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Uniruledness of $\mathcal{M}_{11}$ and prime Fano 3-folds V_{22} of genus 12 — genus 11 and two poristic neighbors —

S. Mukai

in occasion of Prof. Mori's 70 th Birthday

§1 mini-history

• G. Fano (1871–1952)

Sulle varietà algebriche a tre dimensioni a curve-sezioni canoniche, 1937

$$C = C_{2g-2} \xrightarrow[\Phi_{|K|}]{} \mathbb{P}^{g-1},\ z \longmapsto \underbrace{(\omega_1 : \cdots : \omega_g)}_{\text{basis of } H^0(C, K_C)}$$

"K3 surface" in this lecture

= surface $[S \subset \mathbb{P}^g]$ w. $[S \cap \mathbb{P}^g] \cap H$ canonical

• Iskovskih (1977, 78) Modernization / Deciphering of classification of prime Fano 3-folds

Answer $g = 2, \cdots, 10, 12$ ← overlooked by Fano. # of moduli = 6

Method Fano's double projection from a line

• Mori Theory of extremal ray, PNAS (1980)

Mori – M. Classification of Fano 3-folds w. $B_2 \geq 2$

1981–82 IHS @ Princeton AG Year

§2. Main subject 1.

$$\mathcal{M}_g := \{\text{curves of genus } g\}/\text{isom.} \qquad \dim = \begin{cases} g & g=0,1 \\ 3g-3 & g\geq 2 \end{cases}$$

Alg. variety X is uniruled $\Leftrightarrow$ $\exists\, Y$ $\exists$ dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow\!\!\!\!\to X$ with $\dim Y = \dim X - 1$

Mori - M. (1983) $\mathcal{M}_{11}$ is uniruled. More precisely
(Work @ Princeton)

$\exists$ dominant rat'l map $\mathbb{P}^{11} \times \mathcal{F}_{11} \dashrightarrow\!\!\!\!\to \mathcal{M}_{11}$. Furthermore,

19-dimensional

$\mathcal{F}_{11}$ is moduli of polarized K3 surface (S, h) of genus 11.

Remark 1. (a) Chang - Ran (1984) $\mathcal{M}_{11}$ is unirational, i.e.,

$\exists\ \mathbb{P}^{30} \dashrightarrow\!\!\!\!\to \mathcal{M}_{11}$.

(b) Harris - Mumford (1983) $\mathcal{M}_g$ is of general type, i.e. $\kappa(\mathcal{M}_g) = \dim \mathcal{M}_g$ when $g \gg 0$.

Problem discussed in MM'83

$$\mathcal{F}_g := \left\{ \begin{array}{c}\text{polarized K3 surface } (S,h) \\ \text{with } (h^2) = 2g-2 \end{array} \right\} \Big/ \text{isom.}$$

$$\Downarrow$$

$$[S_{2g-2} \subset \mathbb{P}^g_{S}] \cap H = [C_{2g-2} \subset_{\Phi_{|K|}} \mathbb{P}^{g-1}]$$

$$H \longmapsto [C = S \cap H]$$

$$\mathcal{P}_g = \coprod_{(S,h)} \mathbb{P}^{g,*}_S \xrightarrow[\text{forgetful map}]{\varphi_g} \mathcal{M}_g$$

$\mathbb{P}^g$-bundle $\downarrow$ $\mathcal{F}_g$

$K_g := \operatorname{Im} \varphi_g$ Locus of "K3 curves"

Obviously, we have

$$\dim K_g \leq \min\{\underset{g+19}{\dim \mathcal{P}_g}, \underset{3g-3}{\dim \mathcal{M}_g}\} := \text{exp-dim.}$$

<u>Theorem</u> 2. ([MM'83] + α) "=" holds except for $g = 10, 12$.

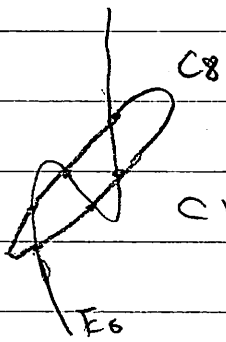
"pf ($g=11$)" C <u>hexagonal</u> curve of $g=11$, i.e. $\exists\, C \longrightarrow \mathbb{P}^1$ of degree 6

$C_{14} \subset \mathbb{P}^5$

$\alpha = \mathcal{O}_C(1)$, $\beta = K_C \otimes \mathcal{O}_C(-1)$ ($BN\# = 11 - 2 \times 6 = -1$)

$h^0 = 2$, degree 6; $h^0 = 6$, 14

$\Phi_{|\beta|} : C_{14} \hookrightarrow \mathbb{P}^5$



$2 \cdot 14 - 10 = 18$ $H^0(\mathcal{O}_C(2)) \longleftarrow H^0(\mathcal{O}_{\mathbb{P}}(2)) = S^2 \mathbb{C}^6$ 21 dim.

