

Polarized K3 surfaces of genus thirteen

Shigeru Mukai

A smooth complete algebraic surface S is of type K3 if S is regular and the canonical class K_S is trivial. A *primitively polarized* K3 surface is a pair (S, h) of a K3 surface S and a primitive ample divisor class $h \in \text{Pic } S$. The integer $g := \frac{1}{2}(h^2) + 1 \geq 2$ is called the *genus* of (S, h) . The moduli space of primitively polarized K3 surfaces of genus g exists as a quasi-projective variety, which we denote by \mathcal{F}_g . As is well known a general polarized K3 surface of genus $2 \leq g \leq 5$ is a complete intersection of hypersurfaces in a weighted projective space: $(6) \subset \mathbf{P}(1112)$, $(4) \subset \mathbf{P}^3$, $(2) \cap (3) \subset \mathbf{P}^4$ and $(2) \cap (2) \cap (2) \subset \mathbf{P}^5$.

In connection with the classification of Fano threefolds, we have studied the system of defining equations of the projective model $S_{2g-2} \subset \mathbf{P}^g$ and shown that a general polarized K3 surface of genus g is a complete intersection with respect to a homogeneous vector bundle \mathcal{V}_{g-2} in a g -dimensional Grassmannian $G(n, r)$, $g = r(n - r)$, in a unique way for the following six values of g :

g	6	8	9	10
r	2	2	3	5
\mathcal{V}_{g-2}	$3\mathcal{O}_G(1) \oplus \mathcal{O}_G(2)$	$6\mathcal{O}_G(1)$	$\wedge^2 \mathcal{E} \oplus 4\mathcal{O}_G(1)$	$\wedge^4 \mathcal{E} \oplus 3\mathcal{O}_G(1)$

12	20
3	4
$3\wedge^2 \mathcal{E} \oplus \mathcal{O}_G(1)$	$3\wedge^2 \mathcal{E}$

Here \mathcal{E} is the universal quotient bundle on $G(n, r)$. See [4] and [5] for the case $g = 6, 8, 9, 10$, [6, §5] for $g = 20$ and §3 for $g = 12$.

By this description, the moduli space \mathcal{F}_g is birationally equivalent to the orbit space $H^0(G(n, r), \mathcal{V}_{g-2}) / (PGL(n) \times \text{Aut}_{G(n, r)} \mathcal{V}_{g-2})$ and

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hence is unirational for these values of g . The uniqueness of the description modulo the automorphism group is essentially due to the *rigidity* of the vector bundle $E := \mathcal{E}|_S$. All the cohomology groups $H^i(\mathit{sl}(E))$ vanish.

A general member $(S, h) \in \mathcal{F}_g$ is a complete intersection with respect to the homogeneous vector bundle $8\mathcal{U}$ in the orthogonal Grassmannian $O\text{-}G(10, 5)$ in the case $g = 7$ ([4]), and with respect to $5\mathcal{U}$ in $O\text{-}G(9, 3)$ in the case 18 ([6]), where \mathcal{U} is the homogeneous vector bundle on the orthogonal Grassmannian such that $H^0(\mathcal{U})$ is a half spinor representation U^{16} . Both descriptions are unique modulo the orthogonal group. Hence \mathcal{F}_7 and \mathcal{F}_{18} are birationally equivalent to $G(8, U^{16})/PSO(10)$ and $G(5, U^{16})/SO(9)$, respectively. The unirationality of \mathcal{F}_{11} is proved in [7] using a non-abelian Brill-Noether locus and the unirationality of \mathcal{M}_{11} , the moduli space of curves of genus 11.

In this article, we shall study the case $g = 13$ and show the following:

Theorem 1. *A general member $(S, h) \in \mathcal{F}_{13}$ is isomorphic to a complete intersection with respect to the homogeneous vector bundle*

$$\mathcal{V} = \bigwedge^2 \mathcal{E} \oplus \bigwedge^2 \mathcal{E} \oplus \bigwedge^3 \mathcal{F}$$

of rank 10 in the 12-dimensional Grassmannian $G(7, 3)$, where \mathcal{F} is the dual of the universal subbundle.

Corollary \mathcal{F}_{13} *is unirational.*

Remark 1. A general complete intersection (S, h) with respect to the homogeneous vector bundle $\bigwedge^4 \mathcal{F} \oplus S^2 \mathcal{E}$ in the 10-dimensional Grassmannian $G(7, 2)$ is also a primitively polarized K3 surface of genus 13. But (S, h) is not a general member of \mathcal{F}_{13} . In fact, S contains 8 mutually disjoint rational curves R_1, \dots, R_7 , which are of degree 3 with respect to h . This will be discussed elsewhere.

