of (11), which is isomorphic to

$$\bigoplus_{a,b \in \mathbf{Z}} H^0(X_G, \mathcal{O}_X(ah - b(e_1 + \cdots + e_n))), \tag{19}$$

in the case where $n = 9$ and $G \subset \mathbf{C}^9$ is of codimension 3. They show that this is not finitely generated if $3D - \sum_{i=1}^{9} c_i \in C$ is of infinite order. The infinite generation of S^G follows from this easily. Note that $S^{T \cdot G}$ becomes finitely generated if $3D - \sum_{i=1}^{9} c_i$ is torsion but still S^G is not finitely generated if the differences $c_i - c_j$ are general. Note also that $\kappa = 3h - \sum_{i=1}^{9} e_i \in \Lambda$ corresponding to $3D - \sum_{i=1}^{9} c_i$ is an imaginary root of the affine root system $\kappa^{\perp}$ of type $T_{2,3,6}$.

Remark 2 Let $\tilde{X} \longrightarrow \mathbf{P}^1$ is an elliptic fibration (with a section) and assume that the Mordell-Weil lattice is isomorphic to E_8. Then there exists a set of nine mutually disjoint sections and the total space $\tilde{X}$ becomes $\mathbf{P}^2$ by blowing down these nine sections. By Shioda [Sh], there exists such an elliptic fibration over a finite field in every positive characteristic p . (In the case $p = 2$, $y^2 + y = x^3 + t^5$, $t \in \mathbf{P}^1$, is such an elliptic fibration.) Hence the original fourteenth problem has a counterexample over a finite field in every positive characteristic.

5 Moduli of parabolic 2-bundles on $\mathbf{P}^1$

Let C be a complete algebraic curve. A pair $(E' \subset E)$ of an (algebraic) vector bundle E of rank 2 on C and its subsheaf E' of rank 2 is called a *quasi-parabolic 2-bundle*. The inclusion $\det E' \subset \det E$ determines an effective divisor on C, which we denote by Δ. E' coincides with E outside the support of D. Let $q_1, \dots, q_n$ be a set of distinct n points on C. $(E' \subset E)$ with $\Delta = q_1 + \cdots + q_n$ is called a quasi-parabolic 2-bundle on the n-pointed curve $(C; q_1, \dots, q_n)$. A pair $(E' \subset E; \alpha)$ of a quasi-parabolic 2-bundle and an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of real numbers in the closed interval $[0, 1]$ is called a *parabolic 2-bundle*.

Definition 1 A parabolic 2-bundle $(E' \subset E; \alpha)$ is *semi-stable* if

$$\deg L - \sum_{i=1}^{n} \alpha_i \mathrm{length}_{p_i} L/(L \cap E') \leq \frac{1}{2}(\deg E - \sum_{i=1}^{n} \alpha_i)$$

holds for every line subbundle $L \subset E$. It is *stable* if the strict inequality holds for every line subbundle $L \subset E$.

We only need the case $C = \mathbf{P}^1$. Let $q_1, \dots, q_n \in \mathbf{P}^1$ and $p_1, \dots, p_n \in \mathbf{P}^{n-3}$ be as in the introduction. We denote by $\mathcal{U}(\alpha)$ the moduli space of semi-stable parabolic 2-bundles $(E' \subset E; \alpha)$ on the n-pointed projective line $(\mathbf{P}^1 : q_1, \dots, q_n)$ with $\det E \simeq \mathcal{O}_{\mathbf{P}}(1)$. Since the 2-bundle E_x in (5) is a subsheaf of the direct sum $\mathcal{O}_{\mathbf{P}}(1) \oplus \mathcal{O}_{\mathbf{P}}$, we obtain a quasi-parabolic 2-bundle $(E_x \subset \mathcal{O}_{\mathbf{P}}(1) \oplus \mathcal{O}_{\mathbf{P}})$ for each $x \in \mathbf{P}^{n-3}$. First we consider the case where the weight α is diagonal, that is, $\alpha = (a, \dots, a)$, for $a \in [0, 1]$. By [B], we have the following:

