Birational Geometry of Algebraic Varieties

Open Problems

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Problems on characterization of the complex projective space

by

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A compact complex manifold X is a Fano manifold if its 1st Chern class $c_1(X) \in H^1(X,\mathbb{Z})$ is positive, or equivalently, the anticanonical class $-K_X$ is ample. The projective space $\mathbb{P}^n$ is the most typical example. In this note, I pose some problems on characterization of $\mathbb{P}^n$ which was conceived during my study on Fano manifolds of coindex 3 [Mu].

1. Characterization by index

For a Fano manifold X, the largest integer r which divides $c_1(X)$ in $H^2(X,\mathbb{Z})$ is called the *index* of X. The index of $\mathbb{P}^n$ is equal to $n+1$.

Theorem 1. ([K-O]). *Let* X *be a Fano manifold. Then* index $X \le \dim X + 1$. *Moreover, the equality holds if and only if* $X \simeq \mathbb{P}^n$.

If X is a Fano manifold of index r, then the vector bundle $\mathcal{O}_X(-K_X/r)^{\oplus r}$ is ample and its first Chern class is equal to $c_1(X)$. So we consider ample vector bundles E on X with $c_1(E) = c_1(X)$. How big can the rank $r(E)$ of E be? By [Mo], there exists a rational curve C on X with $(C \cdot c_1(X)) \le \dim X + 1$. Since every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles, we have $r(E) = r(E|_C) \le \dim X + 1$.

Conjecture 1. Let X be a compact complex manifold and E an ample vector bundle on it with $c_1(E) = c_1(X)$. If $r(E) = \dim X + 1$, then $(X,E) \simeq (\mathbb{P}^n, \mathcal{O}(1)^{\oplus(n+1)})$.

2. Characterization by the tangent bundle

The following was conjectured by [Ha].

Theorem 2. ([Mo]). *A compact complex manifold* X *with ample tangent bundle* T_X *is isomorphic to* $\mathbb{P}^n$.

The tangent bundle T_X is a vector bundle on X with $r(T_X) = \dim X$ and $c_1(T_X) = c_1(X)$. The vector bundles $\mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2)$ over $\mathbb{P}^n$ and $\mathcal{O}(1)^{\oplus n}$ over a hyperquadric $Q^n \subset \mathbb{P}^{n+1}$ also satisfy these conditions.

Conjecture 2. Let E be an ample vector bundle on X with $\mathrm{rk}\, E = \dim X$ and $c_1(E) = c_1(X)$. Then the pair (X, E) is isomorphic to $(\mathbb{P}^n, T_{\mathbb{P}})$, $(\mathbb{P}^n, \mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2))$ or $(Q^n, \mathcal{O}(1)^{\oplus n})$.

3. The logarithmic version of Hartshorne conjecture

The "log analogue" of the tangent bundle T_X is the sheaf of vector fields with logarithmic zeroes along D, which is denoted by $T_X(-\log D)$. $T_X(-\log D)$ is characterized by the natural exact sequence

$$0 \to T_X(-\log D) \to T_X \to N_{D/X} \to 0,$$

where $N_{D/X}$ is the normal bundle $\mathcal{O}_D(D)$ of D and we regard it as a sheaf on X with support on D. If $X = \mathbb{P}^n$ and D is a hyperplane, then $T_X(-\log D)$ is isomorphic to $\mathcal{O}_{\mathbb{P}}(1)^{\oplus n}$.

Conjecture 3.(*) Let X be a compact complex manifold and D a nonzero reduced effective divisor on it. If the logarithmic tangent bundle $T_X(-\log D)$ is ample, then $(X,D) \simeq (\mathbb{P}^n, \text{hyperplane})$.

(*) In the problem session, Mori said that this would be proved by essentially the same argument as in [Mo].

The tangent bundle T_X is ample if the bisectional curvature is positive.

Problem. Find a sufficient condition on the curvature for $T_X(-\log D)$ to be ample, that is, formulate a logarithmic version of the Frankel conjecture which characterizes $\mathbb{C}^n$.

4. Relation with the classification of Fano manifolds

Let E be a rank r vector bundle on X with $c_1(E) = c_1(X)$ and put $Y = \mathbb{P}(E)$. Then $c_1(Y)$ is r times the tautological line bundle $\mathcal{O}_Y(1)$. Hence if E is ample then Y is a Fano manifold of index r. If $r = n+1$, $n = \dim X$, then Y is a Fano $2n$-fold of index $n+1$. We note $\rho(Y) = \rho(X)+1 \geq 2$, where ρ denotes the Picard number. The following is a refinement of Theorem 1.

Conjecture 4. If Y is a Fano manifold with Picard number ρ, then $\text{index } Y \leq \dim Y/\rho + 1$. Moreover, the equality holds iff $Y \simeq (\mathbb{P}^{\text{index } Y-1})^{\rho}$.

For a Fano manifold Y, we define the coindex by $\dim Y - \text{index } Y + 1$, which is nonnegative by Theorem 1. Conjecture 4 implies

Conjecture 4'. If Y is a Fano manifold with Picard number ≥ 2, then $\dim Y \leq 2\cdot\text{coindex } Y$. Moreover, the equality holds iff $Y \simeq \mathbb{P}^{\text{coindex } Y} \times \mathbb{P}^{\text{coindex } Y}$.

This conjecture implies Conjecture 1. In the case $\text{coindex } Y \leq 3$, Conjecture 4' is easily obtained from the following;

Proposition. *Let* Y *be a Fano manifold of coindex* $c \leq 3$ *and* R *an extremal ray of* Y. *Let* $f : Y \to Z$ *be the contraction morphism of* R. *Then we have either* $\dim Z = \dim Y$ *or* $\dim Z \leq c$.

In the former case, f is birational and contracts a divisor to a point or to a curve.

(This proposition is also observed in [Fuj].)

Proof of Conjecture 4' in the case coindex 3:

In the case $\dim Y \geq 4$, Y has a nef extremal ray R_1. Since $\rho(Y) \geq 2$, Y has another extremal ray R_2. Let F_2 be a fiber of maximal dimension of cont_{R_2}. By the proposition, $\dim F_2 \geq \dim Y - 3$. Since the restriction of cont_{R_1} to F_1 is finite, we have $\dim Y - 3 \leq 3$. Moreover, if the equality holds, then both cont_{R_1} and cont_{R_2} are $\mathbb{P}^3$-bundles over 3-folds. Hence we have $Y \simeq \mathbb{P}^3 \times \mathbb{P}^3$.

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