Unlike the known cases described above, the vector bundle $E = \mathcal{E}|_S$ is not rigid. Hence the theorem does not give a birational equivalence between \mathcal{F}_{13} and an orbit space. But E is *semi-rigid*, that is, $H^0(\mathit{sl}(E)) = 0$ and $\dim H^1(\mathit{sl}(E)) = 2$. Instead of \mathcal{F}_{13} , the theorem gives a birational equivalence between the universal family over it and an orbit space.

Let $S \subset G(7, 3)$ be a general complete intersection with respect to \mathcal{V} . Then S is the common zero locus of the two global sections of $\bigwedge^2 \mathcal{E}$ corresponding to general bivectors $\sigma_1, \sigma_2 \in \bigwedge^2 \mathbf{C}^7$ and one global section of $\bigwedge^3 \mathcal{F}$ corresponding to a general $\tau \in \bigwedge^3 \mathbf{C}^{7, \vee}$. The 2-dimensional

subspace $P = \langle \sigma_1, \sigma_2 \rangle \subset \wedge^2 \mathbf{C}^7$ is uniquely determined by S . Let $\overline{P \wedge P}$ be the subspace of $\wedge^3 \mathbf{C}^{7,\vee}$ corresponding to $P \wedge P \subset \wedge^4 \mathbf{C}^7$. Then \mathbf{C}^7 modulo $\overline{P \wedge P}$ is also uniquely determined by S . It is known that the natural action of $PGL(7)$ on $G(2, \wedge^2 \mathbf{C}^7)$ has an open dense orbit (Sato-Kimura[9, p. 94]). Hence we obtain the natural birational map

$$(1) \quad \psi : \mathbf{P}_*(\wedge^4 \mathbf{C}^7 / (P \wedge P)) / G \cdots \rightarrow \mathcal{F}_{13},$$

which is dominant by the theorem, where G is the (10-dimensional) stabilizer group of the action at $P \in G(2, \wedge^2 \mathbf{C}^7)$.

Theorem 2. *For every general member $p = (S, h) \in \mathcal{F}_{13}$, the fiber of ψ at p is birationally equivalent to the moduli K3 surface $M_S(3, h, 4)$ of semi-rigid rank three vector bundles with $c_1 = h$ and $\chi = 3 + 4$.*

As is shown in [8], $\hat{S} := M_S(3, h, 4)$ carries a natural ample divisor class \hat{h} of the same genus (=13) and $(S, h) \mapsto (\hat{S}, \hat{h})$ induces an isomorphism of \mathcal{F}_{13} . (In fact, this is an involution.) Hence we have

Corollary *The orbit space $\mathbf{P}^*(\wedge^4 \mathbf{C}^7 / (P \wedge P)) / G$ is birationally equivalent to the universal family over \mathcal{F}_{13} , or the coarse moduli space of one pointed polarized K3 surfaces (S, h, p) of genus 13.*

Remark 2. Kondō[3] proves that the Kodaira dimension of \mathcal{F}_g is non-negative for the following 17 values:

$$g = 41, 42, 50, 52, 54, 56, 58, 60, 65, 66, 68, 73, 82, 84, 104, 118, 132.$$

The Kodaira dimension of $\mathcal{F}_{m^2(g-1)+1}$ is non-negative for these values of g and for every $m \geq 2$ since it is a finite covering of \mathcal{F}_g .

Notations and convention. Algebraic varieties and vector bundles are considered over the complex number field \mathbf{C} . The dual of a vector bundle (or a vector space) E is denoted by E^\vee . Its Euler-Poincaré characteristic $\sum_i (-)^i h^i(E)$ is denoted by $\chi(E)$. The vector bundles of traceless endomorphisms of E is denoted by $sl(E)$. For a vector space V , $G(V, r)$ is the Grassmannian of r -dimensional quotient spaces of V and $G(r, V)$ that of r -dimensional subspaces. The isomorphism class of $G(V, r)$ with $\dim V = n$ is denoted by $G(n, r)$. $G(V, 1)$ and $G(1, V)$ is denoted by $\mathbf{P}^*(V)$ and $\mathbf{P}_*(V)$, respectively. $\mathcal{O}_G(1)$ is the pull-back of the tautological line bundle by the Plücker embedding $G(V, r) \hookrightarrow \mathbf{P}^*(\wedge^r V)$.