Proposition 1 (1) *If $1/n < a < 1/(n-2)$, then $(E_x \subset \mathcal{O}_{\mathbf{P}}(1) \oplus \mathcal{O}_{\mathbf{P}})$ is stable for every $x \in \mathbf{P}^{n-3}$ and the classification morphism*

$$\mathbf{P}_* H^1(\mathcal{O}_{\mathbf{P}}(1) \otimes I_{q_1, \dots, q_n}) \simeq \mathbf{P}^{n-3} \longrightarrow \mathcal{U}(a, \dots, a), \quad x \mapsto (E_x \subset \mathcal{O}_{\mathbf{P}}(1) \oplus \mathcal{O}_{\mathbf{P}})$$

is an isomorphism. (The moduli space is empty if $0 \le a < 1/n$ and consists of one point if $a = 1/n$.)

(2) *$\mathcal{U}(a, \dots, a)$ is isomorphic to the blow-up $X_G = Bl_{p_1, \dots, p_n} \mathbf{P}^{n-3}$ if $n \ge 5$ and $1/(n-2) < a < 1/(n-4)$.*

In order to describe the moduli space $\mathcal{U}(\alpha)$ for a general weight α, we need the family of hyperplanes

$$H_{I,k} : \sum_{j \notin I} \alpha_j + \sum_{i \in I} (1 - \alpha_i) = k$$

in the hypercube $[0, 1]^n$, where I is a subset of $\{1, \dots, n\}$ and k is an integer with $|I| \equiv k + 1 \bmod 2$. A connected component of the complement of the union of all these hyperplanes is called a *chamber*. The hyperplane $H_{I,k}$ coincides with $H_{I^c, n-k}$, where I^c is the complement of I. Hence we assume $k \le n/2$ in the sequel. We recall some results of [B, §2] for our proof.

Proposition 2 (1) *Let $\mathcal{C}$ be a chamber. Then the moduli space $\mathcal{U}(\beta)$ with $\beta \in \mathcal{C}$ is smooth of dimension $n-3$. Moreover, their isomorphism classes do not depend on β. We denote the isomorphism class by $\mathcal{U}_{\mathcal{C}}$.*

(2) *For each $\alpha \in \overline{\mathcal{C}}$, there exists a (contraction) morphism $f_{\mathcal{C},\alpha} : \mathcal{U}_{\mathcal{C}} \longrightarrow \mathcal{U}(\alpha)$.*

(3) *Let $\mathcal{C}$ and $\mathcal{C}'$ be two adjacent chambers separated by the hyperplane $H_{I,k}$ Assume that $\sum_{j \notin I} \alpha_j + \sum_{i \in I}(1 - \alpha_i) - k$ non-positive on $\mathcal{C}$ and non-negative on $\mathcal{C}'$. Then the two moduli spaces $\mathcal{U}_{\mathcal{C}}$ and $\mathcal{U}_{\mathcal{C}'}$ are related in the following way.*

i) If $k = 2$, then $\mathcal{U}_{\mathcal{C}'}$ is the blow-up of $\mathcal{U}_{\mathcal{C}}$ at a point.

ii) If $3 \le k (\le n/2)$, then $\mathcal{U}_{\mathcal{C}'}$ is a flop of $\mathcal{U}_{\mathcal{C}}$. Let α_0 be a general point of $\overline{\mathcal{C}} \cap \overline{\mathcal{C}'}$. The morphism $f_{\mathcal{C},\alpha_0} : \mathcal{U}_{\mathcal{C}} \longrightarrow \mathcal{U}(\alpha_0)$ contracts a subvariety isomorphic to $\mathbf{P}^{k-2}$ to a singular point and $f_{\mathcal{C}',\alpha_0}$ contracts a subvariety $\simeq \mathbf{P}^{n-k-2}$ to the same point. Both $f_{\mathcal{C},\alpha_0}$ and $f_{\mathcal{C}',\alpha_0}$ are isomorphisms outside the subvarieties.

