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MODULI OF VECTOR BUNDLES ON K3 SURFACES, AND SYMPLECTIC MANIFOLDS

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K3 surfaces have been studied from old times as quartic surfaces or as Kummer surfaces. The name 'K3' itself was introduced only a quarter of a century ago. Since then remarkable progress has been made in its study. In the sixties, the foundations were layed for the modern study on their position in the classification of surfaces, on their moduli space, and on their period mapping, etc. In the seventies, the Torelli type theorem was established, which is the main source of further progress. Now a generalization to higher dimensions is tried and geometries (singularity, automorphism, degeneration, etc.) of K3 surfaces are studied in detail by combining with theories in other fields.

The concept of moduli has long been known. For example, it has been well known that the number of moduli of Riemann surfaces of genus g is equal to $3g-3$. It has been widely understood that the automorphic function is nothing but the function on the moduli space of elliptic curves. In a broad sense, a moduli space is the set of equivalence classes (isomorphism classes in most cases) of a certain type of geometric *objects*, endowed with a suitable *structure*. Among geometric objects are manifolds, submanifolds in a fixed manifold, vector bundles on a manifold, etc. Among structures are topology, differentiable structure, complex structure, etc. For each type of geometric object and for each structure, we can study the moduli problem. In this article, we restrict ourselves to the moduli of vector bundles[1]. But, even under this restriction, we meet various situations depending on which vector bundles we consider on which manifolds. As an example, let us consider complex topological vector bundles on topological spaces (in the category of CW complexes). In this case, there exists a vector bundle $\mathscr{E}$ on a topological space B with the following

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[1] The reader may consult Seshadri's survey article [113] for a more rigorous introduction to the algebro-geometric moduli problem. There he discusses the (local and global) moduli problem of (sub)varieties and of vector bundles and the construction of the moduli by means of the geometric invariant theory.

universal property: For every topological space X, the mapping

$$\{\text{continuous mapping from } X \text{ to } B\}/\text{homotopy equiv.} \to \{\text{vector bundle on } X\}/\text{isom.}$$
$$[f: X \to B] \mapsto [f^*\mathscr{E}]$$

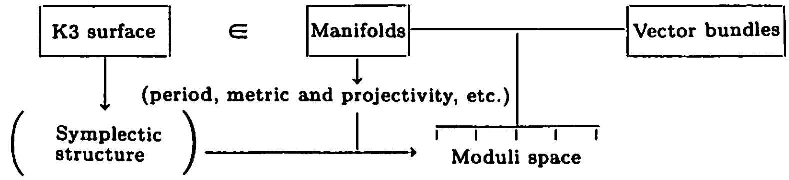
is bijective. B is called the classifying space and $\mathscr{E}$ the universal vector bundle. In the moduli problem discussed in the sequel, we always fix a complex manifold X and study the set V_X of isomorphism classes of holomorphic vector bundles on X. Though it seldom exists and we are forced to make a concession and a modification, the set V_X with a structure of complex analytic space is a *moduli space* in the most ideal sense if there exists a holomorphic vector bundle $\mathscr{E}_X$ on the product $X \times V_X$ with the following universal property: For every analytic space S, the mapping

$$\{\text{holom. mapping from } S \text{ to } V_X\} \to \{\text{holom. vector bundle on } X \times S\}/\text{equiv.}^2$$
$$[f: S \to V_X] \mapsto [(1_X \times f)^*\mathscr{E}_X]$$

is bijective. To a vector bundle F on the product $X \times S$ there is associated a set $\{F|_{X\times s}\}_{s\in S}$ of vector bundles on X. This is regarded as a family of vector bundles on X which vary holomorphically on the parameter s. The holomorphic mapping $f: S \to V_X$ corresponding to F as above is called the classification mapping of F. Thus, the moduli space V_X controls how holomorphic vector bundles on X vary holomorphically.

A moduli space parametrizes geometric objects of a certain type. Once it is constructed, the moduli space itself becomes an interesting geometric object of study. Absolute moduli spaces, such as the above classification space B and the moduli spaces of abelian varieties and curves, have rich geometric structures and plenty of symmetries. For relative moduli spaces, such as the V_X above, we are interested in how V_X inherits various properties (cohomology group, Riemannian metric, structure of algebraic (projective) variety and the field of definition, etc.) from X. In this article, we study this problem in the case of K3 surfaces. We are especially interested in how the moduli space of vector bundles inherits the symplectic structure and the period from K3 surfaces. We also discuss the relation with the theory of (holomorphic) symplectic manifolds, higher dimensional analogues of K3 surfaces.

[2] Two vector bundles F and F' on the product $X \times S$ are *equivalent* if there exists a line bundle L on S such that $F \otimes \pi_S^* L \simeq F'$. Equivalent vector bundles F and F' induce the same family $\{F|_{X\times s}\}_{s\in S} = \{F'|_{X\times s}\}_{s\in S}$ of vector bundles on X.



Notation. An (exterior) differential form of degree r is simply called an r-form [3]. Let X be a complex manifold. The sheaf of holomorphic r-forms on X is denoted by Ω_X^r. In the case $r = \dim X$, an r-form is called a canonical form and the sheaf Ω_X^r is called the canonical (line) bundle. Holomorphic 0-forms are simply holomorphic functions. The sheaf Ω_X^0 is denoted by $\mathcal{O}_X$ and called the structure sheaf of X.

For a vector space or a vector bundle E, we denote its dual by $E^{\vee}$.

1. K3 SURFACES

In a word, K3 surfaces are 2-dimensional analogues of elliptic curves. K3 surfaces and 2-dimensional complex tori have many common properties and their position in all complex surfaces is almost the same as that of elliptic curves in all curves (i.e., compact Riemann surfaces). On one hand, every elliptic curve E is expressed in the following way:

$$\text{(A)} \qquad E = \mathbf{C}/\Gamma, \qquad \Gamma \simeq \mathbf{Z} \oplus \mathbf{Z}$$

as a one dimensional complex torus. On the other hand, it has many projective models. Among them the Weierstrass standard form is the most famous. By using the $\wp$-function [4], E is expressed in the form

$$\text{(B.1)} \qquad \begin{aligned} &E\colon Y^2 = 4X^3 - g_2X - g_3, \qquad X = \wp(z),\ Y = \wp'(z), \\ &g_2 = 60 \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^4}, \qquad g_3 = 140 \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^6}. \end{aligned}$$

This shows that E is a double cover of the (complex) projective line $\mathbf{P}^1$ branching at four points. This also shows that E is a smooth cubic curve in the projective plane $\mathbf{P}^2$. If we use the ϑ-functions [5], then we obtain Jacobi's standard

[3] Do not confuse the r-form with the following: If $\{X_0, \dots, X_n\}$ is a system of homogeneous coordinates of the projective space $\mathbf{P}^n$, then a homogeneous polynomial $F(X_0, \dots, X_n)$ of degree d is called a *form of degree* d on $\mathbf{P}^n$.

[4] For a suitable coordinate z of the universal covering $\mathbf{C}$ of E, the $\wp$-function is defined by $\wp(z) \equiv 1/z^2 + \sum_{0 \neq \gamma \in \Gamma}(1/(z-\gamma)^2 - 1/\gamma^2)$.

[5] Replacing by a suitable affine transformation, we may assume that $\Gamma = \mathbf{Z} + \mathbf{Z}\tau$ and $\operatorname{Im}\tau > 0$. Put $q = e^{i\pi\tau}$. Then ϑ-functions of E are defined by

$$\vartheta_3(z) \equiv \sum_{n \in \mathbf{Z}} q^{n^2} e^{2n\pi i z} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2n\pi z,$$

$$\vartheta_2(z) \equiv e^{i\pi\tau/4} e^{\pi i z} \vartheta_3(z + \tfrac{\tau}{2}) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos(2n+1)\pi z,$$

$$\vartheta_1(z) \equiv \vartheta_2(z + \tfrac{1}{2}) \quad \text{and} \quad \vartheta_0(z) \equiv \vartheta_3(z + \tfrac{1}{2}).$$

form

$$\text{(B.2)} \qquad E: \begin{cases} X_0^2 = kX_1^2 + k'X_3^2, & X_i = \vartheta_i(z), \\ X_1^2 = kX_0^2 - k'X_2^2, & i = 0, 1, 2, 3, \end{cases}$$
$$k = \vartheta_2^2(0)/\vartheta_3^2(0), \quad k' = \vartheta_0^2(0)/\vartheta_3^2(0).$$

This shows that E is a complete intersection [6] of two quadratic surfaces in the projective space $\mathbf{P}^3$.

Among all the curves, the elliptic curves are characterized by the property that they have nowhere zero holomorphic canonical forms. There are exactly three types of surfaces with this property (Kodaira [46, I, §6]). One is the 2-dimensional complex tori and another is the K3 surfaces [7]. They inherit (A) and (B) from the elliptic curve, respectively.

Definition (1.1). A surface (i.e., 2-dimensional compact complex manifold) is a $K3$ *surface* if it satisfies

(1) there exists a holomorphic 2-form $\omega \in H^0(S, \Omega^2)$ without zeroes, and
(2) the first Betti number B_1 is equal to zero.

By (1), K3 surfaces are symplectic manifolds in the following sense.

Definition (1.2). A closed holomorphic 2-form ω on a complex manifold X is a (*holomorphic*) *symplectic structure* if ω is nowhere degenerate, i.e., the skew-symmetric bilinear form $\omega_x: t_{X,x} \times t_{X,x} \to \mathbf{C}$ on the tangent space $t_{X,x}$ of X is nondegenerate at every point $x \in X$.

Every two K3 surfaces can be deformed to each other (ibid., §5). The isomorphism classes of all K3 surfaces are locally parametrized by a 20-dimensional complex manifold. It is known that every K3 surface has a Kähler metric [8] (Siu [86], cf. [8]). In this article, we do not treat nonalgebraic K3 surfaces. All algebraic K3 surfaces [9] are parametrized by a countable union of 19-dimensional algebraic varieties. This relationship between all K3 surfaces and algebraic K3 surfaces is similar to that between 2-dimensional complex tori and abelian surfaces.

Now we give some examples of (algebraic) K3 surfaces.

Example (1.3) (Quartic surface). Let $\{X_0, X_1, X_2, X_3\}$ be a system of homogeneous coordinates of the projective space $\mathbf{P}^3$ and f a homogeneous polynomial

[6] An intersection $Y = Y_1 \cap \cdots \cap Y_n \subset X$ of subvarieties $Y_1, \ldots, Y_n$ in X is a *complete intersection* if the codimension of Y in X is equal to $\sum_{i=1}^n \operatorname{codim}_X Y_i$ at every point of Y.

[7] The third type of surfaces with trivial canonical bundles is called Kodaira surfaces. They are neither algebraic nor Kähler. Their first Betti numbers are equal to 3.

[8] A Hermitian metric $\sum_{\alpha,\beta=1}^n g_{\alpha\bar\beta}\, dz^\alpha\, d\bar z^\beta$ of a complex manifold is a Kähler metric if its fundamental form $(\sqrt{-1}/2)\sum_{\alpha,\beta=1}^n g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$ is closed. A smooth projective variety always has a Kähler metric.

[9] Every (smooth compact) algebraic surface has a projective embedding. In particular it has a Kähler metric.

of degree 4 in the variables X_0, X_1, X_2, and X_3. The set of zeroes

$$S: f(X_0, X_1, X_2, X_3) = 0 \quad \text{in } \mathbf{P}^3$$

of f is a K3 surface if it is smooth at every point $s \in S$, that is, if the partial derivatives $\partial f/\partial X_i$ ($i = 0, 1, 2$, and 3) have no common zeroes.

The quartic surface S satisfies (2) in Definition (1.1) by the theorem of Lefschetz[10]. We show that S also satisfies (1). Let U_0 be the open subset of $\mathbf{P}^3$ defined by $X_0 \neq 0$. Then U_0 is an affine 3-space $\mathbf{C}^3$ with the system of coordinates X_1/X_0, X_2/X_0, and X_3/X_0. We expand the 3-form

$$\Psi_0 = d(X_1/X_0) \wedge d(X_2/X_0) \wedge d(X_3/X_0)$$

on U_0 formally and obtain $\Psi_0 = \Psi/X_0{}^4$, where we put

$$\begin{aligned}\Psi = {} & X_0 dX_1 \wedge dX_2 \wedge dX_3 - X_1 dX_0 \wedge dX_2 \wedge dX_3 \\ & + X_2 dX_0 \wedge dX_1 \wedge dX_3 - X_3 dX_0 \wedge dX_1 \wedge dX_2 .\end{aligned}$$

Hence the 3-form

$$\Psi/f(X_0, X_1, X_2, X_3) = \Psi_0/f(1, X_1/X_0, X_2/X_0, X_3/X_0)$$

has simple poles along the intersection $S \cap U_0$ and is holomorphic on $U_0 \backslash S$. This is the same for the other open subsets U_i: $X_i \neq 0$ ($i = 1, 2$, and 3). Hence $\Psi/f(X_0, X_1, X_2, X_3)$ is a meromorphic 3-form on $\mathbf{P}^3$ with simple poles along S. The residue $\mathrm{Res}_S(\Psi/f)$ of Ψ/f along S is defined as a meromorphic 2-form ω on S. Since S is smooth, ω has no zeroes or poles. So we have proved (1.3).

The above argument also works for higher dimensional projective spaces $\mathbf{P}^n$. We put

$$\Psi = \sum_{i=0}^{n} (-1)^i X_i dX_0 \wedge \cdots \wedge dX_{i-1} \wedge dX_{i+1} \wedge \cdots \wedge dX_n$$

for a system of homogeneous coordinates $X_0, \ldots, X_n$ of $\mathbf{P}^n$. If $f(X_0, \ldots, X_n)$ is a homogeneous polynomial of degree $n+1$, then Ψ/f is an n-form (or canonical form) holomorphic on $f \neq 0$ and has simple poles along $f = 0$. By this fact, we obtain another example of a K3 surface.

