Fano 3-folds

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Abstract: In the beginning of this century, G. Fano initiated the study of 3-dimensional projective varieties $X_{2g-2} \subset \mathbf{P}^{g+1}$ with canonical curve sections in connection with the Lüroth problem.[1] After a quick review of a modern treatment of Fano's approach (§1), we discuss a new approach to Fano 3-folds via vector bundles, which has revealed their relation to certain homogeneous spaces (§§2 and 3) and varieties of sums of powers (§§5 and 6). We also give a new proof of the gunus bound of prime Fano 3-folds (§4). In the maximum genus ($g = 12$) case, Fano 3-folds $X_{22} \subset \mathbf{P}^{13}$ yield a 4-dimensional family of compactifications of $\mathbf{C}^3$ (§8).

A compact complex manifold X is *Fano* if its first Chern class $c_1(X)$ is positive, or equivalently, its anticanonical line bundle $\mathcal{O}_X(-K_X)$ is ample. If $\mathcal{O}_X(-K_X)$ is generated by global sections and $\Phi_{|-K_X|}$ is birational, then its image is called the anticanonical model of X. In the case $\dim X = 3$, every smooth curve section $C = X \cap H_1 \cap H_2 \subset \mathbf{P}^{g-1}$ of the anticanonical model $X \subset \mathbf{P}^{g+1}$ is canonical, that is, embedded by the canonical linear system $|K_C|$. Conversely, every projective 3-fold $X_{2g-2} \subset \mathbf{P}^{g+1}$ with a canonical curve section is obtained in this way. The integer $\frac{1}{2}(-K_X)^3 + 1$ is called the *genus* of a Fano 3-fold X since it is equal to the genus g of a curve section of the anticanonical model.

A projective 3-fold $X_{2g-2} \subset \mathbf{P}^{g+1}$ with a canonical curve section is a complete intersection of hypersurfaces if $g \leq 5$. In particular, the Picard group of X is generated by $\mathcal{O}_X(-K_X)$. We call such a Fano 3-fold *prime*. If a Fano 3-fold X is not prime, then either $-K_X$ is divisible by an integer ≥ 2 or the Picard number ρ of X is greater than one. See [15], [7] and [9] for the classification in the former case and [24] and [25] in the latter case.

§1 Double projection The anticanonical line bundle $\mathcal{O}_X(-K_X)$ is very ample if X is a prime Fano 3-fold of genus ≥ 5 (*cf.* [15] and [41]). To classify prime Fano 3-folds $X_{2g-2} \subset \mathbf{P}^{g+1}$ of genus $g \geq 6$, Fano investigated the *double projection* from a line[2] ℓ on X_{2g-2}, that is, the rational map associated to the linear system $|H - 2\ell|$ of hyperplane sections singular along ℓ.

Example 1 *Let $X_{16} \subset \mathbf{P}^{10}$ be a prime Fano 3-fold of genus 9. Then the double projection $\pi_{2\ell}$ from a line $\ell \subset X_{16}$ is a birational map onto $\mathbf{P}^3$. The union D of conics which intersects ℓ is a divisor of X and contracted to a space curve $C \subset \mathbf{P}^3$ of genus 3 and degree 7. The inverse rational map $\mathbf{P}^3 - \to X_{16} \subset \mathbf{P}^{10}$ is given by the linear system $|7H - 2C|$ of surfaces of degree 7 which are singular along C.*

The key for the analysis of $\pi_{2\ell}$ is the notion of flop. Let X^- be the blow-up of X along ℓ. Since other lines intersect ℓ, X^- is not Fano. But X^- is almost Fano in the sense that $|-K_{X^-}|$ is free and gives a birational morphism contracting no divisors. The anticanonical model $\bar{X}$ of X^- is the image of the projection $X - \to \mathbf{P}^8$ from ℓ. The strict transform $D^- \subset X^-$ of D is relatively negative over $\bar{X}$. By the theory of flops ([33], [19]), there exists another almost Fano 3-fold X^+ which has the same anticanonical model as X^- and such that the strict transform $D^+ \subset X^+$ of D^- is relatively ample over $\bar{X}$. X^+ is called the *D^--flop*[3] of X^-.

[1] A surface dominated by a rational variety is rational by Castelnuovo's criterion. But this does not hold any more for 3-folds. See [5], [44] and [18].

[2] The existence of a line is proved by Shokurov [42].