§1. Vanishing

We prepare the vanishing of cohomology groups of homogeneous vector bundles on the Grassmannian $G(n, r)$, which is the quotient

of $SL(n)$ by a parabolic subgroup P . The reductive part P_{red} of P is the intersection of $GL(r) \times GL(n-r)$ and $SL(n)$ in $GL(n)$. We take $\{(a_1, \dots, a_r; a_{r+1}, \dots, a_n) \mid \sum_1^n a_i = 0\} \subset \mathbf{Z}^n$ as root lattice and $\mathbf{Z}^n/\mathbf{Z}(1, 1, \dots, 1)$ as the common weight lattice of $SL(n)$ and P_{red} . We take $\{e_i - e_{i+1} \mid 1 \leq i \leq n-1\}$ as standard root basis. The half of the sum of all positive roots is equal to

$$\delta = (n-1, n-3, n-5, \dots, -n+3, -n+1)/2.$$

Let ρ be an irreducible representation of P_{red} and $w \in \mathbf{Z}^n/\mathbf{Z}(1, 1, \dots, 1)$ its weight. We denote the homogeneous vector bundle on $G(n, r)$ induced from ρ by E_w . w is *singular* if a number appears more than once in $w + \delta$. If w is not singular and $w + \delta = (a_1, a_2, \dots, a_n)$, then there is a unique (Grassmann) permutation $\alpha = \alpha_w$ such that $a_{\alpha(1)} > a_{\alpha(2)} > \dots > a_{\alpha(n)}$. We denote the length of α_w , that is, the cardinality of the set $\{(i, j) \mid 1 \leq i < j \leq n, a_i < a_j\}$, by $l(w)$.

- Theorem 3** (Borel-Hirzebruch[2]). (a) *If w is singular, then all the cohomology groups $H^i(G(n, r), \mathcal{E}_w)$ vanish.*
 (b) *If w is not, then all the cohomology groups $H^i(G(n, r), \mathcal{E}_w)$ vanish except for $i = l(w)$. Moreover, $H^{l(w)}(G(n, r), \mathcal{E}_w)$ is an irreducible representation of $SL(n)$ with highest weight*

$$(a_{\alpha(1)}, a_{\alpha(2)}, \dots, a_{\alpha(n)}) - \delta.$$

The dimension of this unique nonzero cohomology group is equal to $\prod_{1 \leq i < j \leq n} |a_i - a_j| / (j - i)$.

$l(w)$ is called the *index* of the homogeneous vector bundle E_w .

Example. In the following table, $-$ means that the weight w is singular.

weight w	homogeneous bundle \mathcal{E}_w	$l(w)$	$H^{l(w)}$
$(1, 0, 0, \dots, 0, 0; 0, \dots, 0, 0)$	\mathcal{E} , universal quotient bundle	0	\mathbf{C}^n
$(0, 0, 0, \dots, -1, 0; 0, \dots, 0, 0)$	\mathcal{E}^\vee	-	
$(1, 1, 0, \dots, 0, 0; 0, \dots, 0, 0)$	$\bigwedge^2 \mathcal{E}$	0	$\bigwedge^2 \mathbf{C}^n$
$(1, 1, 1, \dots, 1, 0; 0, \dots, 0, 0)$	$\mathcal{O}_G(1) = \det \mathcal{E} = \det \mathcal{F}$	0	$\bigwedge^r \mathbf{C}^n$
$(0, 0, 0, \dots, 0, 0; -1, \dots, -1)$			
$(0, 0, 0, \dots, 0, 0; 1, \dots, 0, 0)$	\mathcal{F}^\vee , universal subbundle	-	
$(0, 0, 0, \dots, 0, 0; 0, \dots, 0, -1)$	\mathcal{F}	0	$\mathbf{C}^{n, \vee}$
$(1, 0, 0, \dots, 0, 0; 0, \dots, 0, -1)$	$T_{G(n, r)}$, tangent bundle	0	$sl(\mathbf{C}^n)$
$(0, 0, 0, \dots, -1; 1, 0, \dots, 0, 0)$	$\Omega_{G(n, r)}$, cotangent bundle	1	\mathbf{C}
$(-s, -s, \dots, -s; r, r, \dots, r)$	$\mathcal{O}_G(-n)$, canonical bundle	dim	\mathbf{C}

We apply the theorem to the 12-dimensional Grassmannian $G(7, 3)$.