Example (1.4). Assume that in the projective 4-space $\mathbf{P}^4$, a quadratic hypersurface

$$Q: q(X_0, X_1, X_2, X_3, X_4) = 0$$

and a cubic hypersurface

$$D: d(X_0, X_1, X_2, X_3, X_4) = 0$$

[10] If Y is a smooth ample divisor of a smooth projective algebraic variety X, then the natural homomorphism $H^i(X, \mathbf{Z}) \to H^i(Y, \mathbf{Z})$ is an isomorphism for every $0 \leq i < \dim X$ (cf. [97] and [98]).

intersect transversally, that is, two vectors $(\partial q/\partial X_0, \ldots, \partial q/\partial X_4)$ and $(\partial d/\partial X_0, \ldots, \partial d/\partial X_4)$ are linearly independent at every point of $Q \cap D$. Then $S = Q \cap D$ is a K3 surface.

In fact, taking residues of the 4-form Ψ/qd first along Q and next along S, we obtain a holomorphic 2-form on S. In a similar way, we also obtain the following two examples.

Example (1.5). Assume that three quadratic hypersurfaces

$$Q_i: q_i(X_0, X_1, X_2, X_3, X_4, X_5) = 0, \qquad i = 0, 1, \text{ and } 2$$

intersect transversally in the projective 5-space $\mathbf{P}^5$. Then the intersection $S = Q_1 \cap Q_2 \cap Q_3$ is a K3 surface.

Example (1.6). Let $C: \gamma(X_0, X_1, X_2) = 0$ be a smooth sextic curve in the projective plane $\mathbf{P}^2$ and let

$$\pi: S \to \mathbf{P}^2, \qquad Y^2 = \gamma(X_0, X_1, X_2)$$

be the double covering which ramifies exactly along C. Then S is a K3 surface, ($\pi^*\Psi/Y$ is a nowhere zero holomorphic 2-form on S.)

Each example above of a K3 surface carries a natural *polarization* (an equivalence class of finite morphisms to projective spaces). The next example is a classical one but has no natural polarization.

Example (1.7) (Kummer surface). Let $T = \mathbf{C}^2/\Gamma$, $\Gamma \simeq \mathbf{Z}^{\oplus 4}$, be a 2-dimensional complex torus and ι the symmetry $t \mapsto -t$ of T with respect to the origin. The fixed point set of ι coincides with the set of 2-torsion points $\frac{1}{2}\Gamma/\Gamma$. Hence the quotient space T/ι has sixteen ordinary double points[11]. Taking the minimal desingularization of T/ι, we obtain a K3 surface. We call this K3 surface the *Kummer surface* associated to T.

2. Vector bundles on K3 surfaces

In this section, we give some examples of vector bundles on K3 surfaces and show the existence of a symplectic structure on the moduli space in two concrete examples.

Definition (2.1). A holomorphic mapping $\pi: E \to X$ between complex manifolds is a *holomorphic vector bundle*[12] *of rank* r if there exist an open covering $\{U_i\}_{i\in I}$ of X and a family of biholomorphic mappings $\varphi_i: \pi^{-1}(U_i) \xrightarrow{\sim} \mathbf{C}^r \times U_i$,

[11] An n-dimensional hypersurface singularity $O \in \{f(X_0, \ldots, X_n) = 0\}$ is an ordinary double point if the initial form of f is quadratic and nondegenerate. The singularity is resolved by a single blowing-up. The exceptional divisor D is isomorphic to a smooth $(n-1)$-dimensional quadric $Q \subset \mathbf{P}^n$ and its normal bundle is isomorphic to $\mathcal{O}_Q(-1)$. In particular, in the case $n = 2$, D is isomorphic to $\mathbf{P}^1$ and the normal bundle is of degree -2.

[12] A vector bundle of rank one is called a line bundle.

$i \in I$, which satisfy

($\natural$) for every pair of i and $j \in I$, the difference of two mappings φ_i and φ_j over the intersection $U_i \cap U_j$ is expressed by a holomorphic function

$$g_{ij}: U_i \cap U_j \to GL(r, \mathbf{C}) \subset \mathbf{C}^{r^2}$$

to $GL(r, \mathbf{C})$, that is,

$$((\varphi_i|_{U_i \cap U_j})^{-1} \circ (\varphi_j|_{U_i \cap U_j}))(v, t) = (g_{ij}(t)v, t)$$

holds for every vector $v \in \mathbf{C}^r$ and $t \in U_i \cap U_j$.

In the above definition, if we assume further that X is an algebraic variety, U_i's are Zariski [13] open subsets, and g_{ij}'s are restrictions of rational functions on X, then E is called an *algebraic* vector bundle on X. If the base manifold X is a complete (or compact) algebraic variety, then by the GAGA principle (Serre [84]), every holomorphic vector bundle on X is algebraic. In the sequel, *vector bundle* (and its section) always means a holomorphic one unless otherwise specified.

First we take a K3 surface S in Example (1.5). Let W be the vector space of quadratic forms $q(X_0, X_1, X_2, X_3, X_4, X_5) = 0$ on $\mathbf{P}^5$ which vanish identically on S. Then W is a 3-dimensional vector space with basis q_0, q_1, and q_2 defining S. In other words, the set N of quadrics [14] of $\mathbf{P}^5$ containing S is a projective plane spanned by Q_0, Q_1, and Q_2. Let A_i be the symmetric 6×6 matrix corresponding to the quadratic form q_i, for $i = 1, 2$, and 3. A quadric $Q: \alpha_0 q_0 + \alpha_1 q_1 + \alpha_2 q_2 = 0$ is smooth if and only if the matrix $\alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2$ is regular. Hence the set of singular members in N coincides with

$$N_0 = \{Q: \alpha_0 q_0 + \alpha_1 q_1 + \alpha_2 q_2 = 0 \text{ in } \mathbf{P}^5 \mid \det(\alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2) = 0\}.$$

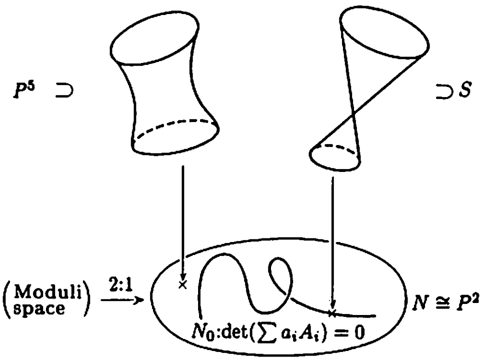
Since $\det(\alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2) = 0$ is a homogeneous polynomial of degree 6 in the variables α_0, α_1, and α_2, N_0 is a sextic curve in $N \simeq \mathbf{P}^2$.

Example (2.2) ([62]). Let S be a K3 surface in Example (1.5) and assume that every quadric containing S is of rank ≥ 5. Let $h \in H^2(S, \mathbf{Z})$ be the cohomology class (i.e., the Poincaré dual of the homology class) of hyperplane sections of $S \subset \mathbf{P}^5$. Then the moduli space of stable [15] (with respect to $S \subset \mathbf{P}^5$)

[13] A subset of an algebraic variety (resp. a compact complex manifold) is *Zariski open* if its complement is a closed algebraic (resp. analytic) subset.

[14] A quadratic hypersurface is simply called a (hyper)quadric. N is called a net of (hyper) quadrics. See [115] for the general theory of nets of quadrics.

[15] Let $X \subset \mathbf{P}^N$ be an n-dimensional projective algebraic variety. We denote the restriction of the tautological line bundle by $\mathcal{O}_X(1)$ and its kth power by $\mathcal{O}_X(k)$. For a coherent sheaf F, there exists a polynomial $P_F(t)$ such that $P_F(k)$ is equal to the dimension of the space of global sections of $F \otimes \mathcal{O}_X(k)$ for $k \gg 0$. $P_F(t)$ is called the Hilbert polynomial of F. $P_F(t)$ is a polynomial of degree $\leq n$ and $n!$ times the coefficient of t^n is equal to the rank $r(E)$ of E. A torsion free coherent sheaf E is (semi-)stable (with respect to $X \subset \mathbf{P}^N$ in the sense of Gieseker [29]) if $P_F(k)/r(F) < P_E(k)/r(E)$, $k \gg 0$ (resp. $\leq$) holds for every proper nonzero subsheaf F of E.



rank 2 vector bundles with $c_1 = h$ and $c_2 = 4$ is a K3 surface described in (1.6). Moreover, it is canonically isomorphic to the double cover of $N \simeq \mathbf{P}^2$ with branch the sextic curve N_0. (Bhosle [12] generalizes this to complete intersections of three quadrics in $\mathbf{P}^n$.)

Now we explain the above relationship between the vector bundles on S and the net of quadrics N. We recall that the Grassmann variety Grass($\mathbf{P}^1 \subset \mathbf{P}^3$) of lines in the projective space $\mathbf{P}^3$ is a smooth quadric $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$ in $\mathbf{P}^5$ by the Plücker coordinates [16]. For a point p (resp. a plane P) in $\mathbf{P}^3$, let L_p (resp. L_P) be the subset of Grass($\mathbf{P}^1 \subset \mathbf{P}^3$) consisting of lines passing through p (resp. contained in P). Both L_p and L_P are planes contained in Grass($\mathbf{P}^1 \subset \mathbf{P}^3$) $\subset \mathbf{P}^5$. The family of planes L_p's are parametrized by $\mathbf{P}^3$ and L_P's by the dual projective space of $\mathbf{P}^3$. All smooth quadrics in $\mathbf{P}^5$ are isomorphic to each other. Hence we have proved that every smooth 4-dimensional quadric Q contains two families of planes parametrized by projective 3-spaces. Take a family of planes on Q and denote it by

$$\{P_t \subset Q \mid P_t \text{ is a plane in } \mathbf{P}^5\}_{t \in \Delta}, \qquad \Delta \simeq \mathbf{P}^3.$$

For every point x of Q, the parameters t with $x \in P_t$ form a line in Δ, which we denote by l_x. Let V be the 4-dimensional vector space of linear forms on Δ. Then we obtain the exact sequence

$$0 \to F_x \to V \to E_x \to 0,$$

where F_x is the space of linear forms that vanish on l_x and E_x is the space of linear forms on l_x. Both F_x and E_x are of dimension 2. So we define a rank 2 subbundle F_Δ and a rank 2 quotient bundle E_Δ of the trivial vector bundle

[16] Let W be an r-dimensional subspace of a vector space V. Then the rth exterior product $\bigwedge^r W$ is a 1-dimensional subspace of $\bigwedge^r V$. The Plücker coordinate of W is the point of $\mathbf{P}_*(\bigwedge^r V)$ corresponding to $\bigwedge^r W$. By the Plücker coordinates, the Grassmann variety of r-dimensional subspaces of V (or $(r-1)$-dimensional subspaces of $\mathbf{P}_*(V)$) is a submanifold of $\mathbf{P}_*(\bigwedge^r V)$. This embedding Grass($\mathbf{P}^{r-1} \subset \mathbf{P}^{N-1}$) $\subset \mathbf{P}^{\binom{N}{r}-1}$ is called the Plücker embedding. In our case, $N = 4$ and $r = 2$, we put $p_{ij} = v_i \wedge v_j$ for a basis $\{v_1, v_2, v_3, v_4\}$ of $V^\vee$.

$V \times S$ by

$$F_\Delta = \bigcup_{s \in S} F_s \times \{s\} \subset V \times S, \qquad E_\Delta = \bigcup_{s \in S} E_s \times \{s\} \leftarrow V \times S.$$

Under the assumption in (2.2), E_Δ is a stable vector bundle satisfying the numerical condition in (2.2). Moreover, the family $\{E_\Delta\}$ is a complete set of representatives of all the isomorphism classes of such vector bundles, where Δ runs over all families of planes in quadrics in N. Associating to each Δ the quadric Q swept out by planes parametrized by it, we obtain a morphism from the moduli space to N. If a quadric Q degenerates and has a singular point, then the two families of planes on it become the same one. Hence this morphism is generically 2 to 1 and ramifies along N_0. This shows (2.2).

Next we consider the K3 surface $S \subset \mathbf{P}^4$ in Example (1.4). Let l be a line that intersects S at exactly two points (counting with the multiplicities) and let s be a point of S. We denote by V_l (resp. $F_{l,s}$) the space of linear forms on $\mathbf{P}^4$ that vanish on l (resp. on l and at s). Put

$$F_l' = \bigcup_{s \in S} F_{l,s} \times \{s\} \subset V_l \times S.$$

Unless s lies on l, $F_{l,s}$ is of dimension 2. Hence F_l' is a vector bundle over $S \backslash (S \cap l)$. F_l' extends a vector bundle F_l on all of S by the following:

Proposition (2.3) ([34]). *Let X be a 2-dimensional complex manifold and E a vector bundle over X minus a point $x \in X$. Then there exists a neighborhood U of x such that E is trivial over $U \backslash \{x\}$. Moreover, there exists a unique vector bundle $\tilde{E}$ on X whose restriction to $X \backslash \{x\}$ is isomorphic to E.*

Example (2.4). Let $S \subset \mathbf{P}^4$ be a K3 surface in Example (1.4) and assume that S contains no lines. Then F_l is a stable (with respect to $S \subset \mathbf{P}^4$) rank 2 vector bundle with $c_1 = -h$ and $c_2 = 4$, where $h \in H^2(S, \mathbf{Z})$ is the cohomology class of hyperplane sections of $S \subset \mathbf{P}^4$. Moreover, for every such stable vector bundle F, there exists a unqiue line l with $\#(l \cap S) = 2$ and such that $F_l \simeq F$.