$\subset \mathbb{P}^5$ $\exists\, Q_1, Q_2, Q_3 \supset C$ $S = Q_1 \cap Q_2 \cap Q_3 \supset C$ K3 surface

"$\varphi_{11}^{-1}[C] = \{S\}$"

inside S_8

Hence φ_{11} is generically finite by "g.e.d." upper $\frac{1}{2}$-cont. of fiber dimension.

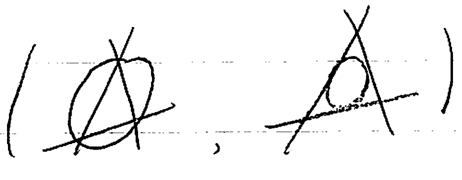
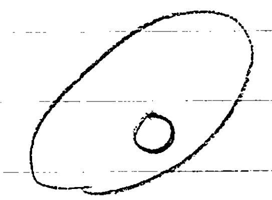
<u>Remark 3</u> Proof in [MM'83] uses DM-stable curve $C_8 + E_6 \subset \mathbb{P}^5$, a degeneration of the above $C_{14} \subset \mathbb{P}^5$, instead.

$g=5$ $g=1$ ... the ... proof ...

§3. Porism

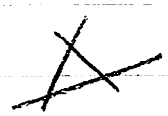
Elementary geometry on $\mathbb{P}^2$

pair of conics

$6+2+2=10$ $\left\{ \left(\right. \right.$  $\left. \left. \right) \right\}$ $\xrightarrow[\varphi]{\text{tangential}}$ $\left\{ \right.$  $\left. \right\}$

$5+5=10$

$\downarrow$

$2+2+2=6$ $\left\{ \right.$ triangle  $\left. \right\}$

Q. Is φ of maximal rk?

A. No, if a pair contains a biscribed triangle, then it contains ∞^1 such pairs (Poncelet's porism, or closure thm).

Poristic two neighbors: $g=10, 12 \Rightarrow \dim \mathcal{K}_g = (\text{exp. dim}) - 1$

$\boxed{\text{Genus } 12}$

$\varphi_{12} : \mathcal{P}_{12} \dashrightarrow \mathcal{M}_{12}$ 31-dim. $\dashrightarrow$ 33-dim.

Existence of a K3-extension (of $C_{22} \subset \mathbb{P}^{11}$, $|K|$) involves that of ∞^1 such extension.

Reason = Existence of Fano 3-folds $V_{22} \subset \mathbb{P}^{13}$ of genus 12 (due to Iskovskikh).

$$[C_{22} \subset \mathbb{P}^{11}] = [V_{22} \subset \mathbb{P}^{13}] \cap H_1 \cap H_2$$

$$\subset [S_t \subset \mathbb{P}^{12}] \cap (aH_1 + bH_2), \quad (a:b) \in \mathbb{P}^1$$

∞^1 K3-extension

$$C \xrightarrow[\Phi_{|\alpha|}]{\text{birat}} \overline{C}_9 \subset \mathbb{P}^2$$

<u>Example 4.</u> $\mathcal{C}_{T_9} = \left\{ \begin{matrix} C \text{ with} \\ \text{a } g^2_9 \end{matrix} \right\} \subset \mathcal{M}_{12}$ codim 3

BN # $= 12 - 3 \times 5 = -3$

$\alpha = \mathcal{O}_C(1)$, $\beta = K_C \otimes \mathcal{O}_C(-1)$

$h^0 = 3$ $\qquad\qquad h^0 = 5$

Consider $\Phi_{|\beta|} : C_{12} \hookrightarrow \mathbb{P}^4$ (as in the case $g=11$).

and $26 - 11 = 15$ $\quad H^0(\mathcal{O}_C(2)) \xleftarrow{r_2} H^0(\mathcal{O}_{\mathbb{P}}(2)) = S^2\mathbb{C}^5$ 15

~~28 $H^0(\mathcal{O}_C(3)) \xleftarrow{r_3} H^0(\mathcal{O}_{\mathbb{P}}(3)) = S^3\mathbb{C}^5$ 35~~

$G_2^{3, sp} := \left\{ \begin{matrix} C \text{ with} \\ \det r_2 = 0 \end{matrix} \right\} \subset G_9^3$ divisor