[3] The smoothness of X^+ follows from [19, 2.4] or from the classification [6, Theorem 15] of the singularity of $\bar{X}$.

Theorem([23], [17]) *Let X, ℓ and D be as in Example 1. Then the D^--flop X^+ of the blow-up X^- of X along ℓ is isomorphic to the blow-up of $\mathbf{P}^3$ along a space curve of genus 3 and degree 7.*

For the proof, the theory of extremal rays ([22]) is applied to the almost Fano 3-fold X^+. If X is a prime Fano 3-fold of genus 10, then X^+ is isomorphic to the blow-up of a smooth 3-dimensional hyperquadric $Q^3 \subset \mathbf{P}^4$ along a curve of genus 2 and degree 7. In the case genus 12, X^+ is the blow-up of a quintic del Pezzo 3-fold[4] $V_5 \subset \mathbf{P}^6$ along a quintic normal rational curve.

§2 Bundle method A line on $X_{2g-2} \subset \mathbf{P}^{g+1}$ can move in a 1-dimensional family. Hence the double projection method does not give a canonical biregular description of $X_{2g-2} \subset \mathbf{P}^{g+1}$. In the case $g = 9$, e.g., there are infinitely many different space curves[5] $C \subset \mathbf{P}^3$ which give the same Fano 3-fold $X_{16} \subset \mathbf{P}^{10}$. By the same reason, the double projection method does not classify $X_{2g-2} \subset \mathbf{P}^{g+1}$ over fields which are not algebraically closed. Even when a Fano 3-fold X is defined over $k \subset \mathbf{C}$, it may not have a line defined over k. Our new classification makes up these defects. It is originated to solve the following:

Problem[6] **:** Classify all projective varieties $X^n_{2g-2} \subset \mathbf{P}^{g+n-2}$ of dimension $n \geq 3$ with a canonical curve section[7] .

We restrict ourselves to the case that every divisor on X is cut out by a hypersurface. In contrast with the case $g \leq 5$, the dimension n cannot be arbitrarily large in the case $g \geq 6$. In each case $7 \leq g \leq 10$, the maximum dimension $n(g)$ is attained by a homogeneous space Σ_{2g-2}.

Table

g	$n(g)$	$\Sigma_{2g-2} \subset \mathbf{P}^{g+n(g)-2}$	$r(E)$	$\chi(E)$	$c_1(E)c_2(E)$
6	6	Hyperquadric section of the cone of the Grassmann variety[8] $G(2,5) \subset \mathbf{P}^9$	2	5	4
7	10	10-dimensional spinor variety $SO(10,\mathbf{C})/P \subset \mathbf{P}^{15}$	5	10	48
8	8	Grassmann variety $G(2,6) \subset \mathbf{P}^{14}$	2	6	5
9	6	$Sp(6,\mathbf{C})/P \subset \mathbf{P}^{13}$	3	6	8
10	5	$G_2/P \subset \mathbf{P}^{13}$	5	7	12
12	3	$G(V,3,N) \subset \mathbf{P}^{13}$ (See Theorem 3.)	3	7	10

We claim that every variety $X \subset \mathbf{P}$ with canonical curve section of genus $g \geq 6$ is a linear section of the above $\Sigma_{2g-2} \subset \mathbf{P}^{g+n(g)-2}$. Since each Σ_{2g-2} has a natural morphism to a Grassmann variety, vector bundles play a crucial role in our classification. Instead of a line, we show the existence of a good vector bundle E on X. Instead of the double projection, we embed X into a Grassmann variety by the linear system $|E|$ and describe its image. The vector bundle is first constructed over a general (K3) surface section S of X and then extended to X applying a Lefschetz type theorem (*cf.* [8]).[9] The numerical invariants of E

[4] A smooth projective variety $V_d \subset \mathbf{P}^{d+n-2}$ with a normal elliptic curve section is called *del Pezzo*. The anticanonical class $-K_V$ is llinearly equivalent to $(n-1)$ times hyperplane section. All quintic del Pezzo 3-folds are isomorphic to each other (see [15] and [9]).

[5] The isomorphism classes of curves C are uniquely determined by the Torelli theorem since the intermediate Jacobian variety of X is isomorphic to the Jacobian variety of C.

[6] Roth [36] [37] studied this problem by generalizing the double projection method.

[7] The anticanonical class of X^n_{2g-2} is $(n-2)$-times hyperplane section. In the case $n = 2$, X^2_{2g-2} is a (polarized) K3 surface. The integer g is called the genus of X.