We show the existence of a symplectic structure on the moduli space in the above case. By our assumption, l is either a line that joins two distinct points x and y on S or a line tangent to S at a point $x \in S$. The latter is the limit of the former as y goes to x inside S (y becomes a 1-dimensional subspace of the tangent space $t_{S,x}$ of S at x). Such a y is called an *infinitely near point* of x. The set of unordered pairs $\{x, y\}$, where x and y are distinct points on S or one is an infinitely near point of the other, is denoted by $\mathrm{Hilb}^2 S$. For every point $\{x, y\}$ of $\mathrm{Hilb}^2 S$, there exists a unique line l that joins x and y. Let $\widetilde{S \times S}$ be the blow-up of the product $S \times S$ of two copies of S along the diagonal. Then $\mathrm{Hilb}^2 S$ is isomorphic to the quotient of $\widetilde{S \times S}$ by the involution induced from the factor change. The natural mapping

$\mathrm{Hilb}^2 S \to \mathrm{Sym}^2 S$ is the minimal resolution of the second symmetric product of S. Moreover, by this description[17] of $\mathrm{Hilb}^2 S$, we have the following.

Proposition (2.5). *If S is a* K3 *surface, then* $\mathrm{Hilb}^2 S$ *has a natural symplectic structure induced from that of S.*

This proposition was first stated by Fujiki and established the existence of higher dimensional simply connected symplectic manifolds, which had been uncertain before[18]. The isomorphism classes of stable vector bundles in (2.4) are parametrized by the open subset

$$(\mathrm{Hilb}^2 S)^0 = \{\{x, y\} \mid l_{x,y} \cap S = \{x, y\}\}$$

of $\mathrm{Hilb}^2 S$. Therefore, we conclude that, in both cases (2.2) and (2.4), the moduli space of stable vector bundles has a symplectic structure. In the next section, we show that this always holds over K3 surfaces.

3. Symplectic structure of the moduli spaces

Let X be a complex manifold. By the moduli space of vector bundles on X we mean the set of their isomorphism classes endowed with a *natural* complex structure. But if we allow all the vector bundles, then we cannot obtain a good moduli *space*[19]. We must choose a nice class of vector bundles carefully according to the property we require of the moduli space. The following are typical examples of nice classes of vector bundles.

(A_{an}) Simple vector bundles on a compact complex manifold. The moduli space is an analytic space that may not be Hausdorff[20].

($\mathrm{A}_{\mathrm{alg}}$) Simple vector bundles on a complete algebraic variety. The moduli space is an algebraic space that may not be separated (Altman-Kleiman [2]). (Consult [48] for algebraic spaces.)

($\mathrm{B}_{\mathrm{alg}}$) Stable vector bundles on a projective algebraic variety $X \subset \mathbf{P}^N$. The moduli space is quasiprojective (in particular Hausdorff[21]). By adding the

[17] Let ω be a symplectic structure of S. Then $\omega^{\oplus 2} := \pi_1^*\omega + \pi_2^*\omega$ is a symplectic structure of $S \times S$. Since $\omega^{\oplus 2}$ is invariant under the factor change involution ι, $\omega^{\oplus 2}|_{S\times S\setminus\Delta}$ descends to a holomorphic 2-form $S^2\omega$ on $(S \times S\setminus\Delta)/\iota \subset \mathrm{Hilb}^2 S$. It is easy to see that $S^2\omega$ extends to a symplectic structure $\mathrm{Hilb}^2\omega$ on $\mathrm{Hilb}^2 S$.

[18] Theorem 2 in [15] is false. (2.5) is its counterexample.

[19] There exists a family of vector bundles $\{E_t\}_{t\in C}$ such that E_t is isomorphic to a vector bundle E for every $t \neq 0$ but E_0 is not. This is called a jumping phenomenon. For example, let L be a line bundle such that $H^1(L) \ni \alpha \neq 0$ and $H^0(L) = 0$. By the canonical isomorphism $\mathrm{Ext}^1(\mathcal{O}, L) \simeq H^1(L)$, every $t\alpha$, $t \in \mathbf{C}$, determines the extension $0 \to L \to E_t \to \mathcal{O} \to 0$. Then the family $\{E_t\}_{t\neq 0}$ jumps to $E_0 = \mathcal{O} \oplus L$ at $t = 0$. If we allowed such $E = E_1$ and E_0 in our moduli problem, then the point $[E]$ would not be closed in the moduli space.

[20] There exists a pair of families of simple vector bundles $\{E_t\}$ and $\{F_t\}$ such that $E_t \simeq F_t$ for every $t \neq 0$ and $E_0 \not\simeq F_0$. An example of such a pair is given over a curve of genus 3 in Narasimhan-Seshadri [72, Remark 12.3].

[21] If a coherent sheaf E is stable, then $P_F(k)/r(F) > P_E(k)/r(E)$, $k \gg 0$, holds for every nonzero quotient sheaf $F \neq E$. Assume that both E and E' are stable and that E' is a deformation of E. Then every nonzero homomorphism from E to E' is an isomorphism. This property of stable sheaves eliminates the jumping and non-Hausdorff phenomena.

set of certain equivalence classes[22] of semistable sheaves, it is compactified and becomes a projective scheme[23] (Mumford [66], Narasimhan-Seshadri [72] ($\dim X = 1$), Gieseker [29] ($\dim X = 2$), Maruyama [56]).

(B_{an}) Vector bundles with Einstein-Hermitian metrics on a compact Kähler manifold (X, g). The moduli space is a Hausdorff analytic space. The Kähler metric g induces a natural Kähler metric[24] on the nonsingular part of the moduli space (Kobayashi [45]).

(B_{YM}) Differentiable[25] vector bundles with anti-self-dual Yang-Mills connections on a compact Riemannian manifold (X, g) of real dimension 4. The moduli space is Hausdorff. The Riemannian metric g induces a Riemannian metric on the moduli space.

A vector bundle E on X is *simple* if every (holomorphic) endomorphism of E is the multiplication by a holomorphic function on X. If X is compact, then every endomorphism of a simple vector bundle is a constant multiplication. A Hermitian metric h of a vector bundle E on (X, g) satisfies the *Einstein condition* if the mean curvature $g^{-1}\bar{\partial}\partial \log h \in C^{\infty}(\mathscr{E}\mathrm{nd}(E))$ is a constant multiplication. For a vector bundle on a projective variety, we have

μ-stable[26]	$\Rightarrow$	stable	$\Rightarrow$	semistable	$\Rightarrow$	μ-semistable
$\Uparrow$		$\Downarrow$				$\Uparrow$
indecomposable E-H		simple	$\Rightarrow$	indecomposable		Einstein-Hermitian

In this section, we show that the moduli space of vector bundles on a K3 surface inherits the symplectic structure. We note that this is generalized in the following form.

(C) Simple vector bundles on a compact symplectic manifold (X, ω). The symplectic structure ω induces a symplectic structure on the smooth part of the moduli space (Kobayashi [44]).

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[22] If E is a semistable sheaf, then there exists a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$ such that each successive quotient $F_i := E_i/E_{i-1}$ is stable and satisfies $P_{F_i}/r(F_i) = P_E/r(E)$. This filtration is called a JHS-filtration of E. The isomorphism class of the direct sum $\mathrm{Gr}(E) := \bigoplus_{i=1}^{s} F_i$ does not depend on the choice of a JHS-filtration. Two semistable sheaves E and E' are *S-equivalent* if $\mathrm{Gr}(E) \simeq \mathrm{Gr}(E')$.

[23] ($\mathrm{B}_{\mathrm{alg}}$) is a beautiful application of the geometric invariant theory developed in Mumford [67] (cf. [77]).

[24] The imaginary part of a Kähler metric induces a real symplectic structure. (B_{an}) can be viewed as a combination of two inheritances of Riemannian metrics and of real symplectic structures [5, p. 46].

[25] If the Riemannian 4-fold is Kählerian, then the vector bundles with anti-self-dual Yang-Mills connections are holomorphic and essentially the same as the Einstein-Hermitian vector bundles in (B_{an}) ([43] and [103]).

[26] A vector bundle E on an n-dimensional projective variety $X \subset \mathbf{P}^N$ is μ-stable or stable in the sense of Mumford and Takemoto [88] (with respect to $X \subset \mathbf{P}^N$) if $(c_1(F) \cdot h^{n-1})/\mathrm{rank}\, F < (c_1(E) \cdot h^{n-1})/\mathrm{rank}\, E$ (resp. $\leq$) holds for every nonzero subsheaf F of E (or $\mathscr{O}(E)$) with $\mathrm{rank}\, F < \mathrm{rank}\, E$, where h is the cohomology class of hyperplane sections of $X \subset \mathbf{P}^N$. Kobayashi [43, 105] proved that every Einstein-Hermitian vector bundle is a direct sum of μ-stable bundles with the same slope and conjectured that the converse holds on projective varieties. In the case $\dim X = 1$, this conjecture is essentially the same as the equivalence of the stable vector bundle and the unitary representation of the fundamental group, which had been proved by Narasimhan and Seshadri [72] (see also [19]). Donaldson [20] has proved this conjecture in the case $\dim X = 2$.

(C_{YM}) Anti-self-dual Yang-Mills connections on a compact Riemannian 4-fold with a covariantly constant quaternion structure. The moduli space also has a covariantly constant quaternion structure (Itoh [39]).

The first (A_{an}) is a consequence of the existence of the Kuranishi space for vector bundles ([22, 25, 87]).

Theorem (3.1). *Let E be a simple vector bundle on a compact complex manifold X. Then there exist an analytic space $M(E)$ with a base point $*$ and a vector bundle $\mathcal{E}$ on the product $X \times M(E)$ which satisfy the following.*

(1) *The restriction $\mathcal{E}|_{X\times *}$ of $\mathcal{E}$ to $X \times *$ is isomorphic to E.*

(2) *Let T be an arbitrary analytic space with a base point $*$. If $\mathcal{E}'$ is a vector bundle on $X \times T$ with $\mathcal{E}'|_{X\times *} \simeq E$, then there exists a holomorphic mapping φ from a neighborhood of the base point of T to $M(E)$ such that $\varphi(*) = *$ and $\mathcal{E}' \simeq (1 \times \varphi)^* \mathcal{E}$.*

(3) *The above mapping φ is unique as a germ of holomorphic mapping from $(T, *)$ to $(M(E), *)$.*

($(M(E), *)$ and $\mathcal{E}$ are called the Kuranishi space and the Kuranishi family of E, respectively.)

Since simpleness is an open condition[27], we may assume that the restriction E_t of $\mathcal{E}$ to $X \times t$ is simple for every point $t \in M(E)$. We define topology and complex structure on the set of isomorphism classes of simple vector bundles on X by those of $M(E)$. We denote by SV_X the analytic space obtained in this manner.

In order to show some local properties of SV_X and an existence of holomorphic 2-forms on it, we consider the infinitesimal deformation of vector bundles. By Definition (2.1), to each vector bundle on X there are associated a pair of an open covering $\{U_i\}_{i\in I}$ of X and a set of holomorphic mappings $g_{ij}: U_i \cap U_j \to GL(r, \mathbf{C})$. We denote by $GL(r, \mathcal{O}_X)$ the sheaf of regular matrices of size r whose entries are holomorphic functions. Then g_{ij}'s are sections of $GL(r, \mathcal{O}_X)$ and satisfy $g_{ij}g_{jk}g_{ki} = 1$ for every $i, j, k \in I$. Hence the set $\{g_{ij}\}_{i,j\in I}$ is a (multiplicative) 1-cocycle with values in $GL(r, \mathcal{O}_X)$. Moreover, the set of isomorphism classes of rank r vector bundles is identified with the cohomology set[28] $H^1(X, GL(r, \mathcal{O}_X))$. Let ε be the infinitely small *number* such that $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. We put $\tilde{g}_{ij} = g_{ij}(1 + \varepsilon a_{ij})$, where a_{ij} is an $r \times r$ matrix whose entries are holomorphic functions on X. The 1-cochain $\{\tilde{g}_{ij}\}_{i,j\in I}$ is considered as a first order infinitesimal deformation of $\{g_{ij}\}_{i,j\in I}$.

[27] For every family $\{E_t\}$ of vector bundles, the function $t \mapsto \dim \mathrm{End}(E_t)$ is upper semi-continuous.

[28] Consult, e.g., [101]. In particular, all the isomorphism classes of line bundles on X are parametrized by the cohomology group $H^1(X, \mathcal{O}_X^*)$. From the exact sequence $0 \to \mathbf{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$, we have the exact sequence $H^1(X, \mathbf{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z})$. By the Hodge theory, the neutral connected component $\mathrm{Coke}[H^1(X, \mathbf{Z}) \to H^1(X, \mathcal{O}_X)]$ of $H^1(X, \mathcal{O}_X^*)$ is a complex torus if X is Kählerian.

It is a 1-cocycle if and only if

$$g_{jk}^{-1} a_{ij} g_{jk} + a_{jk} = a_{ik} \tag{3.2}$$

holds for every $i, j, k \in I$. This is the same as saying that $\{a_{ij}\}_{i,j\in I}$ is an (additive) 1-cocycle with values in the sheaf $\mathcal{E}nd(E)$ of (local) endomorphisms of E. By this correspondence we obtain the canonical isomorphism

(3.3) $\{$first order infinitesimal deformation of $E\}$/isom. $\simeq H^1(X, \mathcal{E}nd(E))$.

If E is simple, by our construction of SV_X and Theorem (3.1), this is equivalent to saying

(3.4) the Zariski tangent space of SV_X at the point $[E]$ is canonically isomorphic to the cohomology group $H^1(X, \mathcal{E}nd(E))$.