$G_2^{3, sp} \subset \mathcal{K}_{12}$ divisor

Because $C \subset \exists Q \subset \mathbb{P}^4$ and $\exists D_1, D_2 \subset \mathbb{P}^4$

~~cubic s.t. $C \subset Q \cap D_2$ K3 surface of degree 6.~~

~~$(35 - 28 - 5 - 1 = 1)$. $\exists$ 1-dim'l family of such K3's (possible)~~

$C \subset Q \cap (aD_2 + bD_1)$,

$(a:b) \in \mathbb{P}^1$

<u>§4. Main subject 2.</u>

$\mathcal{K}_{12} \subset \mathcal{M}_{12}$ contains 4 divisors $\mathcal{D}^{(i)}$, $i = 0, 2, 3, 4$.

corresponding to 1-nodal degeneration $V_{22}^{(i)}$ of smooth

Fano 3-folds. i = # of lines l, $(l \cdot -K_V) = 1$, passing

through the unique node.

node = O.D.P.

$\sim (x^2+y^2+z^2+t^2=0)$

analytic.

~~The node is algebraically non-factorial~~

~~and $V_{22}^{(i)}$ has two small resolutions~~.

V_{22}^a and V_{22}^b. R_a, R_b ext. ray

$i \neq 0$

$\Rightarrow i = -$(BN # of gen. member of $\mathcal{D}^{(i)}$)

i	0	2	3	4
$\mathcal{D}^{(i)}$	G^1_5	$G^{1,sp}_6$	$G^{2,sp}_9$	G^3_{11}
V_{22}^a and cont. R_a	$\mathbb{P}(\mathcal{E})/\mathbb{P}^3$ 2-bundle $c_1=0$, $c_2=4$ (Barth, 1977)	$Bl_{R_4} V_5$	$Bl_{R_5} Q^3$	$Bl_{R_5} \mathbb{P}^3$
V_{22}^b and cont. R_b	$V_{22}^b \longrightarrow \mathbb{P}^1$ dP_5	dP_5	conic bundle $/\mathbb{P}^2$ w. cubic discriminant	$Bl_{R_5'} \mathbb{P}^3$

Table 5. $\mathcal{D}^{(i)}$ and $V_{22}^{(i)}$

(Linear section) Thm 6 for $[C] \in \mathcal{K}_{12}$, $C_{22} \underset{\Phi_{|K|}}{\hookrightarrow} \mathbb{P}^{11}$

(1) $C \notin \bigcup_{i=0,2,3,4} \mathcal{D}^{(i)} \Rightarrow [C_{22} \subset \mathbb{P}^{11}]$

$$\overset{\exists !}{\cong} [\underset{\text{smooth}}{V_{22}} \subset \mathbb{P}^{13}] \cap H_1 \cap H_2$$

(2) $[C] \in \mathcal{D}^{(i)}$ general member

$\Rightarrow C \cong V_{22}^{(i)} \cap H_1 \cap H_2$

Example 4 (cont'd). $[C] \in \mathcal{D}^{(3)} = G^{2,sp}_9$

$\Phi_{|\beta|}: C_{13} \hookrightarrow Q^3 \subset \mathbb{P}^4 \qquad C_{13} \subset Q \cap D_1 \cap D_2$

$Q \cap D_1 \cap D_2 = C_{13} \cup R_5$ rat'l quintic

$R_5 \subset Q^3 \subset \mathbb{P}^4$ has a unique 3-secant line

$\ell \subset Q^3$, $\#(R_5 \cap \ell) = 3$

$$\begin{tikzcd}
\ell' \subset Bl_R Q^3 = V_{22}^a \arrow[d, "?"'] \arrow[ddrr, bend right=60] \arrow[r, dashed, leftrightarrow, "\text{flop}"] \arrow[dr] & V_{22}^b \arrow[d] \arrow[dr, "\text{conic bundle}"] & \\
\ell \subset Q^3 \supset R_5 & V_{22}^{(3)} \arrow[d, phantom, "\ni"] & \Delta_3 \subset \mathbb{P}^2 \\
& \text{ODP} &
\end{tikzcd}$$

Remark 7. Similar results hold for $g = 7, 8, 9, 10$ replacing $V_{22} \subset \mathbb{P}^{13}$ by rat'l homogeneous spaces

g	7	8	9	10
G/P	$OG(5,10)^+ \subset \mathbb{P}^{15}$	$G(2,6) \subset \mathbb{P}^{14}$	$SpG(3,6) \subset \mathbb{P}^{13}$	$G_2/P \subset \mathbb{P}(\mathfrak{g}_2)$

and their nodal degenerations.