- Lemma 4.** (a) All cohomology groups of the homogeneous vector bundle $\Lambda^p(2\mathcal{E}(-1)) \otimes \Lambda^q(\mathcal{F}(-1))$ on $G(7,3)$ vanish except for the following:
- i) $p = q = 0, \quad h^0(\mathcal{O}_G) = 1, \text{ and}$
 - ii) $p = 6, q = 4, \quad h^{12}(\mathcal{O}_G(-7)) = 1.$
- (b) All cohomology groups of $\mathcal{O}_G(1) \otimes \Lambda^p(2\mathcal{E}(-1)) \otimes \Lambda^q(\mathcal{F}(-1))$ vanish except for the following:
- i) $p = q = 0, \quad h^0(\mathcal{O}_G(1)) = 35,$
 - ii) $p = 1, q = 0, \quad h^0(2\mathcal{E}) = 2 \cdot 7 = 14, \text{ and}$
 - iii) $p = 0, q = 1, \quad h^0(\mathcal{F}) = 7.$
- (c) All cohomology groups of $\mathcal{E} \otimes \Lambda^p(2\mathcal{E}(-1)) \otimes \Lambda^q(\mathcal{F}(-1))$ vanish except for $h^0(\mathcal{E}) = 7$ with $p = q = 0$.
- (d) All cohomology groups of $\mathcal{F} \otimes \Lambda^p(2\mathcal{E}(-1)) \otimes \Lambda^q(\mathcal{F}(-1))$ vanish except for $h^0(\mathcal{F}) = 7$ with $p = q = 0$.
- (e) All cohomology groups of $\Lambda^2 \mathcal{E} \otimes \Lambda^p(2\mathcal{E}(-1)) \otimes \Lambda^q(\mathcal{F}(-1))$ vanish except for the following:
- i) $p = q = 0, \quad h^0(\Lambda^2 \mathcal{E}) = 21, \text{ and}$
 - ii) $p = 1, q = 0, \quad h^0(\Lambda^2 \mathcal{E} \otimes (2\mathcal{E}(-1))) = 2.$
- (f) All cohomology groups of $\Lambda^3 \mathcal{F} \otimes \Lambda^p(2\mathcal{E}(-1)) \otimes \Lambda^q(\mathcal{F}(-1))$ vanish except for the following:
- i) $p = q = 0, \quad h^0(\Lambda^3 \mathcal{F}) = 35,$
 - ii) $p = 0, q = 1, \quad h^0(\Lambda^3 \mathcal{F} \otimes \mathcal{F}(-1)) = 1, \text{ and}$
 - iii) $p = 2, q = 0, \quad h^1(\Lambda^3 \mathcal{F} \otimes \Lambda^2(2\mathcal{E}(-1))) = 3h^1(\Lambda^3 \mathcal{F} \otimes \Lambda^2 \mathcal{E}^\vee) = 3.$
- (g) All cohomology groups of $sl(\mathcal{E}) \otimes \Lambda^p(2\mathcal{E}(-1)) \otimes \Lambda^q(\mathcal{F}(-1))$ vanish except for $h^6 = 2$ with $p = 3, q = 2$.

Proof. The following table describes the decomposition of $\Lambda^p(2\mathcal{E}(-1))$ into indecomposable homogeneous vector bundles.

p	decomposition	weight w'	$w' + \delta'$
0	\mathcal{O}_G	$(0, 0, 0)$	$(3, 2, 1)$
1	$2\mathcal{E}(-1)$	$2(0, -1, -1)$	$(3, 1, 0)$
2	$3(\Lambda^2 \mathcal{E})(-2)$	$3(-1, -1, -2)$	$(2, 1, -1),$
	$\oplus S^2 \mathcal{E}(-2)$	$\oplus(0, -2, -2)$	$(3, 0, -1)$
3	$4\mathcal{O}_G(-2)$	$4(-2, -2, -2)$	$(1, 0, -1),$
	$\oplus 2sl(\mathcal{E})(-2)$	$\oplus 2(-1, -2, -3)$	$(2, 0, -2)$
4	$3\mathcal{E}(-3)$	$3(-2, -3, -3)$	$(1, -1, -2),$
	$\oplus (S^2 \Lambda^2 \mathcal{E})(-4)$	$\oplus(-2, -2, -4)$	$(1, 0, -3)$
5	$2(\Lambda^2 \mathcal{E})(-4)$	$2(-3, -3, -4)$	$(0, -1, -3)$
6	$\mathcal{O}_G(-4)$	$(-4, -4, -4)$	$(-1, -2, -3)$

Here $\delta' = (3, 2, 1)$ is the *head* of $\delta = (3, 2, 1; 0, -1, -2, -3)$.

$\bigwedge^q(\mathcal{F}(-1))$ is indecomposable. The following lists their weights w'' and $w'' + \delta''$, where $\delta'' = (0, -1, -2, -3)$ is the *tail* of δ .

q	bundle	weight w''	$w'' + \delta''$
0	\mathcal{O}_G	$(0, 0, 0, 0)$	$(0, -1, -2, -3)$
1	$\mathcal{F}(-1)$	$(1, 1, 1, 0)$	$(1, 0, -1, -3)$
2	$(\bigwedge^2 \mathcal{F})(-2)$	$(2, 2, 1, 1)$	$(2, 1, -1, -2)$
3	$(\bigwedge^3 \mathcal{F})(-3)$	$(3, 2, 2, 2)$	$(3, 1, 0, -1)$
4	$\mathcal{O}_G(-3)$	$(3, 3, 3, 3)$	$(3, 2, 1, 0)$

We prove (a), (f) and (g) applying Theorem 3. The other cases are similar.