Here the Zariski tangent space at the point p is the dual vector space of the quotient $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal at p. In particular, we have the inequality

$$\dim_{[E]} SV_X \leq \dim H^1(X, \mathcal{E}nd(E)). \tag{3.5}$$

The equality holds if and only if SV_X is smooth at the point $[E]$.

For an endomorphism of a vector bundle, its trace is a scalar. Hence we have the trace homomorphism Tr: $\mathcal{E}nd(E) \to \mathcal{O}_X$. Associating $\mathrm{Tr}(f \circ g)$ for each pair (f, g) of endomorphisms, we obtain the bihomomorphism

$$\mathcal{E}nd(E) \times \mathcal{E}nd(E) \to \mathcal{O}_X.$$

Since this is symmetrical in f and g, the induced bilinear mapping

$$H^1(X, \mathcal{E}nd(E)) \times H^1(X, \mathcal{E}nd(E)) \to H^2(X, \mathcal{O}_X) \tag{3.6}$$

is skew-symmetric. Combining with (3.4), we have (3.7).

(3.7) The Zariski tangent space of SV_X has a natural skew-symmetric bilinear form with values in $H^2(X, \mathcal{O}_X)$ at each point.

Let F and G be vector bundles on a compact complex n-fold X and

$$F \times G \to \Omega_X^n \tag{3.8}$$

a bihomomorphism with values in the canonical line bundle Ω_X^n. This induces a bilinear mapping

$$H^i(X, F) \times H^{n-i}(X, G) \to H^n(X, \Omega_X^n) \tag{3.9}$$

for every i. The duality theorem of Serre [82] claims that $H^n(X, \Omega_X^n)$ is 1-dimensional and that (3.9) is nondegenerate if (3.8) is nondegenerate at every point of X. Applying this fact to our situation ($F = G = \mathcal{E}nd(E)$), we have that if X is a K3 surface, then (3.6) is nondegenerate. This and (3.7) are the reasons why the moduli space of vector bundles on a K3 surface has a symplectic structure. To prove it we need to show the following.

(3.10) When is SV_X nonsingular?

(3.11) Does the bilinear mapping (3.6) vary holomorphically as $[E]$ moves in SV_X? Is the 2-form obtained in this way always closed?

First we consider (3.10). If SV_X is smooth at the point $[E]$, then the Kuranishi space $M(E)$ of E is smooth at $*$ and its tangent space is isomorphic to $H^1(X, \mathscr{E}nd(E))$. Hence shrinking it if necessary, we may assume that $M(E)$ is an open neighborhood of 0 in $\mathbf{C}^N$, $N = \dim H^1(X, \mathscr{E}nd(E))$. The Kuranishi family $\mathscr{E}$ (or $\{E_t\}_{t \in M(E)}$, $E_t = \mathscr{E}|_{X \times t}$, see Theorem (3.1)) satisfies the following:

($\Diamond$) The infinitesimal deformations $\alpha_\nu = \partial E_t/\partial t_\nu|_{t=0}$, $\nu = 1, 2, \dots, N$, along t_ν at $t = (t_1, \dots, t_N) = 0$ form a basis of $H^1(X, \mathscr{E}nd(E))$.

So, we try to construct a family of deformations $\{E_t\}_{t \in T}$ of E with ($\Diamond$) for a neighborhood T of 0 in $\mathbf{C}^N$ in search of a condition for the smoothness of SV_X. We assume that the vector bundle E is given by a 1-cocycle $\{g_{ij}\}_{i,j \in I}$ for a sufficiently fine open covering $\{U_i\}_{i \in I}$ of X. We deform E by finding a family of 1-cocycles $\{G_{ij}(t)\}_{i,j \in I}$ parametrized by T such that $G_{ij}(0) = g_{ij}$ for every $i, j \in I$. We expand $G_{ij}(t)$ in a power series of $t = (t_1, \dots, t_N)$.

$$G_{ij}(t) = \sum_{\mu = (\mu_1, \dots, \mu_N)} g_{ij}^{(\mu)} t^\mu .$$

Let $\{\alpha^{(1)}, \dots, \alpha^{(N)}\}$ be a basis of $H^1(X, \mathscr{E}nd(E))$. We assume that $\alpha^{(1)}, \dots, \alpha^{(n)}$ are represented by 1-cocycles $\{a_{ij}^{(1)}\}_{i,j \in I}, \dots, \{a_{ij}^{(N)}\}_{i,j \in I}$. By ($\Diamond$), we may put $\sum_{|\mu|=1} g_{ij}^{(\mu)} t^\mu = g_{ij} \sum_{\gamma=1}^N a_{ij}^{(\gamma)} t_\gamma$. Hence $G_{ij}(t)$ satisfies

$$G_{ij}(t) \equiv g_{ij} + g_{ij} \sum_{\gamma=1}^{N} a_{ij}^{(\gamma)} t_\gamma \quad \mathrm{mod}(t_1, \dots, t_N)^2 .$$

When $\{g_{ij}^{(\mu)}\}_{i,j \in I}$ are defined for $|\mu| \le n$ so that $\{G_{ij}(t)\}_{i,j \in I}$ satisfies the 1-cocycle condition modulo $(t_1, \dots, t_N)^{n+1}$, we ask whether $\{g_{ij}^{(\mu)}\}_{i,j \in I}$ ($|\mu| = n+1$) can be chosen so that $\{G_{ij}(t)\}_{i,j \in I}$ is a 1-cocycle modulo $(t_1, \dots, t_N)^{n+2}$. An easy analysis[29] leads us to define the 2-cocycles $\{\mathrm{ob}_{ijk}^{(\mu)}\}_{i,j,k \in I}$ with coefficients in the sheaf $\mathscr{E}nd(E)$. Their cohomology classes are denoted by $\mathrm{ob}^{(\mu)}$ and are called *obstructions.* The above is possible if and only if its cohomology class $\mathrm{ob}^{(\mu)} \in H^2(X, \mathscr{E}nd(E))$ vanishes for every μ with $|\mu| = n + 1$. In particular, SV_X is smooth[30] at $[E]$ if $H^2(X, \mathscr{E}nd(E)) = 0$.

[29] If $\{G_{ij}(t)\}_{i,j \in I}$ is a 1-cocycle modulo $(t_1, \dots, t_N)^{n+1}$, then there exists a family of matrices $\mathrm{ob}_{ijk}^{(\mu)}$ whose entries are holomorphic functions on $U_i \cap U_j \cap U_k$ such that $G_{ij}(t) G_{jk}(t) G_{ki}(t) \equiv 1 + \sum_{|\mu|=n+1} \mathrm{ob}_{ijk}^{(\mu)} t^\mu \bmod (t_1, \dots, t_N)^{n+2}$. For every μ with $|\mu| = n + 1$, $\{\mathrm{ob}_{ijk}^{(\mu)}\}_{i,j,k \in I}$ is a 2-cocycle with coefficients in $\mathscr{E}nd(E)$.

[30] Assume that all the cohomology classes $\mathrm{ob}^{(\mu)} \in H^2(X, \mathscr{E}nd(E))$ vanish. We can choose $g_{ij}^{(\mu)}$'s so that the power series $G_{ij}(t) = \sum g_{ij}^{(\mu)} t^\mu$ converges in a neighborhood of 0. $\{G_{ij}(t)\}_{i,j \in I}$ defines the Kuranishi family $\mathscr{E}$ of E.

Now we look for a better sufficient condition for the smoothness of SV_X. Note that if $\{G_{ij}(t)\}_{i,j\in I}$ is a 1-cocycle modulo $(t_1, \ldots, t_N)^{n+1}$, then so is $\{\det(G_{ij}(t))\}_{i,j\in I}$. The key observation is this. The trace $\{\mathrm{Tr}(\mathrm{ob}^{(\mu)}_{ijk})\}_{i,j,k\in I}$ of an obstruction cocycle $\{\mathrm{ob}^{(\mu)}_{ijk}\}_{i,j,k\in I}$ is an obstruction cocycle for $\{\det(G_{ij}(t))\}_{i,j\in I}$ to extend a 1-cocycle modulo $(t_1, \ldots, t_N)^{n+2}$. We denote by $\det E$ the line bundle defined by the 1-cocycle $\{\det g_{ij}\}_{i,j\in I}$. Then the trace $\mathrm{Tr}(\mathrm{ob}^{(\mu)})$ is an obstruction for the moduli space (of line bundles) to be smooth at the point $[\det E]$. In other words, the following diagram is commutative.

$$\begin{array}{ccc} \{\text{obstruction for deformation of } E\} & \to & \{\text{obstruction for deformation of } \det E\} \\ \cap & & \cap \\ H^2(X, \mathscr{E}nd(E)) & \xrightarrow{\mathrm{Tr}} & H^2(X, \mathscr{O}_X) \end{array}$$

But every infinitesimal deformation $\alpha \in H^1(X, \mathscr{O}_X) = H^1(X, \mathscr{E}nd(L))$ of a line bundle L can be integrated by $\exp(\alpha) \in H^1(X, \mathscr{O}_X^*)$ on a complex manifold. Hence every obstruction vanishes for deformation of $\det E$. It follows that $\mathrm{Tr}(\mathrm{ob}^{(\mu)})$ vanishes. So we have

Proposition (3.12). *Let E be a vector bundle on a compact complex manifold X. Then every obstruction for the moduli space SV_X to be smooth at $[E]$ lies in the kernel of the natural linear mapping*

$$H^2(\mathrm{Tr})\colon H^2(X, \mathscr{E}nd(E)) \to H^2(X, \mathscr{O}_X).$$

In particular, SV_X is smooth at $[E]$ if $H^2(\mathrm{Tr})$ is injective.

Since E is a vector bundle, the sheaf $\mathscr{E}nd(E)$ is the direct sum of a structure sheaf $\mathscr{O}_X$ and the sheaf $\mathscr{E}nd^0(E)$ of trace zero endomorphisms of E. So the kernel of $H^2(\mathrm{Tr})$ is isomorphic to $H^2(X, \mathscr{E}nd^0(E))$. Proposition (3.12) and its proof have the advantage of being easily generalized to the case that E is a sheaf that may not be locally free (cf. [62]).

Corollary (3.13). *If X is a* K3 *surface, then SV_X is smooth.*

In fact, since the canonical line bundle is trivial, the injectivity of $H^2(\mathrm{Tr})$ is equivalent to the surjectivity of the linear mapping $\mathbf{C} \simeq H^0(X, \mathscr{O}_X) \to H^0(X, \mathscr{E}nd(E))$ by virtue of the Serre duality. The latter holds since E is simple.

Next we discuss (3.11). Let $\{G_{ij}(t)\}_{i,j\in I}$ be a family of 1-cocycles parametrized by an open subset T of $\mathbf{C}^N$. $\{G_{ij}(t)\}_{i,j\in I}$ defines a family of vector bundles on X. We denote it by $\{E_t\}_{t\in T}$. Let φ be its classification mapping from T to the moduli space. For every $a \in T$, $(d\varphi)_a\colon t_{T,a} \to H^1(X, \mathscr{E}nd(E_a))$ maps the tangent vector $\partial/\partial t_\alpha|_{t=a}$ to the cohomology class of the (additive) 1-cocycle $\{G_{ij}(a)^{-1}(\partial G_{ij}(t)/\partial t_\alpha)|_{t=a}\}_{i,j\in I}$. Let $\omega_{[E]}$ be the bilinear mapping in

(3.7) and put $\omega = \{\omega_{[E]}\}_{[E]\in \mathrm{Spl}_X}$. Then we have

$$\omega\left((d\varphi)\left(\frac{\partial}{\partial t_\alpha}\right), (d\varphi)\left(\frac{\partial}{\partial t_\beta}\right)\right)$$
$$= \mathrm{Tr}\left(\left\{G_{ij}^{-1}\frac{\partial}{\partial t_\alpha}G_{ij}\right\}_{i,j\in I} \cup \left\{G_{jk}^{-1}\frac{\partial}{\partial t_\beta}G_{jk}\right\}_{j,k\in I}\right)$$
$$= \left\{\mathrm{Tr}\left(G_{ij}^{-1}\frac{\partial G_{ij}}{\partial t_\alpha}\frac{\partial G_{jk}}{\partial t_\beta}G_{jk}^{-1}\right)\right\}_{i,j,k\in I} \in H^2(X, \mathscr{O}_X).$$

By the cocycle condition $G_{ij}G_{jk}G_{ki} = 1$, the pull-back of ω by φ is expressed as follows:

$$\varphi^*\omega = \frac{1}{2}\sum_{\alpha,\beta}\left\{\mathrm{Tr}\left(G_{ki}\frac{\partial G_{ij}}{\partial t_\alpha}\frac{\partial G_{jk}}{\partial t_\beta}\right)\right\}_{i,j,k\in I} dt_\alpha\wedge dt_\beta. \tag{3.14}$$

Hence ω is holomorphic. The pull-back of its exterior derivative is equal to

$$d(\varphi^*\omega) = \frac{1}{6}\sum_{\alpha,\beta,\gamma}\left\{\mathrm{Tr}\left(\frac{\partial G_{ij}}{\partial t_\alpha}\frac{\partial G_{jk}}{\partial t_\beta}\frac{\partial G_{ki}}{\partial t_\gamma}\right)\right\}_{i,j,k\in I} dt_\alpha\wedge dt_\beta\wedge dt_\gamma$$
$$\in H^2(X, \mathscr{O}_X)\otimes \Omega_T^3.$$

By the cocycle condition of $\{G_{ij}\}_{i,j\in I}$, we have

$$\frac{\partial G_{ki}}{\partial t_\gamma} = \frac{\partial}{\partial t_\gamma}(G_{jk}^{-1}G_{ij}^{-1}) = G_{jk}^{-1}\frac{\partial G_{jk}}{\partial t_\gamma}G_{jk}^{-1}G_{ij}^{-1} + G_{jk}^{-1}G_{ij}^{-1}\frac{\partial G_{ij}}{\partial t_\gamma}G_{ij}^{-1}.$$

Hence it follows that

$$d(\varphi^*\omega) = \frac{1}{6}\sum_{\alpha,\beta,\gamma}\mathrm{Tr}\left[\left\{G_{ij}^{-1}\frac{\partial G_{ij}}{\partial t_\alpha}\left(\frac{\partial G_{jk}}{\partial t_\beta}G_{jk}^{-1}\right)\left(\frac{\partial G_{jk}}{\partial t_\gamma}G_{jk}^{-1}\right)\right\}\right.$$
$$\left.+\left\{\left(G_{ij}^{-1}\frac{\partial G_{ij}}{\partial t_\alpha}\right)\frac{\partial G_{jk}}{\partial t_\beta}G_{jk}^{-1}\left(G_{ij}^{-1}\frac{\partial G_{ij}}{\partial t_\gamma}\right)\right\}\right] dt_\alpha\wedge dt_\beta\wedge dt_\gamma = 0.$$

Thus we have proved the following

Proposition (3.15). *Let SV_X be the moduli space of simple vector bundles on a compact complex manifold. Then the smooth part $(SV_X)_{\mathrm{reg}}$ of SV_X has a closed holomorphic 2-form ω with coefficients in $H^2(X, \mathscr{O}_X)$ such that $\omega_{[E]}$ coincides with* (3.6) *for every $[E]\in (SV_X)_{\mathrm{reg}}$.*

If X is a K3 surface, then (3.6) is nondegenerate. Combining with (3.13), we have

Theorem (3.16). *If X is a* K3 *surface, then the moduli space SV_X is smooth and has a natural symplectic structure.*

As is easily seen from its proof, the theorem also holds for 2-dimensional complex tori.