We also need the behavior of $\mathcal{U}(\alpha)$ in the neighborhood of the facets of $[0, 1]^n$, which is described by the neglect of the parabolic structure at a (parabolic) point. Let $(E' \subset E)$ be a parabolic 2-bundle on $(\mathbf{P}^1 : q_1, \dots, q_n)$ and E_i the subsheaf of E which is E' outside q_i and E itself in the neighborhood of q_i. Then $(E_i \subset E)$ is a parabolic 2-bundle on the $(n-1)$-pointed projective line $(\mathbf{P}^1 : q_1, \dots, \check{q}_i, \dots, q_n)$. Similarly, let E^i be the subsheaf of E which is E outside q_i and E' in the neighborhood of q_i. Then $(E' \subset E^i)$ is also a parabolic 2-bundle.

Proposition 3 *Let C be a chamber with $\alpha_i = 0$ as its supporting hyperplane. Then the neglect $(E' \subset E) \mapsto (E_i \subset E)$ defines a morphism $\mathcal{U}_C \longrightarrow \mathcal{U}'$ onto a moduli spaces of parabolic 2-bundles on $(\mathbf{P}^1 : q_1, \dots, \check{q}_i, \dots, q_n)$. A general fiber is isomorphic to $\mathbf{P}^1$. Similarly if C has $\alpha_i = 1$ as its supporting hyperplane, then $(E' \subset E) \mapsto (E' \subset E^i)$ defines a morphism $\mathcal{U}_C \longrightarrow \mathcal{U}''$ whose general fiber is also $\mathbf{P}^1$.*

This is a moduli theoretic interpretation of the following birational geometry in the case $s = 2$:

Example 4 The projection $\mathbf{P}^{r-1} \cdots \to \mathbf{P}^{r-2}$ with center p_n induces a rational map $X_G = Bl_n\mathbf{P}^{r-1} \cdots \to Bl_{n-1}\mathbf{P}^{r-2}$ to the blow-up of $\mathbf{P}^{r-2}$ at the image of $(n-1)$ points $p_1, \dots, p_{n-1}$. This image is the Gale transform of $q_1, \dots, q_{n-1} \in \mathbf{P}^{s-1}$. The indeterminacy of this rational map is resolved by the flop with center the strict transforms of the $n-1$ lines joining p_n and p_i, $1 \le i \le n-1$. The resulting morphism is a $\mathbf{P}^1$-bundle.

Let $\overline{\Pi}$ be the polytope in $[0,1]^n$ defined by the system of 2^{n-1} inequalities $\sum_{j \notin I} \alpha_j + \sum_{i \in I}(1-\alpha_i) \ge 2$ for the subsets $I \subset \{1, \dots, n\}$ with $|I|$ odd. Let Π be its interior. By virtue of (3) of Proposition 2, $\mathcal{U}(\beta)$'s with $\beta \in \Pi$ are isomorphic to each other in codimension one. So they have the common Picard group and the common total coordinate ring.

The polytope Π is empty if $n = 3$ and consists of one point $(1/2, \cdots, 1/2)$ if $n = 4$. So we assume $n \ge 5$. The diagonal weight $(a, \dots, a)$ with $1/(n-2) < a < 1/(n-4)$ is contained in Π. Hence, by Proposition 1, $\mathcal{U}(\beta)$ is isomorphic to X_G in codimension one for every interior point β of Π.

