Abstracts

Birational automorphism group of the Jacobian of a general Kummer quartic

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The lattice $II_{1,17}(2^{+6})$ is reflective and its orthogonal group has a fundamental domain with 896+64 facets by Borcherds [4, §12]. This is the Picard lattice (with respect to the Beauville form) of the Jacobian symplectic 6-fold

$$\mathrm{Jac}^2|h| := \coprod_{D\in|h|} \mathrm{Jac}^2 D$$

of a very general Kummer quartic surface $(S,h) \subset (\mathbb{P}^3, \mathcal{O}(1))$.

Main Theorem *The birational automorphism group of* $\mathrm{Jac}^2|h|$ *is generated by* 864 *modified reflections with respect to the above* $896-32$ *facets and a group of order* 2^{10} *whose center is Rapagnetta's involution (*[13]*).*

1. Lattices

Let $(L, \langle\ ,\ \rangle)$ be a *lattice*, i.e., a free $\mathbb{Z}$-module with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$.

1.1. **Reflective lattices.** We consider the hyperbolic case, namely, the signature is $(1,*)$ and the orthogonal group $O^+(L)$ of L preserving a positive cone C^+. The orthogonal transformation

$$r_m : x \to x - \frac{2\langle x,m\rangle}{\langle m^2\rangle} m$$

defined for a primitive element $m \in L$ with $\langle m^2\rangle < 0$ is called a *reflection of* L if it preserves L, i.e., $r_m(L) \subset L$. $m \in L$ (or $-m$) is called the *center* of the reflection. A lattice L is called *reflective* if the subgroup generated by all reflections of L is of finite index in $O^+(L)$.

By Esselmann [7], reflective lattices exist only in the range $\operatorname{rank} L \le 22$. For example, $U+D_{20}, U+D_{18}, U+D_{17}, \cdots$ are reflective lattices belonging to the Conway-Vinberg chain ([5] = [6, Chap. 28]).

1.2. **Notations and convention.** Throughout this abstract, we work over the complex number field $\mathbb{C}$.

- A_n, D_n and $E_{6,7,8}$ denote the negative definite root lattices of ADE-type, generated by (-2)-vectors.
- U denotes the hyperbolic lattice $\left(\mathbb{Z}^2, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$ of rank two.
- $\mathrm{Disc}(L) := \mathrm{Coker}[L \to \mathrm{Hom}(L,\mathbb{Z})]$ (with $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form)
- $II_{a,b}(**)$ The genus of even lattices of signature (a,b) with discriminant type $**$. For example, $** = 2^{+2a}$ means that $\mathrm{Disc}(L)$ is a 2-elementary group of length $2a$ and the discriminant form is of even integral type ([15]).

- $G = L \rtimes R$ means the semi-direct product with normal subgroup L (in the left).

1.3. **Kummer lattice.** Let A be an abelian surface and $\mathrm{Km}(A)$ be its Kummer surface, i.e., the minimal resolution of the quotient $A/\pm 1_A$. The Picard lattice of $\mathrm{Km}(A)$ contains the lattice $16A_1$ as sub-lattice. The Kummer lattice **Kum** is the primitive hull of $16A_1$ in the Picard lattice. The isomorphism class of **Kum** does not depend on A. In fact, it is explicitly described by the Reed-Muller code $[16, 8, 4]$, the binary code of length 16, minimal weight 8 and dimension 4. The code is generated by four words $c_1 = (0000000011111111), c_2 = (0000111100001111)$ $c_3 = (0011001100110011)$ and $c_4 = (01010101010101)$. Then we have,

$$\mathbf{Kum} = 16A_1 + \sum_{i=1}^{4} \mathbb{Z}\frac{c_i}{2} + \mathbb{Z}\frac{(1111111111111111)}{2}.$$

Kum belongs to $II_{0,16}(2^{+6})$ and hence $U + \mathbf{Kum}$ belongs to $II_{1,17}(2^{+6})$. By the uniqueness theorem of indefinite lattices (Nikulin [15]), $II_{1,17}(2^{+2a})$ consists of the unique lattice. Hence we use $U + \mathbf{Kum}$ and $II_{1,17}(2^{+6})$ interchangeably.

2. Known geometric realization ($\rho = 18$)

For $a = 0, 1, 2$, the lattice $II_{1,17}(2^{+2a})$ is realized as the Picard lattice of a K3 surface S, and gives an explicit description of the automorphism group $\mathrm{Aut}(S)$.