(a) Take w' and w'' from the tables (2) and (3), respectively, and combine into $w = (w'; w'')$. Then w is singular except for the two cases

$$w + \delta = (3, 2, 1; 0, -1, -2, -3) \quad \text{with} \quad p = q = 0$$

and

$$w + \delta = (-1, -2, -3; 3, 2, 1, 0) \quad \text{with} \quad p = 6, q = 4.$$

The indices $l(w)$ are equal to 0 and 12, respectively.

(f) The homogeneous vector bundle $\bigwedge^3 \mathcal{F} \otimes \bigwedge^q(\mathcal{F}(-1))$ decomposes in the following way:

q	weight w''	$w'' + \delta''$
0	$(0, -1, -1, -1)$	$(0, -2, -3, -4)$
1	$(1, 0, 0, -1) \oplus (0, 0, 0, 0)$	$(1, -1, -2, -4), (0, -1, -2, -3)$
2	$(2, 1, 0, 0) \oplus (1, 1, 1, 0)$	$(2, 0, -2, -3), (1, 0, -1, -3)$
3	$(3, 1, 1, 1) \oplus (2, 2, 1, 1)$	$(3, 0, -1, -2), (2, 1, -1, -2)$
4	$(3, 2, 2, 2)$	$(3, 1, 0, -1)$

Take w' and w'' from the table (2) and this table, respectively, and consider $w = (w'; w'')$. Then w is singular except for the following three cases.

- i) $p = q = 0, w + \delta = (3, 2, 1; 0, -2, -3, -4), l(w) = 0,$
- ii) $p = 0, q = 1, w + \delta = (3, 2, 1; 0, -1, -2, -3), l(w) = 0,$ and
- iii) $p = 2, q = 0, w + \delta = (2, 1, -1; 0, -2, -3, -4), l(w) = 1.$

(g) The following table shows the indecomposable components of $sl(\mathcal{E}) \otimes \wedge^p(2\mathcal{E}(-1))$ which do not appear in that of $\wedge^p(2\mathcal{E}(-1))$.

p	weight w' other than Table (2)	$w' + \delta'$
0	(1, 0, -1)	(4, 2, 0)
1	$2(1, -1, -2) \oplus 2(0, 0, -2)$	(4, 1, -1), (3, 2, -1)
2	$4(0, -1, -3) \oplus (1, -2, -3)$	(3, 1, -2), (4, 0, -2)
(5) 3	$2(0, -2, -4) \oplus 2(-1, -1, -4)$ $\oplus 2(0, -3, -3)$	(3, 0, -3), (2, 1, -3) (3, -1, -2)
4	$(-1, -2, -5) \oplus 4(-1, -3, -4)$	(2, 0, -4), (2, -1, -3)
5	$2(-2, -3, -5) \oplus 2(-2, -4, -4)$	(1, -1, -4), (1, -2, -3)
6	(-3, -4, -5)	(0, -2, -4)

Take w' and w'' from the table (2) and this table, respectively, and consider $w = (w'; w'')$. Then w is singular except for the case $w + \delta = (3, 0, -3; 2, 1, -1, -2)$ with $p = 3$ and $q = 2$. The index is equal to 6. Q.E.D.

Let $S \subset G(7, 3)$ be a complete intersection with respect to $\mathcal{V} = 2\wedge^2 \mathcal{E} \oplus \wedge^3 \mathcal{F}$. The Koszul complex

$$\mathbf{K} : \mathcal{O}_G \leftarrow \mathcal{V}^\vee \leftarrow \bigwedge^2 \mathcal{V}^\vee \leftarrow \dots \leftarrow \bigwedge^9 \mathcal{V}^\vee \leftarrow \bigwedge^{10} \mathcal{V}^\vee \leftarrow 0$$

gives a resolution of the structure sheaf \mathcal{O}_S . $\bigwedge^n \mathcal{V}^\vee$ is isomorphic to $\bigoplus_{p+q=n} \wedge^p(2\mathcal{E}(-1)) \otimes \wedge^q(\mathcal{F}(-1))$.

- Proposition 5.**
- (a) $H^0(S, \mathcal{O}_S) = \mathbf{C}, H^1(S, \mathcal{O}_S) = 0$.
 - (b) The restriction map $H^0(G(7, 3), \mathcal{O}_G(1)) \rightarrow H^0(S, \mathcal{O}_S(1))$ is surjective, $H^0(S, \mathcal{O}_S(1))$ is of dimension 14 and $H^1(S, \mathcal{O}_S(1)) = H^2(S, \mathcal{O}_S(1)) = 0$.
 - (c) The restriction map $H^0(G(7, 3), \mathcal{E}) \rightarrow H^0(S, E)$ is an isomorphism and $H^1(S, E) = H^2(S, E) = 0$.
 - (d) $H^0(G(7, 3), \mathcal{F}) \rightarrow H^0(S, F)$ is an isomorphism.
 - (e) $H^0(G(7, 3), \wedge^2 \mathcal{E}) \rightarrow H^0(S, \wedge^2 E)$ is surjective and the kernel is of dimension 2.
 - (f) $H^0(G(7, 3), \wedge^3 \mathcal{F}) \rightarrow H^0(S, \wedge^3 F)$ is surjective and the kernel is of dimension 4.
 - (g) E is simple and semi-rigid, that is, $H^0(sl(E)) = 0$ and $h^1(sl(E)) = 2$.