For our proof we need a fact from the construction in [MS] also. The moduli space $\mathcal{U}_{(C:q_1,\dots,q_n)}(\alpha)$ is a GIT quotient of the product of a suitable Quot scheme and Grassmannians by suitable linearization. Since $\mathcal{U}(\alpha)$ is the projective spectrum $\operatorname{Proj} R$ of a graded ring R, it carries a natural ample (Cartier) divisor, which we regard as a divisor on X_G by Proposition 2 and denote by D_α. The choice of linearization in [MS] is linear with respect to the weight α. Hence we have

Lemma 6 *If weights $\alpha, \alpha', \alpha'' \in \Pi$ are colinear, then the divisors $D_\alpha, D_{\alpha'}, D_{\alpha''} \in \operatorname{Pic} X_G$ are linearly dependent.*

Proof of 'only if' part of Theorem. Let $\tilde{\Pi}$ be the cone generated by D_α with $\alpha \in \overline{\Pi}$ in $\operatorname{Pic} X_G \otimes \mathbf{R}$. For a chamber C, we denote the subcone generated by D_α with $\alpha \in \overline{C}$ by $\tilde{C}$. Then D_α is semi-ample on the moduli space $\mathcal{U}_C$ by (2) of Proposition 2. Since C is finitely generated, so is $\tilde{C} \cap \operatorname{Pic} X_G$ by Lemma 6. Therefore, by a lemma of Zariski ([HK, Lemma 2.8]), the $\tilde{C}$-part $\bigoplus_{L \in \tilde{C} \cap \operatorname{Pic} X_G} H^0(L)$ of the total coordinate ring $\mathcal{TC}(X_G)$ is finitely generated (over $\mathbf{C}$). Since $\overline{\Pi}$ is the union of finitely many $\overline{C}$, the $\tilde{\Pi}$-part of $\mathcal{TC}(X_G)$ is also finitely generated.

The supporting hyperplanes of the polytope $\overline{\Pi}$ are $H_{I,2}$'s and $\alpha_i = 0, 1$ for $1 \le i \le n$. Let $C \subset \Pi$ be a chamber with $H_{I,2}$ as its supporting hyperplane. Let β_I be a general point of the intersection $\overline{C} \cap H_{I,2}$. Then $\mathcal{U}_C \to \mathcal{U}(\beta_I)$ is a one-point blow-up by Proposition 2. Let e_I be the exceptional divisor and Z_I the line in it. Then $(D_\alpha . Z_I)$ is positive for every $\alpha \in C$ and zero for $\alpha \in \overline{C} \cap H_{I,2}$ by (3) of Proposition 2. Therefore, by Lemma 6, the intersection number $(D.Z_I)$ is non-negative for every $D \in \tilde{\Pi}$ and $(D.Z_I) = 0$ is a supporting hyperplane of $\tilde{\Pi}$.

Let $C \subset \Pi$ be as in Proposition 3 and let F_i be a general fiber of the morphism $\mathcal{U}_C \longrightarrow \mathcal{U}'$. The intersection number $(D_\alpha.F)$ is positive for every $\alpha \in C$ and zero for $\alpha \in \overline{C} \cap \{\alpha_i = 0\}$. Therefore, by Lemma 6, the intersection number $(D.F_i)$ is non-negative for every $D \in \tilde{\Pi}$ and $(D_\alpha.F_i) = 0$ is a supporting hyperplane of $\tilde{\Pi}$.

Now let D be a divisor of X_G. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{R}^n$ does not belong to Π, then either $\sum_{j \notin I} \alpha_j + \sum_{i \in I}(1 - \alpha_i) < 2$ holds for a subset I of $\{1, \dots, n\}$ or $\alpha_i < 0$ or $\alpha_i > 1$ holds for $1 \leq i \leq n$. By Lemma 6, if D does not belong to $\tilde{\Pi}$, then either $(D.Z_I) < 0$ holds for some I or $(D.F_i) < 0$ or $(D.F_i') < 0$ holds for $1 \leq i \leq n$, where F_i' is a general fiber of the morphism $\mathcal{U}_C \longrightarrow \mathcal{U}''$ in Proposition 3. Assume that D is effective. Then the latter is impossible. Hence an effective divisor $D \notin \tilde{\Pi}$ contains the exceptional divisor e_I as irreducible component for some I. Therefore, $TC(X_G)$ is generated as ring by its $\tilde{\Pi}$-part and the canonical global sections $1 \in H^0(\mathcal{O}_X(e_I))$ of the 2^{n-1} exceptional divisors e_I's.