- $a = 0$ (The unimodular lattice) $II_{1,17} \simeq U + E_8 + E_8$ is realized by an elliptic K3 surface with 2 reducible fibers of (Kodaira) type $\tilde{E}_8$. In this case $\mathrm{Aut}(S)$ is finite (Vinberg [17]).
- $a = 1$ $II_{1,17}(2^{+2}) \simeq U + D_{16}$. S is the minimal resolution of the double $\mathbb{P}^1 \times \mathbb{P}^1$ studied by Horikawa, Dolgachev, Barth-Peters, M.-Namikawa etc. in 80's. The automorphism group $\mathrm{Aut}(S)$ is virtually $\mathbb{Z}$.
- $a = 2$ $II_{1,17}(2^{+4}) \simeq U + D_8 + D_8$. S is the Kummer surface $\mathrm{Km}(E_1 \times E_2)$ of product type. The orthogonal group $O^+(II_{1,17}(2^{+4}))$ is the semi-direct product

$$\langle 24\ (-2)\text{−reflections}, 24\ (-4)\text{−reflections}\rangle \rtimes (2^4.\mathfrak{S}_{4,4}).$$

The centers of (-2)-reflections are represented by $\mathbb{P}^1$'s and all (-4)-reflections are geometrically realized by involutions after suitable modification.

Theorem(Keum-Kondo [11]) *The automorphism group of* $\mathrm{Km}(E_1 \times E_2)$ *for general elliptic cuves* E_1, E_2 *is the semi-direct product*

$$\langle 24 \text{ modified } (-4)\text{−reflections}\rangle \rtimes 2^4.$$

- $a = 3$ The next is our lattice $II_{1,17}(2^{+6}) \simeq U + \mathbf{Kum}$ but no more realized as Picard lattice of a K3 surface S since the sum of the rank and the length of discriminant group exceeds 22, the second Betti number of S.

But ...

3. Main theorem

... our lattice $U + \mathbf{Kum}$ is realized as the Picard lattice of a holomorphic symplectic 6-fold associated with a general Kummer quartic surface. More precisely, let $\bar{S} \subset \mathbb{P}^3$ be a Kummer quartic surface of a general curve C of genus 2. $\bar{S}$ is the image of the Jacobian surface $\operatorname{Jac} C$ by the linear system $|2\Theta|$. The minimal resolution S is the Kummer surface of $\operatorname{Jac} C$. We denote the hyperplane section class of $S \to \bar{S} \subset \mathbb{P}^3$ by h. Consider the (compactified) Jacobian fibration $\operatorname{Jac}^d |h| \to |h|$, whose fiber at $[D] \in |h|$ is $\operatorname{Jac}^d D$. In terms of standard notation of moduli of sheaves, $\operatorname{Jac}^d |h|$ is $M_S(0, h, d-2)$. By general theory, $\operatorname{Jac}^d |h|$ is a holomorphic symplectic 6-fold of deformation type $K3^{[3]}$. The birational class does not depend on d but only on its parity. When d is odd, $\operatorname{Jac}^d |h|$ is birationally equivalent to the Hilbert square $\operatorname{Km}(C)^{[2]}$.

Now we restrict ourselves to the even case, i.e., the symplectic 6-fold $\operatorname{Jac}^2 |h|$. By general theory again, its Picard lattice (with respect to the Beauville-Bogomolov-Fujiki form) is the orthogonal complement of $(0, h, 0)$ in $U + \operatorname{Pic}(S)$. Since the orthogonal complement of h in $\operatorname{Pic}(S)$ is the Kummer lattice $\mathbf{Kum}$, we have $(0, h, 0)^{\perp} \simeq U + \mathbf{Kum}$.

Theorem(Borcherds [4, p. 346]) *The orthogonal group $O^+(II_{1,17}(2^{+6}))$ is the semi-direct product*

$$\langle 64\ (-2)\text{−reflections}, 896\ (-4)\text{−reflections}\rangle \rtimes G.\mathfrak{A}_8,$$

where G is the extended extra-special 2-group $2^{1+8}.2$ of order 2^{10}.

The centers of all (-2)-reflections and 32 (-4)-reflections are represented by effective divisors.

Main Theorem *The birational automorphism group of $\operatorname{Jac}^2 |h|$ is the semi-direct product*

$$\langle 864 \text{ modified } (-4)\text{−reflections}\rangle \rtimes G,$$

and the central involution of G is the Mongardi-Rapagnetta-Saccà involution.