Proof. We prove (a) and (f) as sample. Other cases are similar.

(a) The restriction map $H^0(G(7, 3), \mathcal{O}_G) \rightarrow H^0(S, \mathcal{O}_S)$ is surjective by the vanishing $H^1(\mathcal{V}^\vee) = H^2(\wedge^2 \mathcal{V}^\vee) = \dots = H^{10}(\wedge^{10} \mathcal{V}^\vee) = 0$ and

the exact sequence $0 \leftarrow \mathcal{O}_S \leftarrow \mathbf{K}$. $H^1(S, \mathcal{O}_S)$ vanishes since $H^1(\mathcal{O}_G) = H^2(\mathcal{V}^\vee) = \dots = H^{11}(\bigwedge^{10} \mathcal{V}^\vee) = 0$.

(f) The restriction map is surjective by the vanishing $H^n(\bigwedge^3 \mathcal{F} \otimes \bigwedge^n \mathcal{V}^\vee)$ for $n = 1, \dots, 10$ and the exact sequence

$$0 \leftarrow \bigwedge^3 F \leftarrow \bigwedge^3 \mathcal{F} \otimes \mathbf{K}.$$

The dimension of the kernel is equal to

$$h^0(\bigwedge^3 \mathcal{F} \otimes \mathcal{V}^\vee) + h^1(\bigwedge^3 \mathcal{F} \otimes \bigwedge^2 \mathcal{V}^\vee) = 1 + 3 = 4$$

since $H^{n+1}(\bigwedge^3 \mathcal{F} \otimes \bigwedge^n \mathcal{V}^\vee) = 0$ for $n = 3, \dots, 10$. Q.E.D.

§2. Proof of Theorems 1 and 2

Let S be a zero locus $(s)_0$ of a general global section s of the homogeneous vector bundle $\mathcal{V} = \bigwedge^2 \mathcal{E} \oplus \bigwedge^2 \mathcal{E} \oplus \bigwedge^3 \mathcal{F}$ in the Grassmanian $G(7, 3)$. Since \mathcal{V} is generated by global sections, S is smooth by [6, Theorem 1.10], the Bertini type theorem for vector bundles. Since $r(\mathcal{V}) = 3 + 3 + 4 = \dim G(7, 3) - 2$ and

$$\det \mathcal{V} \simeq \mathcal{O}_G(2) \otimes \mathcal{O}_G(2) \otimes \mathcal{O}_G(3) \simeq \det T_{G(7,3)},$$

S is of dimension two and the canonical line bundle is trivial. By (a) of Proposition 5, S is connected and regular. Hence S is a K3 surface. We denote the class of hyperplane section by h . Then, by (b) of Proposition 5, we have $\chi(\mathcal{O}_S(h)) = 14$, which implies $(h^2) = 24$ by the Riemann-Roch theorem. Hence we obtain the rational map

$$\Psi : \mathbf{P}_* H^0(G(7, 3), \mathcal{V}) \cdots \rightarrow \mathcal{F}'_{13} \quad s \mapsto ((s)_0, h)$$

to the moduli space \mathcal{F}'_{13} of polarized K3 surfaces which are not necessarily primitive.

By (g) of Proposition 5, the vector bundle $E = \mathcal{E}|_S$ is simple. Let (S', h') be a small deformation of (S, h) . Then there is a vector bundle E' on S' which is a deformation of E by Proposition 4.1 of [6]. E' enjoys many properties satisfied by E : E' is simple, generated by global sections, $h^0(E') = 7$, $\bigwedge^3 H^0(E') \rightarrow H^0(\bigwedge^3 E')$ is surjective, etc. Therefore, E' embeds S' into $G(7, 3)$ and S' is also a complete intersection with respect to \mathcal{V} . Hence the rational map Ψ is dominant onto an irreducible component of \mathcal{F}'_{13} and Theorem 1 follows from the following:

Proposition 6. *The polarization h of (S, h) , a complete intersection with respect to \mathcal{V} in $G(7, 3)$, is primitive.*

In the local deformation space of (S, h) , the deformations (S', h') 's with Picard number one form a dense subset. More precisely, it is the complement of an infinite but countable union of divisors. Hence we have

Lemma 7. *There exists a smooth complete intersection S with respect to \mathcal{V} whose Picard number is equal to one.*

Proof of Proposition 6. Since the assertion is topological it suffices to show it for one such (S, h) . We take (S, h) as in this lemma. Assume that h is not primitive. Since $(h^2) = 24$, h is linearly equivalent to $2l$ for a divisor class l with $(l^2) = 6$. The Picard group $\text{Pic } S$ is generated by l . By the Riemann-Roch theorem and the (Kodaira) vanishing, we have $h^0(\mathcal{O}_S(nl)) = 3n^2 + 2$ for $n \geq 1$.