6 Moduli of certain 2-bundles on a del Pezzo surface

Let $p_1, \dots, p_n \in \mathbf{P}^{r-1}$ and $q_1, \dots, q_n \in \mathbf{P}^{s-1}$, $r + s = n$, be as in the introduction. They are the Gale transform of each other. Let $X = X_G$ and $S = S_G$ be their blow-ups. We need a certain linear isomorphism between $\operatorname{Pic} X \otimes \mathbf{Q}$ and $\operatorname{Pic} S \otimes \mathbf{Q}$ for our proof.

Generally the correspondence $e_i - e_{i+1} \mapsto e_{n+1-i} - e_{n-i}$ for $1 \leq i \leq n$ and $h - \sum_1^r e_i \mapsto h - \sum_1^s e_i$ gives an isomorphism from the Dynkyn diagram $T_{2,r,n-r}$ of X to $T_{2,s,n-s}$ of S, and hence an isometry φ_0 between two lattices $(-K_X)^\perp \subset \operatorname{Pic} X$ and $(-K_S)^\perp \subset \operatorname{Pic} S$ with respect to the inner product defined in §3. We identify the two Weyl groups $W(T_{2,s,n-s})$ and $W(T_{2,r,n-r})$ by this correspondence. The following is easily verified:

Proposition 4 *Let Ψ be the standard Cremona transformation of $\mathbf{P}^{s-1}$ with center the s points $q_1, \dots, q_s$ and Ψ' that of $\mathbf{P}^{r-1}$ with center the r points $p_{s+1}, \dots, p_n$. Then*

$$q_1, \dots, q_s, \Psi(q_{s+1}), \dots, \Psi(q_n) \in \mathbf{P}^{s-1}$$

and

$$\Psi'(p_1), \dots, \Psi'(p_s), p_{s+1}, \dots, p_n \in \mathbf{P}^{r-1}$$

are the Gale transform of each other.

Now we assume that $s = 3$ and extend the isometry φ_0 to a linear isomorphism $\varphi : \operatorname{Pic} X \otimes \mathbf{Q} \longrightarrow \operatorname{Pic} S \otimes \mathbf{Q}$ by setting $\varphi(K_X) = 2K_S$. The following is easily calculated:

$$\varphi(e_i) = h - e_i, \quad \varphi(h) = (n-2)h - e. \tag{20}$$

Remark 3 Though φ is not an isometry, $(\varphi(D)^2) = (D^2) - (K_S.D)^2/4$ holds for every $D \in \operatorname{Pic} S$.

The main tool of our proof is vector bundle as in previous section. More precisely we consider torsion free sheaves E on S with

$$r(E) = 2, c_1(E) = -K_S \quad \text{and} \quad c_2(E) = 2. \tag{21}$$

For an ample divisor L on S, we denote by $\overline{M}_{S,L}$ the moduli space of such torsion free sheaves E which are semi-stable with respect to L in the sense of Gieseker [G]. It contains the moduli space $M_{S,L}$ of stable bundles as an open set. $M_{S,L}$ is smooth of dimension $n-4$ by the general theory. We study the variation of $M_{S,L}$ as L moves. See [EG], [FQ] and [MW] for the general theory.

We further assume that $n = (6,)7, 8$. Then S is a del Pezzo surface, that is, a surface with ample $-K_S$. The degree (K_S^2) is equal to $9-n$.