4. Sketch of proof

The proof is similar to the case of $\operatorname{Km}(E_1 \times E_2)$ in §2 and the case of general Jacobian Kummer surfaces in [12].

Firstly the centers of 64 (-2)-reflections are represented by effective irreducible divisors. 32 appear in the reducible fibers of the Lagrangian fibration $\operatorname{Jac}^2 |h| \to |h| \simeq \mathbb{P}^3$ in pairs $(16\tilde{A}_1)$. The remaining 32 appear in the dual fibration $\operatorname{Jac}^2 |\hat{h}| \to |\hat{h}| \simeq \mathbb{P}^{3,\vee}$ (another $16\tilde{A}_1$). (A Kummer quartic is self dual.)

The (in-)effectivity is subtle for the centers of 896 (-4)-reflections. The answer is that most of (-4)-centers are not but special 32 are represented by effective irreducible divisors!

We recall the general theory of MMP for $M_S(v)$ with $v = (r, *, s)$ from Bayer-Macri [2] and Hassett-Tschinkel [9]. Divisorial contractions are classified into the following three types modulo flops:

(BN) (-2)-contraction induced from a rigid (or spherical) object, e.g., $(-2)\mathbb{P}^1$ on S. BN stands for Brill-Noether.

(HC) The minimal resolution $S^{[n]} \to S^{(n)}$ of a symmetric product is typical. In this typical case the diagonal divisor (with Beauville norm $2-2n$) is contracted. HC stands for Hilbert-Chow.

(LGU) The original one is the contraction from the moduli of Giesekser-semi-stable rank 2 sheaves to the moduli of Uhlenbeck-Yau compactification of μ-stable moduli. LGU stands for Li-Gieseker-Uhlenbeck.

In our case all 64 (-2)-divisors above are all (BN), and (HC) does not occur since $v = (0, h, 0)$ has divisibility 2. (LGU) happens for 32 (-4)-centers in the following way:

We need a preparation on $\mathrm{Jac}^0|h|$ instead of $\mathrm{Jac}^2|h|$. Consider the difference divisor $D - D$ in the abelian 3-fold $\mathrm{Jac}^0 D$. Moving $[D]$ over the linear system $|h| \simeq \mathbb{P}^3$, we obtain the difference divisor $\mathcal{D} - \mathcal{D}$ in $\mathrm{Jac}^0|h|$. We have a forgetful morphism $\mathcal{D} - \mathcal{D} \to S \times S$ by definition. Then this divisor $\mathcal{D} - \mathcal{D}$ is contracted to the 4-fold $S \times S$ by an extremal contraction of $\mathrm{Jac}^0|h|$ of (LGU)-type.

Now we return to $\mathrm{Jac}^2|h|$. Recall that the Kummer quartic $\bar{S} \subset \mathbb{P}^3$ has 16 tropes, that is, double conic plane sections. Let $t \subset S$ be the reduced part of one of them and consider the line bundle $\mathcal{O}_S(-t)$. Its tensor gives rise to an isomorphism from $\mathrm{Jac}^2|h|$ to $\mathrm{Jac}^0|h|$. Hence we have 16 divisorial contractions of (LGU)-type of $\mathrm{Jac}^2|h|$. We have another set of 16 divisorial contractions by duality, i.e., by changing h by $\hat{h}$. In total we obtain 32 contractible divisors with Beauville norm -4.

The remaining 864 (-4)-reflections, whose centers are ineffective, are realized by involutions after suitable modification, whose details we omit here.

5. Other realization

The realization of $U + \mathbf{Kum}$ in §3 is generalized to non-principally polarized abelian surface A of type $(1, d)$. The twice polarization descends to a polarization h of degree $4d$ on the Kummer surface $S = \mathrm{Kum}(A)$. By the same computation as in §3, the Picard lattice of $(4d+2)$-fold $M_S(0, h, 0)$ contains $U + \mathbf{Kum}$. In the case $d = 2$, $\mathrm{Kum}(A) \subset \mathbb{P}^5$ is described explicitly by Barth [1] (see also [8]). Hence $K3^{[5]}$-type Jacobian symplectic 10-folds $\mathrm{Jac}^4|h| = M_S(0, h, 0)$ are also interesting.