SV_X has an infinite number of connected components. Here we calculate their dimensions. For a pair of vector bundles E and F, we denote by $\mathcal{H}om(E, F)$ the sheaf of (local) homomorphisms from E to F. ($\mathcal{H}om(E, F)$ is a sheaf of (local) sections of the vector bundle $E^{\vee} \otimes F$.) We define the Euler-Poincaré characteristic of the pair (E, F) by

$$(3.17) \qquad \chi(E, F) = \sum_i (-1)^i \dim H^i(X, \mathcal{H}om(E, F)).$$

Let $r(E)$ and $c(E) = \sum c_i(E)$ be the rank and the Chern class of E. The Euler-Poincaré characteristic $\chi(E, F)$ is expressed in terms of $r(E)$, $r(F)$, $c(E)$, and $c(F)$ by virtue of the Riemann-Roch type theorem. If X is a K3 or an abelian surface, then we have

$$(3.18) \qquad \chi(E, F) = r(E)s(E) - (c_1(E) \cdot c_1(F)) + s(E)r(F),$$

where we put $s(E) = \varepsilon r(E) + \frac{1}{2}(c_1(E)^2) - c_2(E)$, ε is equal to 1 or 0 according as X is of type K3 or abelian, and $(\cdot)$ is the intersection pairing on $H^2(X, \mathbf{Z})$.

For a better understanding of (3.18), we introduce a *lattice*, i.e., a free $\mathbf{Z}$-module with an integral bilinear form. For a K3 or abelian surface X, put

$$\widetilde{H}(X, \mathbf{Z}) = \mathbf{Z} \oplus H^2(X, \mathbf{Z}) \oplus \mathbf{Z}.$$

We extend the inner product $(\cdot)$ by

$$(3.19) \qquad ((r, l, s) \cdot (r', l', s')) = -rs' + (l \cdot l') - sr'$$

and call $\widetilde{H}(X, \mathbf{Z})$ with $(\cdot)$ the *extended* K3 *lattice*. Moreover, for a vector bundle E on X, we define the vector $v(E)$ associated to E by

$$(3.20) \qquad v(E) = (r(E), c_1(E), s(E)) \in \widetilde{H}(X, \mathbf{Z}).$$

Then (3.17) becomes the following simple form:

$$(3.21) \qquad \chi(E, F) = -(v(E) \cdot v(F)).$$

If E is simple, then by the Serre duality theorem ([82], see (3.9)), we have

$$\dim H^2(X, \mathcal{E}nd(E)) = \dim H^0(X, \mathcal{E}nd(E)) = 1.$$

Hence, combining with (3.4) and (3.13), we have

$$(3.22) \qquad \dim_{[E]} SV_X = \dim H^1(X, \mathcal{E}nd(E)) = (v(E)^2) + 2.$$

For a vector v in $\widetilde{H}(X, \mathbf{Z})$, we denote by $SV_X(v)$ the set of isomorphism classes of simple vector bundles E with $v(E) = v$. Since the rank and the Chern class are invariant under deformation, $SV_X(v)$ is open and closed in SV_X. By (3.13) and (3.22), $SV_X(v)$ is of pure dimension $(v^2) + 2$.

Here we explain a generalization of Theorem (3.16) to coherent sheaves. The existence of Kuranishi spaces for (coherent) sheaves (of $\mathcal{O}_X$-modules) is proved by Siu and Trautman [87]. Hence, in the same way as vector bundles, a complex structure is defined on the set of isomorphism classes of simple sheaves on X.

We denote by Spl_X the analytic space obtained in this way. For a vector bundle E, we denote the sheaf of its sections by $\mathcal{O}_X(E)$. $\mathcal{O}_X(E)$ is locally free and the mapping $E \mapsto \mathcal{O}_X(E)$ gives an open immersion[31] of SV_X into Spl_X. So we identify SV_X with its image in Spl_X. All assertions so far for SV_X remain true and are proved by improving the above arguments if we replace SV_X with Spl_X and $H^i(X, \mathcal{E}nd(E))$ with $\mathrm{Ext}^i(E, E)$.

Theorem (3.23) ([62]). *If X is a* K3 *or an abelian surface, then the moduli space* Spl_X *of simple sheaves on X is smooth and has a natural symplectic structure. Moreover, for every simple sheaf E on X, the dimension of* Spl_X *at the point* $[E]$ *is equal to* $(v(E)^2) + 2$.

This honest generalization of (3.16) yields two important corollaries. We denote by $\mathrm{Hilb}^n X$ the set of 0-dimensional subschemes N of length n of X. $\mathrm{Hilb}^n X$ has a natural complex structure as a connected component of the Hilbert scheme Hilb_X of X (Grothendieck [100]). $\mathrm{Hilb}^n X$ is compact if X is compact. By forgetting the scheme structure of N, we obtain the 0-cycle $[N] = \sum_p m_p(N)(p)$ of length n, where p runs the support of N and $m_p(N)$ is the dimension (or multiplicity) of N at p. $[N]$ is regarded as a point of the nth symmetric product $\mathrm{Sym}^n X$ of X. The mapping φ: $\mathrm{Hilb}^n X \to \mathrm{Sym}^n X$, $N \mapsto [N]$ is holomorphic. If $\dim X \leq 2$, $\mathrm{Hilb}^n X$ is smooth and connected[32]. Hence the mapping φ is a desingularization of $\mathrm{Sym}^n X$. So we call $\mathrm{Hilb}^n X$ the nth Hilbert product of X in the case $\dim X = 2$. For a 0-dimensional subscheme N of X, let $\mathcal{I}_N$ be the sheaf of ideals defining N. The sheaf $\mathcal{I}_N \otimes L$ is a simple sheaf of rank 1 for every line bundle L on X. Every small deformation of $\mathcal{I}_N \otimes L$ is also of the form $\mathcal{I}_{N'} \otimes L'$. Hence the isomorphism classes of all $\mathcal{I}_N \otimes L$ with length $N = n$ form an open subset U in Spl_X. If $\dim X \geq 2$, $\mathcal{I}_N \otimes L \simeq \mathcal{I}_{N'} \otimes L'$ implies $N = N'$ and $L \simeq L'$. Hence U is isomorphic to the product of $\mathrm{Hilb}^n X$ and the Picard variety[33] $\mathrm{Pic}\, X$ of X. If X is a K3 surface, then $\mathrm{Pic}\, X$ is discrete. If X is a complex torus, then every connected component of $\mathrm{Pic}\, X$ is isomorphic to the dual torus $\hat{X}$ of X. Hence (3.23) implies the following generalization of (2.5), which was first proved by a different method in Beauville [9].

Corollary (3.24). *If X is a* K3 *(resp. an abelian) surface, then the Hilbert product* $\mathrm{Hilb}^n X$ *(resp. the product* $\hat{X} \times \mathrm{Hilb}^n X$*) has a natural symplectic structure.*

Thus we have obtained compact symplectic manifolds as open subsets of the moduli of rank 1 simple sheaves. Now we consider the case of rank ≥ 2.

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[31] By Definition 2.1, the sheaf $\mathcal{O}_X(E)$ is isomorphic to $\mathcal{O}_X{}^{\oplus r}$ on each open subset U_i. A sheaf with such an open covering $\{U_i\}_{i \in I}$ is called a locally free sheaf (of $\mathcal{O}_X$-modules) of rank r. A family of vector bundles E is recovered from a family of locally free sheaves $\mathcal{O}_X(E)$.

[32] See Fogarty [26]. In contrast to this fact, $\mathrm{Hilb}^n X$ can be reducible if $\dim X \geq 3$ (see [102]).

[33] The moduli space of line bundles on X is denoted by $\mathrm{Pic}\, X$ and called the Picard variety of X. Since the tensor product $\otimes$ induces a group structure, $\mathrm{Pic}\, X$ is also called the Picard group. If X is projective, then $\mathrm{Pic}\, X$ is an abelian variety (cf. [112] and footnote 28).

The moduli space $SV_X(v)$ (see (3.22)) very rarely contains a compact open subset[34]. In contrast with this, the moduli space $\mathrm{Spl}_X(v)$ of simple sheaves E on X with $v(E) = v$ often contains a compact open subset by virtue of $(\mathrm{B}_{\mathrm{alg}})$. We fix a projective embedding $X \subset \mathbf{P}^N$. For a vector $v = (r, l, s)$ of the extended lattice $\widetilde{H}(X, \mathbf{Z})$, let $M_X(v)$ be the set of isomorphism classes[35] of stable (with respect to $X \subset \mathbf{P}^N$) sheaves on X. A stable sheaf is simple and semistable. Since stability is an open condition (Maruyama [107]), $M_X(v)$ is an open subset of $\mathrm{Spl}_X(v)$. By $(\mathrm{B}_{\mathrm{alg}})$, $M_X(v)$ is naturally compactified by adding the equivalence classes of nonstable, semistable sheaves E with $v(E) = v$. Therefore, if it happens that every semistable sheaf E with $v(E) = v$ is stable, then $M_X(v)$ is compact. This happens, for example, if the greatest common divisor of the three integers r, $(l \cdot h)$, and s is equal to one[36], where h is the cohomology class of hyperplane sections of $X \subset \mathbf{P}^N$.

Corollary (3.25). *Let X be a K3 or abelian surface and $v = (r, l, s)$ a vector of the extended lattice $\widetilde{H}(X \cdot \mathbf{Z})$. If $\mathrm{GCD}(r, (l \cdot h), s) = 1$, then every connected component of the moduli space $M_X(v)$ is a smooth projective variety of dimension $(v^2) + 2$ with a natural symplectic structure.*

Corollary (3.24) is the special case of (3.25) with $v = (1, 0, \varepsilon - n)$. The moduli space $M_X(v)$ is connected in many cases, e.g., if $r \leq 2$.

Conjecture. The moduli space $M_X(v)$ is connected for every v if X is a K3 or an abelian surface.

4. Higher dimensional symplectic manifolds

In this section, we recall the general theory of compact Kähler manifolds with (holomorphic) symplectic structures. We give some examples of them and pose some problems concerning them.

If ω is a symplectic structure of $2n$-dimensional complex manifolds, then its Pfaffian

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$$

is a (holomorphic) canonical form without zeroes. Hence the canonical line bundle of a symplectic manifold is trivial. Conversely, let X be a compact complex manifold with trivial canonical line bundle. We further assume that

[34] As an example we consider the moduli space of vector bundles in Example 2.4. The K3 surface S is contained in a quadric Q and every line in Q meets S at three points. Hence the moduli space $\simeq (\mathrm{Hilb}^2 S)^0$ of stable vector bundles is not compact. Take a curve $\{x_t, y_t\}_{t \in \Delta}$ in $\mathrm{Hilb}^2 S$ such that $\{x_t, y_t\} \in (\mathrm{Hilb}^2 S)^0$ for every $0 \neq t \in \Delta \subset \mathbf{C}$ and $\{x_0, y_0\} \notin (\mathrm{Hilb}^2 S)^0$, then $\lim_{t \to 0} E_{l_t}$ is not a vector bundle at the third point of $l_0 \cap S$, where l_t is a line joining x_t and y_t.

[35] See [29], [55], or [57, Part I] for a more precise definition of M_X.

[36] If E is semistable and not stable, then there exists a subsheaf F of E with $(1/r(F))(r(F), (c_1(F) \cdot h), s(F)) = (1/r(E))(r(E), (c_1(E) \cdot h), s(E))$ and $0 < r(F) < r(E)$. Hence $r(E), (c_1(E) \cdot h)$, and $s(E)$ have a common divisor greater than one.

X has a Kähler metric. (This is the case if X is a projective algebraic variety.) By virtue of Yau's [95, 96] solution of Calabi's conjecture, X has a Kähler metric $g = (g_{ij})$ whose Ricci curvature (R_{ij}) is identically zero. Let $\widetilde{X}$ be the universal covering of X. The decomposition of the holonomy representation (with respect to g) into irreducible ones induces a decomposition of $\widetilde{X}$ into the product of a complex Euclidean space and Kähler manifolds with irreducible holonomy representations. (This is called the de Rham decomposition.)