Lemma 7 *Every member of $E \in \overline{M}_{S,L}$ has a nonzero global section.*

Proof. By the Riemann-Roch formula, we have $\chi(E) = 9 - n \geq 1$. Since $H^2(E) \simeq \mathrm{Hom}\,(E, \mathcal{O}_S(K_S))^\vee = 0$, we have $H^0(E) \neq 0$. $\square$

Let l be a *line*, *i.e.*, a smooth rational curve $l \subset S$ with $(l. -K_S) = 1$. When L crosses crosses the hyperplane $H_{l,1} : (2l + K_S.L) = 0$ from the positive side to the negative, the non-trivial extensions

$$0 \longrightarrow \mathcal{O}_S(-K_S - l) \longrightarrow E \longrightarrow \mathcal{O}_S(l) \longrightarrow 0,$$

which are parameterized by $\mathbf{P}^{n-6}$, are replaced by the opposite non-trivial extensions

$$0 \longrightarrow \mathcal{O}_S(l) \longrightarrow E' \longrightarrow \mathcal{O}_S(-K_S - l) \longrightarrow 0,$$

which are parameterized by $\mathbf{P}^1$, in the moduli spaces. We denote this $\mathbf{P}^1$ by Z_l. In the case $n = 8$, $-K_S$ belongs to the positive side and the moduli space is flipped when L crosses the hyperplane $H_{l,1}$.

Similarly, let C be a *conic*, *i.e.*, a smooth rational curve C with $(C. -K_S) = 2$. When L crosses the hyperplane $H_{C,1} : (2C + K_S.L) = 0$ from the positive side, the family of non-trivial extensions E of $\mathcal{O}_S(C)$ by $\mathcal{O}_S(-K_S - C)$ parameterized by $\mathbf{P}^{n-5}$ is replaced by the unique non-trivial opposite extension E_C. In fact, the moduli space is blow down to the point $[E_C]$. We denote the exceptional divisor $\simeq \mathbf{P}^{n-5}$ parameterizing E's in the moduli space by e_C.

Let $\Pi \subset \mathrm{Pic}\, S \otimes \mathbf{R}$ be the cone of ample divisor classes L on S such that $(L.2C + K_S) > 0$ for every conic $C \subset S$.

Lemma 8 *If $E \in \overline{M}_{S,L}$ is strictly μ-semi-stable with respect to an ample divisor $L \in \overline{\Pi}$, then we have either $(2l + K_S.L) = 0$ for a line l or $(2C + K_S.L) = 0$ for a conic C.*

Proof. E is an extension of a line bundle by another line bundle of the same degree outside a finite set of points. By Lemma 7, one of these two line bundles has a nonzero global section and is isomorphic to $\mathcal{O}_S(D)$ for an effective divisor D. By the strict μ-semi-stability, we have $(2D + K_S.L) = 0$. Assume that $h^0(\mathcal{O}_S(D)) = 1$. Then D is supported by a disjoint union of lines $l_1, \ldots, l_n$. Since $2 = (l_1. -K_S - l_1) \leq (D. -K_S - D) \leq c_2(E) = 2$, we have $D = l_1$. Assume that $h^0(\mathcal{O}_S(D)) \geq 2$. Then either $|D+K_S| \neq \emptyset$ or $|D-C| \neq \emptyset$ for a conic C. But the former contradicts to $(2D + K_S.L) = 0$. The latter implies $D - C = 0$ since $L \in \overline{\Pi}$. □

Let $\mathcal{C}$ be a *chamber* of Π, that is, a connected component of the complement of $\bigcup_{l\,:\,\text{line}} H_{l,1}$ in Π. For every $L \in \mathcal{C}$, every member $E \in \overline{M}_{S,L}$ is stable. Hence all $M_{S,L}$ ($= \overline{M}_{S,L}$), $L \in \mathcal{C}$, are isomorphic to each other. We denote this isomorphism class by $M_{S,\mathcal{C}}$. In particular, $M_{S,L}$'s, $L \in \Pi$, are isomorphic to each other in codimension one.