$U + \mathbf{Kum}$ seems also realized in the Picard lattice of symplectic manifolds of (deformation) type OG10. There are two candidates of *pseudo Kummer surfaces*:

(A) 4-dimensional family of K3 surface $\bar{S}$ of degree 6 in $\mathbb{P}^4$ with spatial Heawood configuration SH of 15 nodes and 15 twisted cubics, where SH is the configuration $(15_7 - 15_7)$ of 15 points and 15 planes in the projective space $\mathbb{P}^3(\mathbb{F}_2)$ over the binary field.

(B) 4-dimensional family of double planes $\bar{S}$ with branch the union of six lines (see e.g., [18], [16]). The minimal resolution S has 21 $\mathbb{P}^1$'s with $(15_2 - 6_5)$ configuration.

It is interesting to study the birational automorphism group of $S^{[OG]}$ of these K3 surfaces S. Here $S^{[OG]}$ denotes originally the minimal resolution of the moduli space $M_S(2, 0, -2)$ of semi-stable 2-bundles on S. But, more generally, it also denotes the minimal resolution of $M_S(2v)$ for v with $(v^2) = 2$. For example, $(0, h, 0)$, h being the pull-back of a line, seems a natural choice for v in the case (B).

In the case (A), the *sum* of a twisted cubic counted twice and the seven nodes on it gives the hyperplane section class h of $\bar{S} \subset \mathbb{P}^4$. Furthermore we have 15 such divisors in $|h|$. This is a very natural analogy of 16 tropes of a Kummer quartic surface. This kind of degree 6 divisor $2R + N_1 + \cdots + N_7$ is observed recently in our study of parabolic version of the description of general polarized K3 surface of genus 13 in [10].

References

[1] W. Barth, *Abelian surface with a $(1, 2)$-polarization*, Adv. Stud. Pure Math. **10**(1987), 41–84.

[2] A. Bayer, and E. Macri, *MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations*, Invent. math. **198**(2014), 505–590.

[3] R. Borcherds, *Automorphism groups of Lorentzian lattices*, J. Algebra **111**(1987), 133–153.

[4] R. Borcherds, *Reflection groups of Lorentzian lattices*, Duke Math. J., **104**(2000),319–366.

[5] J.H. Conway and N.J.A. Sloane, *Leech roots and Vinberg groups*, Proc. Roy. Soc. London, **A384**(1983), 233–258. (Chap. 28 of [5])

[6] J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, 3rd edition, Grundlehren der Mathematischen Wissenschaften **290**, Springer–Verlag, New York 1999.

[7] F. Esselmann, *Über die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe*, J. Number Th., **61**(1996), 103–144.

[8] A. Garbagnati and A. Sarti, Kummer surfaces and K3 surfaces with $(\mathbb{Z}/2\mathbb{Z})^4$ symplectic action, Rocky Mountain J. Math. **46**(2016), 1141–1205.

[9] B. Hassett and Y. Tshinkel, *Extremal rays and automorphisms of holomorphic symplectic varieties*, in "K3 surfaces and their moduli", Progr. in Math., 2016, pp. 73-95.

[10] A. Kanemitsu and S. Mukai, *Polarized K3 surfaces of genus thirteen and curves of genus three*, arXiv:2310.02078.

[11] J. Keum and S. Kondo, *The automorphism groups of Kummer surfaces associated with the product of two elliptic curves*, Trans. AMS., **353**(2000), 1469–1487.

[12] S. Kondo, *The automorphism group of a generic Jacobian Kummer surface*, J. Alg. Geom. **7**(1998), 589–609.

[13] G. Mongardi, A. Rapagnetta and G. Saccà, *The Hodge diamond of O'Grady's six-dimensional example*, Comp. Math. **154**(2018), 984–1013.

[14] V. Nikulin, *On Kummer surfaces,* Math. USSR Izv., **9**(1975), 261–275.

[15] V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izv., **14**(1980), 105–167.

[16] K. Utsumi, Weierstrass equations for Jacobian fibrations on a certain K3 surface. Hiroshima Math. J. **42**(2012), 353–383.

[17] E. B. Vinberg, *Some arithmetic discrete groups in Lobachevskii spaces*, in "Discrete subgroups of Lie groups and applications to Moduli", Tata–Oxford,1975, 323–348.

[18] M. Yoshida, *Hypergeometric functions, my love. Modular interpretations of configuration spaces*, Aspects of Mathematics, **E32**. Friedr. Vieweg & Sohn, Braunschweig, 1997.

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