[8] $G(s,n)$ denotes the Grassmann variety of s-dimensional subspaces of a fixed n-dimensional vector space.

[9] By our assumption on X and [21], there exists a surface section with Picard number one. Hence every member of $|\mathcal{O}_S(-K_X)|$ is irreducible. We use this property to analyze $\Phi_{|E|}$.

are as in the above table.[10] All higher cohomology groups of E vanish and E is generated by its global sections. The morphism[11] $\Phi_{|E|} : X \longrightarrow G(H^0(E), r(E))$ is an embedding if $g \geq 7$. The first Chern class $c_1(E)$ is equal to $2c_1(X)$ if $g = 7$ and equal to $c_1(X)$ otherwise. E is characterized by the following two properties:

1) $r(E)$, $c_1(E)$ and $c_2(E)$ are as above, and
2) the restriction[12] of E to a general surface section is stable.

In the case $g = 9$, $|E|$ embeds X into the 9-dimensional Grassmann variety $G(V, 3)$, where $V = H^0(X, E)$. Consider the natural map

$$\lambda_2 : \bigwedge^2 H^0(X, E) \longrightarrow H^0(X, \bigwedge^2 E).$$

The kernel is generated by a nondegenerate bivector σ on V. Hence the image of X is contained in the zero locus $G(V, 3, \sigma)$ of the global section of $\bigwedge^2 \mathcal{E}$ corresponding to σ, where $\mathcal{E}$ is the universal quotient bundle on $G(V, 3)$. $G(V, 3, \sigma)$ is a 6-dimensional homogeneous space of $Sp(V, \sigma)$ and a projective variety $\Sigma_{16} \subset \mathbf{P}^{13}$ with a canonical curve section of genus 9. In the case $\dim X = 3$, we have

Theorem 2 *A prime Fano 3-fold $X_{16} \subset \mathbf{P}^{10}$ of genus 9 is isomophic to the intersection of Σ_{16} and a linear subspace $\mathbf{P}^{10}$ in $\mathbf{P}^{13}$.*

By the above characterization, E is defined over $k \subset \mathbf{C}$ if X is so. Hence the theorem holds true for every Fano 3-fold $X_{16} \subset \mathbf{P}^{10}_k$ over $k \subset \mathbf{C}$ such that $X \otimes \mathbf{C}$ is prime.

The results are similar for $g = 7, 8$ and 10. In the case $g = 7$ and 10, the natural mappings $\sigma_2 : S^2 H^0(X, E) \longrightarrow H^0(X, S^2 E)$ and $\lambda_4 : \bigwedge^4 H^0(X, E) \longrightarrow H^0(X, \bigwedge^4 E)$ are considered instead of λ_2. In the case $g = 6$, X is a double cover of a linear section of $G(2, 5) \subset \mathbf{P}^9$ if the linear subspace P passes through the vertex of the Grassmann cone. Otherwise, X is isomorphic to the complete intersection of a 6-dimensional hyperquadric $Q \subset P$ and $G(2, 5) \subset \mathbf{P}^9$.

§3 Fano 3-fold of genus 12 A prime Fano 3-fold[13] X of genus 12 cannot be an ample divisor of a 4-fold. But the vector bundle E gives a canonical description of X in the 12-dimensional Grassmann variety $G(V, 3), V = H^0(X, E)$. Consider the natural map $\lambda_2 : \bigwedge^2 H^0(X, E) \longrightarrow H^0(X, \bigwedge^2 E)$ as in the case $g = 9$. Its kernel N is of dimension 3. Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be a basis of N.

Theorem 3 *A prime Fano 3-fold $X_{22} \subset \mathbf{P}^{12}$ of genus 12 is isomorphic to the common zero locus $G(V, 3, N)$ of the three global sections of $\bigwedge^2 \mathcal{E}$ corresponding to σ_1, σ_2 and σ_3, where $\mathcal{E}$ is the universal quotient bundle on $G(V, 3)$.*

The third Chern number $\deg c_3(E)$ is equal to 2. Hence every general global section of E vanishes at two points. Conversely, since V is of dimension 7, there exists a nonzero global section $s_{x,y}$ vanishing at x and y for every pair of distinct points x and y. If x and y are general, then $s_{x,y}$ is unique up to constant multiplications. The correspondence $(x, y) \mapsto [s_{x,y}]$ gives the birational mappings $\Pi : S^2 X - \to \mathbf{P}_*(V) \simeq \mathbf{P}^6$ and $\Pi_x : X - \to \mathbf{P}_*(V_x) \simeq \mathbf{P}^3$ for general x, where $V_x \subset V$ is the space of global sections of E which vanish at x. In particular, X is rational. The birational mapping Π_x is the same as the triple projection of $X_{22} \subset \mathbf{P}^{13}$ from x.