We relate $M_{S,L}$ with the blow-up X_G. By the Riemann-Roch formula, we have $\chi(\mathcal{H}om\,(E, \mathcal{O}_S(h))) = 1$. Since $H^2(S, \mathcal{H}om\,(E, \mathcal{O}_S(h))) \simeq \text{Hom}\,(\mathcal{O}_S(h), E(K_S))^\vee = 0$, we have $\dim \text{Hom}\,(E, \mathcal{O}_S(h)) \geq 1$ for every semi-stable bundle $E \in \overline{M}_{S,L}$. In particular, if $(L. - K_S)/2 > (L.h)$, then the moduli space $\overline{M}_{S,L}$ is empty. For example, this applies if $L = ah - K_S$ and if $a > n - 3$. In the range $n - 5 < a < n - 3$, a nonzero homomorphism $f : E \longrightarrow \mathcal{O}_S(h)$ is surjective and unique up to constant multiplication. Hence $M_{S,L}$ is isomorphic to the $(n-4)$-dimensional projective space $\mathbf{P}_*\text{Ext}^1(\mathcal{O}_S(h), \mathcal{O}_S(2h-e)) \simeq \mathbf{P}_* H^1(\mathbf{P}^2, I_{q_1,\dots,q_n}(1))$, where we put $e = \sum_1^n e_i$. This identification is nothing but (6) in the introduction.

Among these extensions E of $\mathcal{O}_S(h)$ by $\mathcal{O}_S(2h-e)$, there is a unique E_i which contains $\mathcal{O}_S(h - e_i)$ as its subsheaf for each $1 \leq i \leq n$. E_i is nothing but $\tilde{E}_{p_i} \otimes \mathcal{O}_S(h)$ in the introduction. Hence $M_{S,L}$ is the blow-up X_G of the $\mathbf{P}^{n-4}$ at the n points $p_1, \dots, p_n$ between $a = n - 5$ and the next critical value ($= n - 7$). Since $ah - K_S$ belongs to Π for $n - 7 < a < n - 5$, $M_{S,\mathcal{C}}$ is isomorphic to X_G in codimension one for every chamber $\mathcal{C} \subset \Pi$. When $a = n - 7$, we have $(2l + K_S.ah - K_S) = 0$ for every $l = h - e_i - e_j$, $1 \leq i < j \leq n$. In fact, at $a = n - 7$ the moduli space $M_{S,ah-K_S}$ is flopped with center the strict transforms of lines joining p_i and p_j.

A line l yields another 1-cycle other than Z_l. Let $\pi : S \longrightarrow S'$ be the blow-down of $l \subset S$ to a point q on a smooth surface S' and assume that an ample divisor L is sufficiently near to the pull-back of an ample divisor L' on S'. The direct image $\pi_* E$ of a member E of $M_{S,L}$, is not locally free at $q \in S'$. But its double dual belongs to $\overline{M}_{S',L'}$ and we get a morphism

$$M_{S,L} \longrightarrow \overline{M}_{S',L'}, \quad E \mapsto (\pi_* E)^{\vee\vee}. \tag{22}$$

This morphism is a $\mathbf{P}^1$-bundle ove the open set $M_{S',L'}$ and interprets Example 4 moduli theoretically in the case $s = 3$. We denote by F_l a general fiber of this morphism.

The following is a substitute for Lemma 6 in the cases [7] and [9].

Lemma 9 *Let l be a line. Then*

$$2(Z_l.D) = -(2l + K_S.\varphi(D)) \quad \text{and} \quad (F_l.D) = (l.\varphi(D))$$

hold for every divisor D on X.