Decomposition Theorem (4.1) (Bogomolov [14], Kobayashi [42], Beauville [9])[37]. *Let X be a compact Kähler manifold and assume that the first Chern class $c_1(X) \in H^2(X, \mathbf{Z})$ is torsion. Then there exists a finite unramified covering X' of X which is isomorphic to the product*

$$T \times \prod_i U_i \times \prod_j V_j ,$$

where

(1) *T is a complex torus,*
(2) *each U_i is a simply connected projective variety such that $H^0(U_i, \Omega^p) = 0$ for every $0 < p < \dim U_i$, and*
(3) *each V_j is a simply connected symplectic manifold such that $\dim H^0(V_j, \Omega^2) = 1$.*

Remark (4.2). The holonomy group is a special unitary group $SU(*)$ for each U_i and a symplectic group $\mathrm{Sp}(*)$ for V_j. The hypersurfaces of degree $n+1$ in the projective spaces $\mathbf{P}^n$ are examples of U_j's. Algebraic K3 surfaces satisfy both (2) and (3).

The manifold V satisfying (3) in the theorem is called an *irreducible* symplectic manifold. The symplectic structure ω of V is unique up to constant multiplications. Moreover, the algebra $\bigoplus_p H^0(V, \Omega^p)$ of holomorphic forms on V is generated by ω.

For a K3 surface S, its Hilbert product $\mathrm{Hilb}^n S$ is an irreducible symplectic manifold (Corollary (3.24)). For a 2-dimensional complex torus T, the fibers of the Albanese mapping $\mathrm{Hilb}^{n+1} T \to T$ are irreducible symplectic manifolds. We denote their isomorphism class by $\mathrm{Kum}^n T$ and call it the nth Kummer product of T. $\mathrm{Kum}^n T$ is a desingularization of the subvariety $\{\{t_0, \dots, t_n\} \mid \sum_i t_i = 0\}$ of the $(n+1)$st symmetric product $\mathrm{Sym}^{n+1} T$ of T. The first Kummer product is nothing but the Kummer surface (1.7) associated to T. $\mathrm{Kum}^n T$ appears as a decomposition factor when we apply (4.1) to the symplectic manifold $\widehat{T} \times \mathrm{Hilb}^{n+1} T$ (see (3.24)). In fact, the mapping

$$T \times \mathrm{Kum}^n T \to \mathrm{Hilb}^{n+1} T, \qquad (t, \{t_0, \dots, t_n\}) \mapsto \{t_0 + t, \dots, t_n + t\}$$

is an unramified Galois covering of degree $(n+1)^4$.

[37] The Decomposition Theorem is also proved by Michelsohn [108]. But Theorem 7.18 in [108] is not correct because an incorrect Theorem 2 in [15] is applied.

Example (4.3) (Donagi-Beauville [11]). Let V be a smooth cubic hypersurface in $\mathbf{P}^5$ and Grass($\mathbf{P}^1 \subset \mathbf{P}^5$) the Grassmann variety of lines in $\mathbf{P}^5$. Let $F(V)$ be the subvariety of Grass($\mathbf{P}^1 \subset \mathbf{P}^5$) consisting of the lines contained in V. Then $F(V)$ is an irreducible symplectic manifold.

It is proved in [1] that $F(V)$ has a trivial canonical line bundle and that $F(V)$ is a 4-dimensional subvariety of degree 108 in $\mathbf{P}^{14}$ by the Plücker coordinates. If V is deformed to another cubic hypersurface V', then $F(V)$ is deformed to $F(V')$. Hence, in view of (4.1), for the proof of (4.3), it suffices to show (4.3) for *one* cubic hypersurface. In [11], this is shown by using a K3 surface of degree 14. Here we prove it by using a K3 surface of degree 6. Let V_0 be a cubic hypersurface in $\mathbf{P}^5$ which has an ordinary double point at $p = (0: 0: 0: 0: 0: 1)$ and is smooth elsewhere. The defining equation of V_0 is of the form

$$V_0: q(X_0, X_1, X_2, X_3, X_4)X_5 + d(X_0, X_1, X_2, X_3, X_4) = 0 \quad \text{in } \mathbf{P}^5$$

for quadratic and cubic forms q and d. Let S be the surface in $\mathbf{P}^4$ defined as the common zero locus of q and d. By our assumption on V_0, the intersection of $q = 0$ and $d = 0$ is transversal. Hence S is a K3 surface by Example (1.4). It is easy to see that for every *pair of points* $\{a, b\} \in \mathrm{Hilb}^2 S$ of S, there exists a unique line $l_{a,b}$ in V_0 that meets the two lines $\overline{pa}$ and $\overline{pb}$. The mapping $\varphi: \mathrm{Hilb}^2 S \to F(V_0)$, $\{a, b\} \mapsto l_{a,b}$ is holomorphic and birational. $F(V_0)$ has ordinary double points along a subvariety isomorphic to S and φ is its minimal resolution. Since $\mathrm{Hilb}^2 S$ is a symplectic manifold and since $F(V)$ is a deformation of $F(V_0)$, $F(V)$ is also a symplectic manifold[38] for every smooth V.

As another example, we explain a way to obtain a new symplectic manifold from an old one. Let X be a $2n$-dimensional complex manifold with a symplectic structure ω and Y a complex submanifold of X. There exists a natural exact sequence

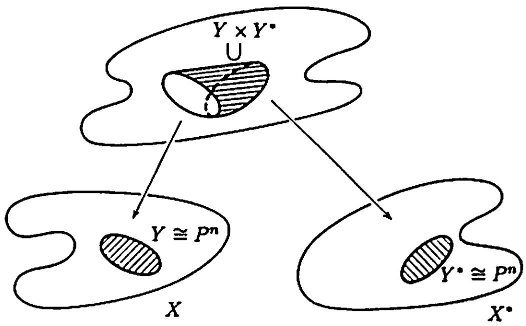
$$0 \to T_Y \to T_X|_Y \to N_{Y/X} \to 0$$

and ω induces a skew-symmetric bilinear form on T_Y. An n-dimensional submanifold Y is called *Lagrangian* (with respect to ω) if the restriction of ω to T_Y is identically zero. Since the restriction of ω to $T_X|_Y$ is nondegenerate, the normal bundle $N_{X/Y}$ and the tangent bundles T_Y of a Lagrangian submanifold are each other's dual. Since a global (holomorphic) 2-form on a rational variety is always zero, an n-dimensional rational submanifold of X is always Lagrangian. Let us consider the special case $Y \simeq \mathbf{P}^n$. We blow up X along Y. The inverse image $\tilde{Y}$ of Y is isomorphic to the projectivization of the normal bundle $N_{Y/X} \simeq T_Y{}^\vee$. Hence $\tilde{Y}$ is isomorphic to the (partial) flag variety

$$\{(p, H) \mid p \in Y \text{ and } H \text{ is a hyperplane passing through } p\} \subset Y \times Y^*$$

[38] If X has ordinary double points along a (smooth) subvariety of codimension 2, then the minimal resolution $\tilde{X}$ is a flat deformation of X.

of Y, where Y^* is the dual projective space of Y. $\tilde{Y}$ is a $\mathbf{P}^{n-1}$-bundle not only over Y but also over Y^*. The situation is symmetrical in Y and Y^*. $\tilde{Y}$ can be blown down in the direction $\tilde{Y} \to Y^*$ in X. We obtain a new complex manifold X^* which contains Y^* and such that $X^*\backslash Y^* \simeq X\backslash Y$. Moreover, X^* has a symplectic structure:



Theorem (4.4) ([62]). *Let X be a $2n$-dimensional symplectic manifold and Y its submanifold isomorphic to $\mathbf{P}^n$. Then there exist a symplectic manifold X^*, its submanifold Y^* canonically isomorphic to the dual projective space of Y, and a birational mapping $\varphi: X \cdots \to X^*$ that satisfy the following:*

(1) *φ (resp. φ^{-1}) is not defined on Y (resp. on Y^*) but an isomorphism outside it, and*
(2) *the indeterminacy of φ (resp. φ^{-1}) is resolved by the blowing up along Y (resp. Y^*).*

This theorem can be easily generalized to the case in which Y is a submanifold of codimension r and is a $\mathbf{P}^r$-bundle over a manifold. The mapping φ (resp. the symplectic manifold X^*) is called the *elementary transformation* (resp. *elementary transform*) of X along Y. The elementary transformation is an example of a birational mapping that is not an isomorphism but an isomorphism in codimension 1. This phenomenon does not occur for manifolds of dimension ≤ 2: Every birational mapping φ between surfaces X and Y is an isomorphism if both φ and φ^{-1} are defined in codimension one. Concerning the elementary transformation, the following problems are interesting.

Problem (4.5). Classify the birational mappings between symplectic manifolds, especially in the 4-dimensional case[39].

[39] The birational mappings between two 3-folds with trivial canonical bundles, more generally between two minimal models of 3-folds, are classified by Kawamata [104] and Kollár [106].

Problem (4.6).[40] Are the elementary transforms X^* of X always a deformation of X?

An irreducible symplectic manifold is a higher dimensional analogue of a K3 surface in many respects, e.g., period, polarization, deformation, etc. In the next section we shall discuss the period. Here we consider torus fibrations, which can be regarded as degenerate polarizations. A K3 surface often has an elliptic fibration $\pi: S \to \mathbf{P}^1$ over $\mathbf{P}^1$. This fibration is very useful for the study of S (see e.g. [114]). A torus fibration is a higher dimensional analogue of an elliptic fibration. A $2n$-dimensional irreducible symplectic manifold often has a morphism $f: X \to \mathbf{P}^n$ onto $\mathbf{P}^n$ whose generic fiber is an n-dimensional complex torus. The following are typical examples.

Example (4.7). The nth Hilbert product

$$f = \mathrm{Hilb}^n \pi : \mathrm{Hilb}^n S \to \mathrm{Hilb}^n \mathbf{P}^1 = \mathrm{Sym}^n \mathbf{P}^1 \simeq \mathbf{P}^n$$

of an elliptic fibration $\pi: S \to \mathbf{P}^1$ of a K3 surface S.

Example (4.8). The nth Kummer product

$$f = \mathrm{Kum}^n \tau : \mathrm{Kum}^n T \to \mathrm{Kum}^n E = (\text{fiber of } \mathrm{Sym}^{n+1} E \to E) \simeq \mathbf{P}^n$$

of an elliptic fibration $\tau: T \to E$ of a 2-dimensional complex torus T.

Example (4.9) ([62]). Let $\rho: S \to \mathbf{P}^2$ be a K3 surface in Example (1.6). For a *pair of points* $\{a, b\} \in \mathrm{Hilb}^2 S$ (see §2) of S, we denote the line joining $\rho(a)$ and $\rho(b)$ by $l_{a,b}$. The line $l_{a,b}$ is uniquely determined unless $\{a, b\} = \rho^{-1}(x)$ for a point $x \in \mathbf{P}^2$. So we obtain the rational mapping $f': \mathrm{Hilb}^2 S \cdots \to \mathbf{P}_2$, $\{a, b\} \mapsto l_{a,b}$ from $\mathrm{Hilb}^2 S$ onto the dual projective plane $\mathbf{P}_2$ of $\mathbf{P}^2$. This mapping f' is not defined on the subvariety $Y = \{\rho^{-1}(x) \mid x \in \mathbf{P}^2\}$. But the indeterminacy of f' is resolved by the elementary transformation φ along Y, that is, the composite $f = f' \circ \varphi^{-1}: (\mathrm{Hilb}^2 S)^* \to \mathbf{P}_2$ is a morphism. If l is a line of $\mathbf{P}^2$, then the fiber of f over $[l] \in \mathbf{P}_2$ is isomorphic to the Picard variety of the curve $\rho^{-1}(l)$ of genus 2. Hence f is a torus fibration.

The study of torus fibrations will be useful for the classification of symplectic manifolds.

Problem (4.10). When does the total space X of a torus fibration $X^{2n} \to \mathbf{P}^n$ over $\mathbf{P}^n$ have a symplectic structure?

5. Period of the moduli space

Both K3 surfaces and curves (= compact Riemann surfaces) have the property that their isomorphism classes are determined uniquely by their periods.

[40] For every 3-fold X, the (pure) Hodge structures $IH_i(X, \mathbf{Z})$ are independent of the choice of a minimal model X of the 3-fold ([106, Corollary 4.12]). *Problem.* Are the Hodge sturctures $H^i(X, \mathbf{Z})$ birational invariants of a symplectic manifold X?

We expect that higher dimensional symplectic manifolds have a similar property. In this section we formulate a Torelli type problem for them and study the periods of symplectic manifolds that are obtained as moduli spaces of vector bundles (Corollary (3.25)).

In a naive sense, a period is the integral of a (holomorphic) r-form on a topological r-cycle or a set of such integrals. Let C be a curve of genus g. We take a basis $\omega_1, \dots, \omega_g$ of the space $H^0(C, \Omega_C)$ of holomorphic 1-forms on C and a basis $\alpha_1, \dots, \alpha_{2g}$ of the first homology group $H_1(C, \mathbf{Z})$ of C. We obtain $2g^2$ integrals $\int_{\alpha_j} \omega_i$ $(1 \leq i \leq g,\ 1 \leq j \leq 2g)$. The $g \times 2g$ matrix $(\int_{\alpha_j} \omega_i)_{1\leq i\leq g,\ 1\leq j\leq 2g}$ is called the period matrix of C. The Torelli theorem asserts that the isomorphism class of C is uniquely determined by a certain equivalence class of its period matrix. To a 1-form ω we associate a homomorphism $f_\omega\colon H_1(C, \mathbf{Z}) \to \mathbf{C}$, $\alpha \mapsto \int_\alpha \omega$. We identify[41] $\omega \in H^0(C, \Omega)$ with $f_\omega \in \mathrm{Hom}(H_1(C, \mathbf{Z}), \mathbf{C})$ and $H^0(C, \Omega)$ with a subspace of $H^1(C, \mathbf{C})$. The natural orientation of C and the cup product induce the intersection pairing

$$(5.1) \qquad (\cdot)\colon H^1(C, \mathbf{Z}) \times H^1(C, \mathbf{Z}) \to \mathbf{Z}$$

on the first cohomology group $H^1(C, \mathbf{Z})$. The following form of the Torelli theorem seems to be most natural in the geometric point of view.