The bundle method gives another canonical description of prime Fano 3-folds of genus 12 in the variety of twisted cubics ([29, §3]). This description is useful to analyze the double projection of $X_{22} \subset \mathbf{P}^{13}$ from a line.

[10] The bundle method works for other values of g, e.g., 18 and 20 and gives a description of polarized K3 surfaces (see [30]).

[11] For a vector space V, $G(V, r)$ denotes the Grassmann variety of r-dimensional quotient space of V.

[12] The restriction of E is rigid and characterized by its numerical invariants and stability ([27, §3]).

[13] Prime Fano 3-folds of genus 12 were omitted in [38, Chap. V, §7] and first constructed by Iskovskih [16].

Remark 4 *The third Betti number of a prime Fano 3-fold of of genus $g \geq 7$ is equal $2(n(g)-3)$. In particular, prime Fano 3-folds of genus 12 have the same homology group as $\mathbf{P}^3$.*

§4 Genus bound The descriptions given in §§2 and 3 complete the classification of prime Fano 3-folds by virtue of Iskovskih's genus bound:

Theorem 5 *The genus g of a prime Fano 3-fold satisfies $g \leq 10$ or $g = 12$.*

This is proved in the course of the classification by the double projection method. Here we sketch a simple proof using a correspondence between the moduli spaces of K3 surfaces and curves. Let $\mathcal{F}_g$ be the moduli space of polarized K3 surfaces (S, h) of degree $2g-2$. A smooth member of $|h|$ is a curve of genus g. Hence we obtain the rational map ϕ_g from the $\mathbf{P}^g$-bundle $\mathcal{P}_g := \coprod_{(S,h)\in\mathcal{F}_g} |h|$ over $\mathcal{F}_g$ to the moduli space $\mathcal{M}_g$ of stable curves of genus g. The key observation is this.

Proposition 6 *If a prime Fano 3-fold of genus g exists, then the rational map $\phi_g : \mathcal{P}_g - \to \mathcal{M}_g$ is not generically finite.*

By a simple deformation argument, we have that the generic hyperplane section (S, h) of the generic prime Fano 3-fold is generic in $\mathcal{F}_g$. Take a generic pencil P of hyperplane sections of $X_{2g-2} \subset \mathbf{P}^{g+1}$. The isomorphism classes of the members of P vary since the pencil P contains a singular member. But every member of P contains the base locus of P, which is a curve of genus g. This shows the proposition.

Since $\dim \mathcal{P}_g = g + 19$ and $\dim \mathcal{M}_g = 3g - 3$, $\dim \mathcal{P}_g \leq \dim \mathcal{M}_g$ holds if and only if $g \geq 11$. We recall the proof of the generic finiteness of ϕ_{11} in [25]. Let $C \subset \mathbf{P}^5$ be a sextic normal elliptic curve and S a smooth complete intersection of three hyperquadrics containing C. Let H be a general hyperplane section of S and put $\Gamma = H \cup C$. The S is a K3 surface and Γ is a stable curve of genus 11.

Theorem([25, (1.2)]) *For every embedding $i : \Gamma \to S'$ of Γ in to a K3 surface S', there exists an isomorphism $I : S \to S'$ whose restriction to Γ coincides with i.*

This implies that the point $\xi \in \mathcal{P}_{11}$ corresponding to (S, Γ) is isolated in $\phi_{11}^{-1}(\phi_{11}(\xi))$. Hence ϕ_{11} is generically finite and a prime Fano 3-fold of genus 11 does not exist. The non-existence of prime Fano 3-folds of genus ≥ 13 is proved in a similar way. Note that the elliptic curve C induces an elliptic fibration of S, which we denote by $\pi : S \to \mathbf{P}^1$. We consider the case in which π has two singular fibers of the following types:

i) $E_1 \cup E_2 \cup E_3$ with $(E_2.E_3) = (E_3.E_1) = (E_1.E_2) = 1$, and

ii) $E_2' \cup E_4$ with $(E_2'.E_4) = 2$,

where E_ν is isomorphis to $\mathbf{P}^1$ and satisfies $(E_\nu.H) = \nu$ for evey $1 \leq \nu \leq 4$. It is easy to construct a stable curve Γ_g of genus ≥ 13 on S from Γ by adding fibres of π. For example, $\Gamma \cup E_3$, $\Gamma \cup E_4$ and $\Gamma \cup E_2 \cup E_3$ are of genus 13, 14 and 15, respectively. Note that to add one general fibre of π increases the genus by 6. Byt the above theorem, it is easy to show that every embedding of Γ_g into a K3 surface S' is extended to an isomorphism from S onto S'. Hence we have

Theorem 7 *The rational map $\phi_g : \mathcal{P}_g - \to \mathcal{M}_g$ is generically finite if and only if $g = 11$ or $g \geq 13$.*

This completes the proof of Theorem 5.

Remark 8 *The map ϕ_g is generically of maximal rank except for $g = 10, 12$. In the case of $g = 10$, the image of ϕ_{10} is is a divisor of $\mathcal{M}_{10}$ (See [28]).*

§5 Theory of polars Prime Fano 3-folds of genus 12 are related to the classical problem on sums of powers, which is a polynomial version of the Warring problem. Let F_d be a homogeneous polynomial of degree d in n variables.

1) Are there N linear forms $f_1, \cdots, f_N$ such that $F_d = \sum_1^N f_i^d$?
2) If so, then how many?

In the following cases, every general F_d is a sum of d-th powers of N linear forms and the expression is unique:

(1) $n = 2$ and $d = 2N$ (Sylvester[43]),
(2) $n = 4$, $d = 3$ and $N = 5$(Sylvester's pentahedral theorem [34] [39]), and
(3) $n = 3$, $d = 5$ and $N = 7$(Hilbert [14, p. 153], Richmond [34] and Palatini [32]).

We consider the case $n = 3$. Let C and Γ be the plane curves defined by F_d and $\prod_1^N f_i$, respectively. Γ is called a *polar N-side* of C if $F_d = \sum_1^N f_i^d$. The name comes from the following:

Example 9 *Let C be a smooth conic and ℓ_1, ℓ_2 and ℓ_3 three distinct lines. Then the following are equivalent:*

(1) $\triangle = \ell_1 + \ell_2 + \ell_3$ is a polar 3-side of C in the above sense, and

(2) the triangle $\triangle$ is self polar with respect to C, that is, each side is the polar of its opposite vertex.

§6 Variety of sums of powers We regard the set of polar N-sides of $C : F_d(x, y, z) = 0$ as a subvariety of the projective space of plane curves of degree N. We denote its closure[14] by $VSP(C, N)$ or $VSP(F_d, N)$. The homogeneous forms of degree N form a vector space of dimension $\frac{1}{2}(d+1)(d+2)$. The N-ples of linear forms form a vector space of dimension $3N$. Hence the dimension of $VSP(C, N)$ is expected to be $3N - \frac{1}{2}(d+1)(d+2)$ for general C. In the case $(d, N) = (2, 3)$, this is true.

Proposition 10 *If C is a smooth conic, then $VSP(C, 3)$ is a smooth quintic del Pezzo 3-fold.*

Let V_2 be the vector space of quadratic forms. If $\triangle : f_1 f_2 f_3 = 0$ is a polar 3-side of C, then the defining equation F_2 of C is contained in the subspace $< f_1^2, f_2^2, f_3^2 >$ of V_2. Therefore, $\triangle$ determines a 2-dimensional subspace W of $V^* := V_2/\mathbf{C}F_2$. Hence we have the morphism from $VSP(C, 3)$ to the 6-dimensional Grassmann variety $G(2, V^*) \subset \mathbf{P}^9$. Let $q : V_2 \longrightarrow \mathbf{C}$ be the linear map associated to the dual conic of C. For a pair of quadratic forms f and g, consider the three minors $J_i(f, g)$, $i = 1, 2, 3$, of the Jacobian matrix

$$\begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix}$$

and put $\sigma_i(f, g) = q(J_i(f, g))$. Then σ_i are skew-symmetric forms on V_2 and F_2 is their common radical. Therefore, each σ_i, $i = 1, 2, 3$, determine three hyperplanes H_i of $\mathbf{P}^9 = \mathbf{P}_*(\bigwedge^2 V^*)$. $VSP(C, 3)$ is isomorphic to the quintic del Pezzo 3-fold $G(2, V^*) \cap H_1 \cap H_2 \cap H_3$.