Proof. We prove the case $n = 8$. Other cases are similar and easier. The isomorphism φ is $W(E_8)$-equivariant and the Weyl group $W(E_8)$ acts transitively on the set of 240 classes of all lines. Hence, by Proposition 4, it suffices to verify the assertion for one line l. For the first formula, we take $h - e_1 - e_2$ as l. As we saw above, Z_l is the strict transform of the line passing through p_1 and p_2. Hence we have $(Z_l.e_1) = (Z_l.e_2) = 1$, $(Z_l.e_i) = 0$ for $3 \leq i \leq 8$

and $(Z_l.-K_X)=-1$. On the other hand we have $(l.h-e_1)=(l.h-e_2)=0$, $(l.h-e_i)=1$ for $3\le i\le 8$ and $(l.-2K_S)=1$. Hence, we have the equality $(Z_l.D)=-(\frac{1}{2}K_S+l.\varphi(D))$ for $D=e_1,\dots,e_8,-K_X$ by (20). Since $e_1,\dots,e_8$ and $-K_X$ generate $\operatorname{Pic} X\otimes\mathbf{Q}$, the equality holds for every D.

For the second formula, we take e_8 as l. By Example 4, F_l is the strict transform of a general line passing through p_8. Hence we have $(F_l.e_i)=0$ for $1\le i\le 7$, $(F_l.e_8)=1$ and $(F_l.-K_X)=2$. These intersection numbers on X are equal to $(e_8.h-e_i)$ and $(e_8.-2K_S)$, respectively. □

By the lemma, the hyperplanes $H_{l,1}$ and $H_{l,0}$ are mapped to those in $\operatorname{Pic} X\otimes\mathbf{R}$ defined by the 1-cycles Z_l and F_l by φ^{-1} respectively. A similar computation shows that $H_{C,1}$ is mapped to the hyperplane defined by Z_C for every conic C.

Proof of 'only if'part of Theorem. We prove the theorem by the induction on $n=(6,)7$ and 8. First we show the finite generation of $\mathcal{TC}(X_G)$ over $\varphi^{-1}\overline{\Pi}\subset\operatorname{Pic} X\otimes\mathbf{R}$. This is equivalent to the following:

Claim. The $\varphi^{-1}\overline{C}$-part of $\mathcal{TC}(X_G)$ is finitely generated for every chamber $\mathcal{C}$ in Π.

Every facet $\overline{\Pi}$ corresponds to either the blow-down of $e_C\simeq\mathbf{P}^{n-5}$ or a generic $\mathbf{P}^1$-bundle over $\overline{M}_{S',L'}$, where S' is the blow-down of a line from S. The blow-down of e_C is isomorphic in codimension one to $Bl_{n-1}\mathbf{P}^{n-4}$. Hence, by induction and by the result of §1, $\varphi^{-1}\mathcal{F}$-part of $\mathcal{TC}(X_G)$ is finitely generated for every facet $\mathcal{F}$ of Π. Let $R_1,\dots,R_n$ be the edges of $\overline{\mathcal{C}}$ contained in Π. We choose an ample divisor L_i on S from each R_i. By the GIT construction, $\overline{M}_{S,L_i}$ carries a natural ample (Cartier) divisor, which we denote by D_i. Then D_i is semi-ample on $\overline{M}_{S,\mathcal{C}}$. By the first formula of Lemma 9, D_i belongs to the ray $\varphi^{-1}R_i$. Therefore, by a lemma of Zariski ([HK, Lemma 2.8]), $\varphi^{-1}\mathcal{C}$-part of $\mathcal{TC}(X_G)$ is finitely generated. Thus the claim is proved.

The cone $\varphi^{-1}\overline{\Pi}$ is defined by two kinds of supporting hyperplanes, $\varphi^{-1}H_{C,1}$'s of divisorial (contraction) type and $\varphi^{-1}H_{l,0}$'s of fiber type. By the same argument as the case [4] in 5, $\mathcal{TC}(X_G)$ is generated by its $\varphi^{-1}\overline{\Pi}$-part and $\bigoplus_{C\,:\,\mathrm{conic}} H^0(\mathcal{O}_X(e_C))$. □

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