Claim 1. $h^0(E(-l)) = 0$.

Assume the contrary. Then E contains a subsheaf isomorphic to $\mathcal{O}_S(nl)$ with $n \geq 1$. Since $h^0(\mathcal{O}_S(nl)) \leq h^0(E) = 7$, we have $n = 1$ and the quotient sheaf $Q = E/\mathcal{O}_S(l)$ is torsion free. Since $5 = h^0(\mathcal{O}_S(l)) < h^0(E) = 7$, we have $H^0(Q) \neq 0$. Since Q is of rank two and $\det Q \simeq \mathcal{O}_S(l)$, we have $\text{Hom}(Q, \mathcal{O}_S(l)) \neq 0$, which contradicts (g) of Proposition 5.

Now we consider the vector bundle $M = (\wedge^2 E)(-l)$. By the claim and the Serre duality, we have $h^2(M) = \dim \text{Hom}(M, \mathcal{O}_S) = h^0(E(-l)) = 0$. Hence we have $h^0(M) \geq \chi(M) = 4$. Take 4 linearly independent global sections of M and we consider the homomorphism $\varphi : 4\mathcal{O}_S \rightarrow M$.

Claim 2. φ is surjective outside a finite set of points on S .

Let r be the rank of the image of φ . Since $\text{Hom}(\mathcal{O}_S(l), M) = H^0(\wedge^2 E)(-h) = H^0(E^\vee) = H^2(E)^\vee = 0$ by (c) of Proposition 5, we have $r \geq 2$. Since $\text{Hom}(M, \mathcal{O}_S) = 0$, $r = 2$ is impossible. Hence we have $r = 3$. Since the image and M have the same determinant line bundle ($\simeq \mathcal{O}_S(l)$), the cokernel of φ is supported by a finite set of points.

The kernel of φ is a line bundle by the claim. It is isomorphic to $\mathcal{O}_S(-l)$. Hence we have the exact sequence

$$0 \rightarrow \mathcal{O}_S(-l) \rightarrow 4\mathcal{O}_S \xrightarrow{\varphi} M.$$

Since $\chi(\text{Coker } \varphi) = 3 < \chi(M)$, φ is not surjective. In fact, the cokernel is a skyscraper sheaf supported at a point. Tensoring $\mathcal{O}_S(l)$, we have

the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow 4\mathcal{O}_S(l) \xrightarrow{\varphi(l)} \bigwedge^2 E \longrightarrow \mathbf{C}(p) \longrightarrow 0.$$

$H^0(\varphi(l))$ is surjective since $h^0(4\mathcal{O}_S(l)) = 20$ and $h^0(\bigwedge^2 E) = 19$. But this contradicts (e) of Proposition 5. Q.E.D.

Proof of Theorem 2. Let $P = \langle \sigma_1, \sigma_2 \rangle$ be a general 2-dimensional subspace of $\bigwedge^2 \mathbf{C}^7$ and $X^6 \subset G(7, 3)$ the common zero locus of the two global sections of $\bigwedge^2 \mathcal{E}$ corresponding to σ_1 and σ_2 . A point q of $\mathbf{P}_*(\bigwedge^3 \mathbf{C}^{7,\vee} / \overline{P \wedge P})$ determines a global section of $\bigwedge^3 \mathcal{F}|_X$. We denote the zero locus by $S_q \subset X^6$.

$$\begin{array}{ccccc} S_q & \subset & X^6 & \subset & G(7, 3) \\ \cap & & \cap & & \cap \\ \mathbf{P}^{13} & \subset & \mathbf{P}^{20} & \subset & \mathbf{P}^{34} \end{array}$$

The restriction of \mathcal{E} to S_q is semi-rigid by (g) of Proposition 5. Let $\Xi^{31} \subset \mathbf{P}_*(\bigwedge^3 \mathbf{C}^{7,\vee} / \overline{P \wedge P})$ be the open subset consisting of points q such that S_q is a K3 surface and the restriction $\mathcal{E}|_{S_q}$ is stable with respect to h .

Lemma 8. Ξ^{31} is not empty.

Proof. Let (S, h) be as in Lemma 7 and put $E = \mathcal{E}|_S$. Then, by Proposition 7, $\text{Pic } S$ is generated by h . Since $h^0(\mathcal{O}_S(h)) = 14 > h^0(E) = 7$, we have $\text{Hom}(\mathcal{O}_S(nh), E) = 0$ for every integer $n \geq 1/3$. Since $c_1(E) = h$ and since $\text{Hom}(E, \mathcal{O}_S(nh)) = 0$ for every integer $n \leq 1/3$, E is stable. Q.E.D.