Torelli Theorem in Strong Form (5.2) (Matsusaka [58]). *Let C and C' be curves of the same genus and let $\varphi\colon H^1(C', \mathbf{Z}) \xrightarrow{\sim} H^1(C, \mathbf{Z})$ be an isomorphism between their first cohomology groups. Assume that φ is compatible with the intersection pairings* (5.1) *and that $\varphi \otimes \mathbf{C}$ maps $H^0(C', \Omega)$ onto $H^0(C, \Omega)$. Then their exists an isomorphism $f\colon C \xrightarrow{\sim} C'$ from C onto C' that induces φ or $-\varphi$ on the first cohomology group.*

Torelli Theorem (5.3) (Weil [94], Andreotti [3]).[42] *A curve C is isomorphic to C' if and only if there exists an isomorphism $\varphi\colon H^1(C', \mathbf{Z}) \xrightarrow{\sim} H^1(C, \mathbf{Z})$ that is compatible with the intersection pairings and such that $\varphi \otimes \mathbf{C}$ maps $H^0(C', \Omega)$ onto $H^0(C, \Omega)$.*

In the case of K3 surfaces, we use 2-forms and 2-cycles instead of 1-forms and 1-cycles. As in the case of curves, we identify the space $H^0(S, \Omega^2)$ of holomorphic 2-forms on S with a subspace of the second cohomology group $H^2(S, \mathbf{C})$. The natural orientation of S and the cup product induce the intersection pairing

$$(5.4) \qquad (\cdot)\colon H^2(S, \mathbf{Z}) \times H^2(S, \mathbf{Z}) \to \mathbf{Z}$$

on the second cohomology group $H^2(S, \mathbf{Z})$. Though the situation is different in that the intersection pairing (5.4) is symmetric while (5.1) is skew-symmetric, an analogue of (5.3) holds for K3 surfaces.

[41] The mapping $H^0(C, \Omega) \to H^1(C, \mathbf{C})$ is injective by the theorems of de Rham and Dolbeault (see e.g., [102]).

[42] The ideas of its several proofs are illustrated in [69, Lecture IV].

Torelli Type Theorem (5.5) (Pijateckiĭ-Šapiro and Šafarevič [81, 16, 52]). *A* K3 *surface* S *is isomorphic to* S' *if and only if there exists an isomorphism* $\varphi: H^2(S', \mathbf{Z}) \xrightarrow{\sim} H^2(S, \mathbf{Z})$ *that is compatible with the intersection pairing* (5.4) *and such that* $\varphi \otimes \mathbf{C}$ *maps* $H^0(S', \Omega^2)$ *onto* $H^0(S, \Omega^2)$.

A Torelli type theorem in strong form, which is an analogue of (5.2), is also proved for K3 surfaces (Looijenga and Peters [52], Burns and Rapoport [16], cf. [7]). We call the pair of the lattice $H^2(S, \mathbf{Z}) \simeq \mathbf{Z}^{\oplus 22}$ and a subspace $H^0(S, \Omega^2) \simeq \mathbf{C}$ of $H^2(S, \mathbf{Z}) \otimes \mathbf{C}$ the *period* of S. The Torelli type theorem claims that the isomorphism class of S is uniquely determined by its period.

Remark (5.6). The *period* of a 2-dimensional complex torus T is also defined in the same way by using 2-forms and 2-cycles. But (5.5) does not hold for complex tori. In fact, T and its dual torus $\hat{T}$ have the same period with respect to 2-forms. Shioda [85] has proved that every 2-dimensional complex torus with the same period as T is isomorphic to T or $\hat{T}$.

Let X be a $2n$-dimensional irreducible compact symplectic Kähler manifold. We identify the space $H^0(X, \Omega^2)$ of holomorphic 2-forms on X with a subspace of $H^2(X, \mathbf{C})$ as in the case of a K3 surface. For a cohomology class $\alpha \in H^2(X, \mathbf{Z})$, we denote by (α^n) its self-intersection number, that is, $\alpha^{2n} \in H^{4n}(X, \mathbf{Z})$ measured by the natural orientation. The self-intersection form $H^2(X, \mathbf{Z}) \to \mathbf{Z}$, $\alpha \mapsto (\alpha^{2n})$ is not quadratic if $n \geq 2$. But the second cohomology group $H^2(X, \mathbf{Z})$ still has a natural inner product $\langle \cdot \rangle$ (Beauville [10], Fujiki [27]). The following is also interesting from the view point of the topology of symplectic manifolds.

Theorem (5.7) ([27]). *Let* X *be as above. Then the self-intersection form on* $H^2(X, \mathbf{Z})$ *is an nth power of a quadratic form. To be precise, there exist an integral bilinear form*

$$\langle \cdot \rangle : H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \to \mathbf{Z}$$

and a rational number r *such that*

$$(\alpha^{2n}) = r\langle \alpha \cdot \alpha \rangle^n$$

holds for every $\alpha \in H^2(X, \mathbf{Z})$.

We normalize the inner product so that $\langle \omega \cdot \overline{\omega} \rangle$ is positive and the greatest common divisor of the $\langle \alpha \cdot \beta \rangle$'s is equal to 1, where α and β run over $H^2(X, \mathbf{Z})$ and ω is a symplectic structure of X. Then the rational number r is equal to $(2n)!/n!2^n$ (resp. $(2n)!(n+1)/n!2^n$) if X is the nth Hilbert (resp. Kummer) product of a K3 surface (resp. a 2-dimensional complex torus). By virtue of the inner product $\langle \cdot \rangle$, we can define periods of similar type as K3 surfaces for symplectic manifolds. It is important that the period thus defined be a birational invariant of X.

Proposition (5.8). *Let X and Y be irreducible compact symplectic Kähler manifolds. Assume that X and Y are birationally equivalent, that is, there exist a complex manifold Z and proper holomorphic mappings $f: Z \to X$ and $g: Z \to Y$ that are isomorphisms over nonempty Zariski open subsets. Then the homomorphism*

$$\Phi^* = g_* \circ f^* : H^2(X, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$$

induced by the rational mapping $\Phi = f \circ g^{-1} : Y \cdots \to X$ is an isomorphism. Moreover, Φ^* is compatible with the (normalized) inner products $\langle \cdot \rangle$ and $\Phi^* \otimes \mathbf{C}$ maps $H^0(X, \Omega^2)$ onto $H^0(Y, \Omega^2)$.

Since the canonical line bundles of X and Y are trivial, the exceptional divisors of f and g are the same. This is a key to the proof of the proposition.

We denote by $NS(X)$ the subgroup of $H^2(X, \mathbf{Z})$ consisting of the integral cohomology classes that are perpendicular to the symplectic structure $\omega \in H^0(X, \Omega^2)$ with respect to the inner product $\langle \cdot \rangle$. We call $NS(X)$ with the restriction of $\langle \cdot \rangle$ the *Néron-Severi lattice* of X. $NS(X)$ is identified with the set of Chern classes of all line bundles on X.

Corollary (5.9). *If two irreducible compact symplectic Kähler manifolds are birationally equivalent, then their Néron-Severi lattices $NS(X)$ and $NS(Y)$ are isomorphic to each other.*

Two symplectic manifolds with the same period are not necessarily isomorphic to each other [17]. By Proposition (5.8), we formulate the Torelli type problem for symplectic manifolds as follows.

Torelli Type Problem (5.10). *Let X and Y be compact irreducible symplectic Kähler manifolds of the same deformation type. Assume that there exists an isomorphism $\varphi: H^2(X, \mathbf{Z}) \xrightarrow{\sim} H^2(Y, \mathbf{Z})$ that is compatible with the (normalized) inner products $\langle \cdot \rangle$ and such that $\varphi \otimes \mathbf{C}$ maps $H^0(X, \Omega^2)$ onto $H^0(Y, \Omega^2)$. Is X birationally equivalent to Y?*

Now we calculate the period of the moduli space of stable sheaves on a K3 surface S. We fix a projective embedding $S \subset \mathbf{P}^N$ of S and a vector $v = (r, l, s)$ of the extended K3 lattice $\widetilde{H}(S, \mathbf{Z})$. Let $M_S(v)$ be the moduli space of stable sheaves E on S with $v(E) = v$ (§3). By Theorem (3.16), $M_S(v)$ has a symplectic structure. By Corollary (3.25), $M_S(v)$ is projective if $\mathrm{GCD}(r, (h \cdot l), s) = 1$. We first consider the case of $\dim M_S(v) = 2$. By Corollary (3.25), this happens if and only if v is *isotropic*, i.e., $(v^2) = 0$. In this case, a period is constructed from the extended K3 lattice (3.19) in the following way.

(1) Let $v^{\perp}$ be the orthogonal complement of v in $\widetilde{H}(X, \mathbf{Z})$. Since v is isotropic, the inner product on $\widetilde{H}(X, \mathbf{Z})$ induces an inner product on the quotient group $v^{\perp}/\mathbf{Z}v$.

(2) Since the Chern class of E is algebraic, $v = v(E)$ is perpendicular to $H^0(S, \Omega^2)$ in $\widetilde{H}(X, \mathbf{Z})$. Hence $H^0(S, \Omega^2)$ is contained in $v^\perp \otimes \mathbf{C}$ and determines a one-dimensional subspace of $(v^\perp/\mathbf{Z}v) \otimes \mathbf{C}$.

Theorem (5.11) ([63]). *Let S be an algebraic* K3 *surface and assume that $M_S(v)$ is compact and of dimension* 2. *Then $M_S(v)$ is a* K3 *surface and its period is equal to $v^\perp/\mathbf{Z}v$ constructed above. (In particular, $M_S(v)$ is connected.) To be precise, there exists an isomorphism $\varphi: v^\perp/\mathbf{Z}v \xrightarrow{\sim} H^2(M_S(v), \mathbf{Z})$ that is compatible with the inner products and such that $\varphi \otimes \mathbf{C}$ maps $H^0(S, \Omega^2)$ to $H^0(M_S(v), \Omega^2)$.*

Define the linear mapping $\psi: H^2(S, \mathbf{Q}) \to \widetilde{H}(S, \mathbf{Q})$ by $\alpha \mapsto (0, \alpha, (l \cdot \alpha)/r)$. Then ψ induces an isometry between $H^2(S, \mathbf{Q})$ and $(v^\perp/\mathbf{Z}v) \otimes \mathbf{Q}$. Since $\psi \otimes \mathbf{C}$ maps $H^0(S, \Omega^2)$ onto itself, we have the following.

Corollary (5.12). *Under the same conditions as in Theorem* (5.11), *the two* K3 *surfaces S and $M_S(v)$ have the same period over* $\mathbf{Q}$.

Remark (5.13). (1) Theorem (5.11) also holds for abelian surfaces T. This follows, e.g., from an explicit description of $M_T(v)$. If $M_T(v)$ is of dimension 2, then it is an abelian surface isogeneous to T. Every member of $M_T(v)$ is a vector bundle and is said to be *semihomogeneous* or projectively flat, in the case $r > 0$. A detailed analysis of semihomogeneous vector bundles shows that $M_T(v)$ is isomorphic to $\widehat{T}/\varphi_l(T_r)$ [59], where $\widehat{T}$ is the dual abelian surface of T, T_r is the group of r-torsion points of T and $\varphi_l: T \to \widehat{T}$ is a homomorphism associated to a line bundle L with $c_1(L) = l$ (cf. [68]).

(2) Applying Theorem (5.11) to Example (2.2), we obtain a relation between periods of two types of K3 surfaces (1.5) and (1.6). A generalization of this relation is studied by O'Grady [79].

A sheaf $\mathcal{E}$ on the product $S \times M_S(v)$ is called a *universal sheaf* if it satisfies the following three conditions:

(a) $\mathcal{E}$ is flat over $M_S(v)$ (this is automatically satisfied if $\mathcal{E}$ is locally free),
(b) the restriction of $\mathcal{E}$ to $S \times [E]$ is isomorphic to E for every member E of $M_S(v)$, and
(c) the restriction of $\mathcal{E}$ to $S \times U_{[E]}$ is isomorphic to the Kuranishi family of E (cf. (3.1) and [87]) for every member E of $M_S(v)$ and for a sufficiently small neighborhood $U_{[E]}$ of $[E]$.

The homomorphism φ in Theorem (5.11) can be constructed from the Chern class of a universal sheaf. But a universal sheaf does not always exist. Hence we need its substitute. A sheaf $\mathcal{E}$ on $S \times M_S(v)$ is called a *quasiuniversal sheaf of similitude* s if it satisfies (a), (b$'$) the restriction of $\mathcal{E}$ to $S \times [E]$ is isomorphic to $E^{\oplus s}$, and (c$'$) the restriction of $\mathcal{E}$ to $S \times U_{[E]}$ is isomorphic to the direct sum of s copies of the Kuranishi family of E, for every member E of $M_S(v)$.