Now we consider plane quartic curves $C : F_4(x, y, z) = 0$. The dimension count

$$3N - 15 \stackrel{?}{=} \dim VSP(C, N)$$

does not hold for $N = 5$:

Let $\{\partial_1 = \partial^2/\partial x^2, \cdots, \partial_6 = \partial^2/\partial z^2\}$ be a basis of the space of homogeneous second order partial differential operators.

[14] The closure is taken in the symmetric product $Sym^N \mathbf{P}^2$. But this is a temporary definition. In practice, we choose a suitable model of $Sym^N \mathbf{P}^2$ to define $VSP(C, N)$.

Theorem (Clebsch [4]) *If a plane quartic curve $C : F_4(x, y, z) = 0$ has a polar 5-side, then*

$$\Omega(F) := \det(\partial_i\partial_j F)_{1\le i,j\le 6} = 0.$$

In particular, general plane quartic curves have no polar 5-sides.

In other words, polar 5-sides are not equally distributed to quartic curves. Once a quartic curve has a polar 5-side, it has a 1-dimensional family of polar 5-sides. (The same happens for polar 2-sides of conics.)

Polar 6-sides of plane quartics was studied by Rosanes [35] and Scorza [40]. The dimension count is correct for $N = 6$ and we obtain 3-folds.

Theorem 11 *(1) If a quartic curve C has no polar 5-sides or no complete quadrangles as its polar 6-sides, then the variety $VSP(C, 6)$ of polar 6-sides of C is a prime Fano 3-fold of genus 12.*

(2) Conversely every prime Fano 3-fold X of genus 12 is obtained in this way. The isomorphism class of C is uniquely determined by that of X.

By virtue of Theorem 3, it suffice to show that $G(V, 3, N)$ is isomorphic to $VSP(C, 6)$. Let V_3 be the vector space of cubic forms. If $\Gamma : f_1 f_2 \cdots f_6 = 0$ is a polar 6-side of C, then the partial derivatives F_x, F_y and F_z of the defining equation F_4 are contained in $< f_1^3, f_2^3, \cdots, f_6^3 >$. Hence Γ determines a 3-dimensional subspace of $V^* := V_3/ < F_x, F_y, F_z >$ and we obtain a morphism ϕ from $VSP(C, 6)$ to $G(3, V^*)$. Three skew-symmetric forms σ_1, σ_2 and σ_3 on V^* are defined as in the case of $VSP(F_2, 3)$ and the image of ϕ is contained in $G(3, V^*, \sigma_1, \sigma_2, \sigma_3)$.

Conversely, let V and $N \subset \bigwedge^2 V$ be as in Theorem 3. The multiplication in the exterior algebra $\bigwedge^\bullet V$ induces the map $\sigma_3 : S^3 N \longrightarrow \bigwedge^6 V$. This is surjective and its kernel is of dimension 3.

Lemma 12 *There exists a quartic polynomial $F(x, y, z) \in S^4 N$ whose partial derivatives F_X, F_Y and F_Z form a basis of the kernel of σ_3, where $\{x, y, z\}$ is a basis of N and $\{X, Y, Z\}$ is its dual.*

The conics on (the anticanonical model of) $G(V, 3, N)$ is parametrized by the projective plane[15] $\mathbf{P}_*(N)$ For every point x of $G(V, 3, N)$, there exist exactly six conics $\{Z_{\lambda_i}\}_{1\le i\le 6}$, $\lambda_i \in \mathbf{P}_*(N)$, passing through x, counted with their multiplicities. . Let Λ_i, $1 \le i \le 6$, be the lines on $\mathbf{P}(N)$ with coordinates λ_i. Then $\Gamma = \sum_1^6 \Lambda_i$ is a polar 6-side of the plane curve C on $\mathbf{P}(N)$ defined by the quartic form $F(x, y, z)$ in the lemma. This correspondence $x \mapsto \Gamma$ gives the inverse of the above morphism ϕ.

Remark 13 *(1) Assume that a plane quartic C' has a polar 5-side and that the 5lines are in general position.*

When C in Theorem 11 deforms to C', the variety $VSP(C, 6)$ deforms to a Fano 3-fold X' with an ordinary double point. X' is isomorphic to the anti canonical model of $\mathbf{P}(E)$, where E is a stable vector bundle on $\mathbf{P}^2$ with $c_1 = 0$ and $c_2 = 4$ (cf. [2]).