The correspondence $q \mapsto \mathcal{E}|_{S_q}$ induces a morphism from a general fiber of $\Xi^{31}/G \cdots \rightarrow \mathcal{F}_{13}$ at $[S_q]$ to the moduli space $M_S(3, h, 4)$ of semi-rigid bundles. Conversely there exists a morphism from a non-empty open subset of $M_S(3, h, 4)$ to the fiber since a small deformation E' of $\mathcal{E}|_{S_q}$ gives an embedding of S_q into $G(7, 3)$ such that the image is a complete intersection with respect to \mathcal{V} .

Remark 3. By (f) of Proposition vanishing, $H^0(X^6, \bigwedge^3 \mathcal{F}|_X)$ is isomorphic to $\bigwedge^3 \mathbf{C}^{7,\vee} / \overline{P \wedge P}$. Hence the rational map ψ in (1) coincides with $\mathbf{P}_*(H^0(X^6, \bigwedge^3 \mathcal{F}|_X))/G \cdots \rightarrow \mathcal{F}_{13}$ induced by $s \mapsto (s)_0$.

§3. K3 surface of genus seven and twelve

We describe two cases $g = 7$ and 12 closely related with Theorems 1 and 2. The proofs are quite similar to the case $g = 18, 13$, respectively, and we omit them.

First a polarized K3 surface of genus 7 has the following description other than that in the orthogonal Grassmannian $O-G(5, 10)$:

Theorem 9. *A general polarized K3 surface (S, h) of genus 7 is a complete intersection with respect to the rank four homogeneous vector bundle $2\mathcal{O}_G(1) \oplus \mathcal{E}(1)$ in the 6-dimensional Grassmannian $G(5, 2)$.*

S is the common zero locus of two hyperplane sections H_1 and H_2 of $G(5, 2) \subset \mathbf{P}^9$ corresponding to $\sigma_1, \sigma_2 \in \wedge^2 \mathbf{C}^5$ and one global section s of $\mathcal{E}(1)$. The 2-dimensional subspace $P = \langle \sigma_1, \sigma_2 \rangle \subset \wedge^2 \mathbf{C}^5$ is uniquely determined by S and $X^4 = G(5, 2) \cap H_1 \cap H_2$ is a quintic del Pezzo fourfold. Let Q be the image of $\mathbf{C}^5 \otimes P$ by the natural linear map $\mathbf{C}^7 \otimes \wedge^2 \mathbf{C}^7 \rightarrow H^0(\mathcal{E}(1))$. Then Q is of dimension 10 and we obtain the natural rational map

$$(6) \quad \mathbf{P}_*(H^0(\mathcal{E}(1))/Q)/G^8 = \mathbf{P}_*(H^0(\mathcal{E}(1)|_X))/G^8 \cdots \rightarrow \mathcal{F}_7$$

as in the case $g = 13$, where G^8 is the general stabilizer group of the action $PGL(5) \curvearrowright G(2, \wedge^2 \mathbf{C}^5)$. $H^0(\mathcal{E}(1))$ is a 40-dimensional irreducible representation of $GL(5)$ by Theorem 3. The fiber of the map (6) at general (S, h) is a surface and birationally equivalent to the moduli K3 surface $M_S(2, h, 3)$ of semi-rigid rank two vector bundles with $c_1 = h$ and $\chi = 2 + 3$.

Secondly, in the 12-dimensional Grassmannian $G(7, 3)$, there is another type of K3 complete intersection other than Theorem 1.

Theorem 10. *A general member $(S, h) \in \mathcal{F}_{12}$ is a complete intersection with respect to $\mathcal{V}_{10} = 3\wedge^2 \mathcal{E} \oplus \mathcal{O}_G(1)$ in $G(7, 3)$.*

S is the common zero locus of the three global sections of $\wedge^2 \mathcal{E}$ corresponding to general bivectors $\sigma_1, \sigma_2, \sigma_3 \in \wedge^2 \mathbf{C}^7$. The 3-dimensional subspace $N = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \subset \wedge^2 \mathbf{C}^7$ is uniquely determined by S . The common zero locus X_N of the global sections of $\wedge^2 \mathcal{E}$ corresponding to N is a Fano threefold and is embedded into \mathbf{P}^{13} anti-canonically. X_N 's are parameterized by an open set Ξ^6 of the orbit space $G(3, \wedge^2 \mathbf{C}^7)/PGL(7)$. See [5] for other descriptions of X_N 's and their moduli spaces. The moduli space \mathcal{F}_{12} is birationally equivalent to a \mathbf{P}^{13} -bundle over this Ξ^6 .

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Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
Japan
E-mail address: mukai@kurims.kyoto-u.ac.jp