A quasiuniversal sheaf of some similitude always exists. If both $\mathscr{E}$ and $\mathscr{E}'$ are quasiuniversal sheaves on $S \times M_S(v)$, then there exist vector bundles V and V' on $M_S(v)$ such that $\mathscr{E} \otimes \pi_M^* V \simeq \mathscr{E}' \otimes \pi_M^* V'$, where $\pi_M \colon S \times M_S(v) \to M_S(v)$ is the projection to the second factor. We construct an algebraic cycle[43] Z_v on $S \times M_s(v)$ from the Chern class of a quasiuniversal sheaf and define the homomorphism

$$\widetilde{H}(S, \mathbf{Q}) \to H^*(M_S(v), \mathbf{Q})$$

by $\alpha \mapsto \pi_{M,*}(Z_v \cdot \pi_S^* \alpha)$. We consider the restriction of its H^2-part to $v^\perp \otimes \mathbf{Q}$:

$$\theta_{v,\mathbf{Q}} \colon v^\perp \otimes \mathbf{Q} \to H^2(M_S(v), \mathbf{Q}). \tag{5.14}$$

By the above uniqueness property, $\theta_{v,\mathbf{Q}}$ is independent of the choice of a quasi-universal sheaf. Since Z_v is algebraic, $\theta_{v,\mathbf{Q}}$ maps $H^0(S, \Omega^2)$ to $H^0(M_S(v), \Omega^2)$. In the case $\dim M_S(v) = 2$, $\theta_{v,\mathbf{Q}}$ is surjective and its kernel is $\mathbf{Q}v$, which is the essential part of Theorem (5.11). In the case $\dim M_S(v) > 2$, we have the following:

Theorem (5.15). *Let S be an algebraic* K3 *surface and fix a projective embedding $S \subset \mathbf{P}^N$. Let v be a vector of rank* 1 *or* 2 *in $\widetilde{H}(X, \mathbf{Z})$ and $M_S(v)$ the moduli space of stable sheaves E on S with $v(E) = v$. Assume that $M_S(v)$ is compact and of dimension ≥ 4. Then the homomorphism* (5.14) *induces an isomorphism*

$$\theta_v \colon v^\perp \xrightarrow{\sim} H^2(M_S(v), \mathbf{Z})$$

between the orthogonal complement of v in $\widetilde{H}(S, \mathbf{Z})$ and the second cohomology group $H^2(M_S(v), \mathbf{Z})$ of $M_S(v)$. Moreover, θ_v is compatible with the inner products $(\cdot)$ and $\langle \cdot \rangle$[44]*, and $\theta_v \otimes \mathbf{C}$ maps $H^0(S, \Omega^2)$ onto $H^0(M_S(v), \Omega^2)$.*

Corollary (5.16). *The Néron-Severi lattice $NS(M_S(v))$ of the moduli space $M_S(v)$ is isomorphic to the intersection of $v^\perp$ and $\mathbf{Z} \oplus NS(S) \oplus \mathbf{Z} \subset \widetilde{H}(S, Z)$.*

In the rank one case, say $v = (1, 0, 1-n)$, every stable sheaf with $v(E) = v$ is an ideal (sheaf) defining a 0-dimensional subscheme of length n. Hence $M_S(v)$ is isomorphic to $\mathrm{Hilb}^n S$. (This does not depend on the choice of the embedding $S \subset \mathbf{P}^N$.) The orthogonal complement $v^\perp$ is generated by $\delta = (1, 0, n-1)$ and $H^2(S, \mathbf{Z})$. Hence the period of $\mathrm{Hilb}^n S$ is isomorphic to the pair of the lattice $\mathbf{Z}\delta \perp H^2(S, \mathbf{Z})$ and $H^0(S, \Omega^2)$. In particular, the Néron-Severi lattice of $\mathrm{Hilb}^n S$ is isomorphic to $\mathbf{Z}\delta \perp NS(S)$ with $\langle \delta^2 \rangle = -2(n-1)$. This can be proved also by direct computation (see Beauville [9]).

Example (5.17). *Let* $\mathrm{Grass}(\mathbf{P}^1 \subset \mathbf{P}^4)$ *be the Grassmann variety of lines in the projective* 4*-space* $\mathbf{P}^4$. *Let*

$$\mathrm{Grass}(\mathbf{P}^1 \subset \mathbf{P}^4) \subset \mathbf{P}^9$$

[43] See [63] for the explicit construction.

[44] Since $M_S(v)$ is connected, $H^2(M_S(v), \mathbf{Z})$ has the inner product $\langle \cdot \rangle$ by (5.7).

be the Plücker embedding and cut it three times by a general hyperplane and once by a general hyperquadric. Then we obtain a (polarized) K3 *surface*[45] , $S \subset \mathbf{P}^6$. *We denote by* h *the cohomology class of hyperplane sections of* $S \subset \mathbf{P}^6$. *We assume that the three hyperplanes and the hyperquadric are sufficiently general*[46] *so that* $NS(S)$ *is generated by* h. *Then we can describe all members of* $M_S(v)$, $v = (2, h, 2)$.

Let Q *be a singular hyperquadric of* $\mathbf{P}^6$ *that contains* S. *The projection from a singular point of* Q *induces a morphism from* S *to* $\mathbf{P}^5$. *The image of* S *is contained in the image* $\overline{Q}$ *of* Q. $\overline{Q}$ *is a hyperquadric in* $\mathbf{P}^5$. *Hence, as in Example* (2.2), *we obtain a rank* 2 *vector bundle from a family* Δ *of planes in* $\overline{Q}$. *Moreover, we can show that all locally free members of* $M_S(2, h, 2)$ *are obtained in this way.*

Since Grass($\mathbf{P}^1 \subset \mathbf{P}^4$) *is a projective variety of degree* 5 *in* $\mathbf{P}^9$, *we have* $(h^2) = 10$. *Hence the moduli space* $M_S(2, h, 2)$ *is a 4-fold by Corollary* (3.25). *This 4-fold is explicitly described as follows: Let* N *be the set of all hyperquadrics of* $\mathbf{P}^6$ *that contain* S *and* N_0 *the subset consisting of singular ones as in Example* (2.2). *Then* N *is isomorphic to* $\mathbf{P}^5$ *and* N_0 *is a hypersurface of degree* 7 *in* N. *But* N_0 *is reducible: Let* N_0' *be the set of restrictions of all hyperquadrics of* $\mathbf{P}^9$ *that contain* Grass($\mathbf{P}^1 \subset \mathbf{P}^4$). N_0' *is a hyperplane*[47] *of* N *contained in* N_0. *Hence* N_0 *is the union of* N_0' *and a sextic hypersurface* N_0'' *of* N. *The moduli space* $M(2, h, 2)$ *is a double cover of the* (*singular*) *sextic hypersurface* N_0'' *in* $N \simeq \mathbf{P}^5$.

In this example, the intersection of the orthogonal complement $v^\perp$ and $\mathbf{Z} \oplus NS(S) \oplus \mathbf{Z}$ is generated by $u = (1, 0, -1)$ and $w = (2, h, 3)$ since $NS(S) = \mathbf{Z}h$. Hence, by Corollary (5.16), the Néron-Severi lattice NS of $M_S(v)$ is isomorphic to the lattice

$$\mathbf{Z}u \oplus \mathbf{Z}w, \quad \text{with } \langle u^2 \rangle = 2, \ \langle u \cdot w \rangle = -1 \text{ and } \langle w^2 \rangle = -2.$$

This is an even integral quadratic form in two variables of discriminant 5. In particular, NS is indecomposable. Hence $M_S(v)$ is not birationally equivalent to $\mathrm{Hilb}^2 S$ for any K3 surface S by Corollary (5.9).[48] Moreover, $M_S(v)$ is not birationally equivalent to any symplectic 4-fold obtained as in Example (4.3),

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[45] This kind of construction of (polarized) K3 surfaces from Grassmann varieties is extensively generalized in [110] and [111].

[46] Moishezon [109] proved the following generalization of Noether's theorem: Let $X \subset \mathbf{P}^N$ be a projective 3-fold. If $\dim H^0(Y_0, \Omega^2) > \dim H^0(X, \Omega^2)$ for a smooth hyperplane section Y_0 of $X \subset \mathbf{P}^N$, then there exists a smooth hyperplane section Y such that the restriction mapping $\mathrm{Pic}\, X \to \mathrm{Pic}\, Y$ is an isomorphism.

[47] Let $\{v_1, \dots, v_N\}$ be a basis of $V^\vee$ and put $p_{ij} = v_i \wedge v_j$. The Grassmann variety $G(2, V) \subset \mathbf{P}_*(\bigwedge^2 V)$ is defined by $\binom{N}{4}$ quadratic forms $q_I \equiv p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} = 0$, where I runs over all 4-element subsets $\{i < j < k < l\}$ of $\{1, \dots, N\}$. In particular, the set of quadratic forms identically zero on Grass($\mathbf{P}^1 \subset \mathbf{P}^4$) $\subset \mathbf{P}^9$ is a 5-dimensional vector space spanned by these Plücker relations.

[48] This will dispell the doubt expressed in [9, p. 781].

since NS has no length 6 vectors.[49] Thus we have obtained a new irreducible symplectic manifold as a moduli space of stable sheaves on S.

Though we can construct many new symplectic manifolds in a similar way, every component of the moduli space of rank 2 stable sheaves on a K3 surface becomes birationally equivalent to a Hilbert product of a K3 surface under a suitable deformation of complex structure.[50] (It is expected that this will also hold in the higher rank case.) Hence the following is still open.

Problem. Is there any irreducible symplectic manifold that is not equivalent to $\mathrm{Hilb}^n S$ or $\mathrm{Kum}^n T$ modulo deformation of complex structures and modulo birational modification?

6. Notes on references

The following is a short guide to references related to the topics on vector bundles, discussed in this article.

The projective line $\mathbf{P}^1$ has the unusual property[51] that every vector bundle over it decomposes into a direct sum of line bundles [30]. Atiyah [4] classified the vector bundles on an elliptic curve on $\mathbf{C}$. Later Oda [78] classified them over an arbitrary algebraically closed field. By their classification, the moduli space of indecomposable vector bundles with fixed rank and degree is always isomorphic to the base elliptic curve. This is also an unusual property. The moduli space of vector bundles on a smooth projective variety X contains arbitrarily higher dimensional subvarieties if X is not a projective line or an elliptic curve. The rank 2 vector bundles on a curve of genus 2 and their moduli space are explicitly described by [71, 74], etc. Desale and Ramanan [18] generalize this result to rank 2 vector bundles over a hyperelliptic curve of an arbitrary genus. For the moduli space of rank 2 vector bundles over an arbitrary curve, its topology and rationality are studied by Newstead [73, 75, 76], Harder [31], Harder and Narasimhan [32]. Mumford and Newstead [70] study the relationship between the periods of a curve and the moduli space of vector bundles on it.[52]

The global property of the moduli space of vector bundles over an abelian surface was first studied by Umemura [91, 92]. The author [61] has found a Fourier transformation for vector bundles on an abelian variety. The Fourier transformation is very useful for the study of vector bundles on abelian varieties

[49] For every cubic 4-fold $V \subset \mathbf{P}^5$, the Néron-Severi lattice of $F(V)$ contains an (integral) vector of length 6. In fact, the cohomology class $h \in H^2(F(V), \mathbf{Z})$ of hyperplane sections of $F(V) \subset \mathrm{Grass}(\mathbf{P}^1 \subset \mathbf{P}^5) \subset \mathbf{P}^{14}$ satisfies $(h^4) = 108$ [1] and $\langle h^2 \rangle = \sqrt{(h^4)/3} = 6$. Let $h^\perp$ be the orthogonal complement of h in $H^2(F(V), \mathbf{Z})$ with respect to $\langle \cdot \rangle$. Then the Hodge structure $h^\perp$ is isomorphic to the orthogonal complement of $c_2(V)$ in $H^4(V, \mathbf{Z})$ with respect to the intersection pairing.

[50] This is also true for the symplectic manifolds in (4.3) as we saw in §4.

[51] Cf. [53, Corollary 3.4.1].

[52] As a corollary to its main theorem, it is proved there that the isomorphism class of a curve C is uniquely determined by that of the moduli space of rank 2 stable vector bundles with a fixed determinant line bundle of an odd degree on C. This Torelli type theorem, together with its generalization to higher rank vector bundles, is also proved in [89] and [90].

([60, 64, 65]).[53] The vector bundles on the projective plane are studied by many authors. In the rank 2 case, the moduli space is explicitly described by Barth [6] (cf. [57, Part II]), Ellingsrud and Strømme [24], and Hulek [35]. The moduli of vector bundles of an arbitrary rank on $\mathbf{P}^2$ is studied by Drezet-Le Potier [23]. Le Potier [50, 51] studies the Picard group of the moduli space. Among varieties of higher dimension, the vector bundles over $\mathbf{P}^n$ are well studied. Consult [33, 80] and their references on this subject.

The moduli space of (μ-stable) vector bundles can be regarded as the moduli space of anti-self-dual Yang-Mills connections or as the moduli space of Einstein-Hermitian metrics. Many problems on the moduli space of vector bundles, e.g., topology, complex structure, Kähler metric, compactification, symplectic structure, etc., can be approached by the differential geometric technique. (See Atiyah and Bott [5], Kobayashi [45], Itoh [36], Donaldson [19], Kirwan [40].) Recently Donaldson [21, 99] constructed an example of two compact complex surfaces that are homeomorphic but not diffeomorphic to each other. He defined new invariants for differentiable manifolds by using the moduli of Yang-Mills connections. The moduli space of stable vector bundles plays an important role in the calculation of his invariants (cf. [28, 93]).

In this article, we have restricted ourselves to the moduli space. Last we note that the geometry of vector bundles themselves on a K3 surface is also interesting. It has applications to special divisors on a curve [49] and to the classification of Fano 3-folds [110, 111].

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