(2) If C is a general plane septic curve, then the variety $VSP(C, 10)$ is a polarized K3 surface of genus 20.

[15] For a vector space V, $\mathbf{P}_*(V)$ is the projective space of 1-dimensional subspaces of V. $\mathbf{P}(V)$, or $\mathbf{P}^*(V)$, is its dual.

§7 Almost homogeneous Fano 3-fold Varieties of sums of powers give two examples of almost homogeneous spaces of $SO(3,\mathbf{C})$ and their compactifications. Apply Theorem 11 to a double conic, say $2C_0 : (XZ+Y^2)^2 = 0$.

The variety $VSP(2C_0, 6)$ is a Fano 3-fold and has an action of $SO(3,\mathbf{C})$. It is easy to check

$$30(XZ+Y^2)^2 = 25Y^4 + \sum_{i=0}^{4}(\zeta^i X + Y + \zeta^{-i}Z)^4,$$

where ζ is a fifth root of unity. The polar 6-side

$$\Gamma : Y\prod_{i=0}^{4}(\zeta^i X + Y + \zeta^{-i}Z) = 0$$

intersects the 2-sphere C_0 at the 12 vertices of a regular icosahedron. The stabilizer group at Γ of $SO(3,\mathbf{C})$ is the icosahedral group $\simeq A_5$. Hence we have

Theorem 14 *The variety $VSP(2C_0, 6)$ is a smooth equivariant compactification of $SO(3,\mathbf{C})/Icosa$.*

Similarly the quintic del Pezzo 3-fold $VSP(C_0, 3)$ is a smooth equivariant compactification of the quotient of $SO(3,\mathbf{C})$ by an octahedral group $\simeq S_4$ by Proposition 10. These two compactifications are described in [31, §§3 and 6] by another method. We remark that Q^3 and $\mathbf{P}^3$ are also almost homogeneous spaces of $SO(3,\mathbf{C})$. The stabilizer groups are tetrahedral group $\simeq A_4$ and a dihedral group of order 6 $\simeq S_3$, respectively.

§8 Compactification of $\mathbf{C}^3$ There are four types of Fano 3-folds with the same homology growps as $\mathbf{P}^3$: $\mathbf{P}^3$ itself, $Q^3 \subset \mathbf{P}^4$, $V_5 \subset \mathbf{P}^6$ and the 6-dimensional family of prime Fano 3-folds $X_{22} \subset \mathbf{P}^{13}$ of genus 12 (see Remark 4). These Fano 3-folds are related to not only $SO(3,\mathbf{C})$ but also $\mathbf{C}^3$, the affine 3-space. It is well-known that $\mathbf{P}^3$ and Q^3 are smooth compactifications of $\mathbf{C}^3$ with irreducible boundary divisors. The quintic del Pezzo 3-fold $V_5 \subset \mathbf{P}^6$ is a compactifications of $\mathbf{C}^3$ in two ways (see [10] and [13]).

Furushima has found that the almost homogeneous Fano 3-fold $U_{22} := VSP(2C_0, 6)$ also is a Compactification of $\mathbf{C}^3$. This fact is proved in three ways using

i) the defining equation ([31] p.506) of $U_{22} \subset \mathbf{P}^{12}$ (see [11]),

ii) the double projection of $U_{22} \subset \mathbf{P}^{13}$ from a line (see [12]), and

iii) the action of a torus $\mathbf{C}^* \subset SO(3,\mathbf{C})$ on U_{22} (see [1] and [20]).

In the last case, U_{22} is decomposed into a disjoint union of affine spaces by virtue of [3]. The four compactifications by $\mathbf{P}^3, Q^3$ and V_5 are rigid but that by U_{22} is not. In fact, by a careful analysis of the double projection of $VSP(C, 6) \subset \mathbf{P}^{13}$ from a line, we have

Theorem 15 *The variety $VSP(C, 6)$ in Theorem 11 is a compactification of $\mathbf{C}^3$ if C has a non-ordinary singular point.*

The variety $VSP(C, 6)$ has a line ℓ (on its anticanonical model) with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(-2)$ corresponding to a non-ordinary singular point of C. Let D be the union of conics which intersect ℓ as in Example 1. Then the complement of D is isomorphic to $\mathbf{C}^3$.

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