Algebraic Geometry and Commutative Algebra
in Honor of Masayoshi NAGATA
pp. 357–377 (1987)

Curves, K3 Surfaces and Fano 3-folds of Genus ≤ 10

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A pair (S, L) of a K3 surface S and a pseudo-ample line bundle L on S with $(L^2) = 2g - 2$ is called a (polarized) K3 surface of genus g. Over the complex number field, the moduli space $\mathcal{F}_g$ of those (S, L)'s is irreducible by the Torelli type theorem for K3 surfaces [12]. If L is very ample, the image S_{2g-2} of $\Phi_{|L|}$ is a surface of degree $2g - 2$ in $\mathbf{P}^g$ and called the projective model of (S, L), [13]. If $g = 3, 4, 5$ and (S, L) is general, then the projective model is a complete intersection of $g - 2$ hypersurfaces in $\mathbf{P}^g$. This fact enables us to give an explicit description of the birational type of $\mathcal{F}_g$ for $g \leq 5$. But the projective model is no more complete intersection in $\mathbf{P}^g$ when $g \geq 6$. In this article, we shall show that a general K3 surface of genus $6 \leq g \leq 10$ is still a complete intersection in a certain homogeneous space and apply this to the discription of birational type of $\mathcal{F}_g$ for $g \leq 10$ and the study of curves and Fano 3-folds. The homogeneous space X is the quotient of a simply connected semi-simple complex Lie group G by a maximal parabolic subgroup P. For the positive generator $\mathcal{O}_X(1)$ of Pic$X \cong \mathbf{Z}$, the natural map $X \to \mathbf{P}(H^0(X, \mathcal{O}_X(1))$ is a G-equivariant embedding and the image coincides with the G-orbit $G \cdot \bar{v}$, where v is a highest weight vector of the irreducible representation $H^0(X, \mathcal{O}_X(1))^\vee$ of G. For each $6 \leq g \leq 10$, G and the representation $U = H^0(X, \mathcal{O}_X(1))$ are given as follows:

(0.1)

g	6	7	8	9	10
G	SL(5)	Spin(10)	SL(6)	Sp(3)	exceptional of type G_2
$\dim G$	24	45	35	21	14
U	$\wedge^2 V^5$	half spinor representation	$\wedge^2 V^6$	$\wedge^3 V^6 / \sigma \wedge V^6$	adjoint representation
$\dim U$	10	16	15	14	14
$\dim X$	6	10	8	6	5

where V^i denotes an i-dimensional vector space and $\sigma \in \wedge^2 V^6$ is a non-degenerate 2-vector of V^6.

*Partically supported by SFB 40 Theoretische Mathematik at Bonn and Educational Projects for Japanese Mathematical Scientists.

Received April 7, 1987.

In the case $7 \leq g \leq 10$, dim U is equal to $g+n-1$, $n = \dim X$. X is of degree $2g-2$ in $\mathbf{P}(U) \cong \mathbf{P}^{g+n-2}$ and the anticanonical (or 1st Chern) class of X is $n-2$ times hyperplane section (cf. (1.5)). Hence a smooth complete intersection of $X = X_{2g-2}$ and $n-2$ hyperplanes is a K3 surface of genus g. (This has been known classically in the case $g = 8$ and is first observed by C. Borcea [1] in the case $g = 10$.)

Theorem 0.2. *If two K3 surfaces S and S' are intersections of X_{2g-2} ($7 \leq g \leq 10$) and g-dimensional linear subspaces P and P', respectively, and if $S \subset P$ and $S' \subset P'$ are projectively equivalent, then P and P' are equivalent under the action of $\bar{G}$ on $\mathbf{P}(U)$, where $\bar{G}$ is the quotient of G by its center.*

By the theorem there exists a nonempty open subset Ξ of the Grassmann variety $G(n-2, U)$ of $n-2$ dimensional subspaces of U such that the natural morphism $\Xi/\bar{G} \to \mathcal{F}_g$ is injective. For each $7 \leq g \leq 10$, it is easily checked that $\dim \Xi/\bar{G} = 19 = \dim \mathcal{F}_g$. Hence the morphism is birational.

Corollary 0.3. *The generic K3 surface of genus $7 \leq g \leq 10$ is a complete intersection of $X_{2g-2} \subset \mathbf{P}(U)$ and a g-dimensional linear subspace in a unique way up to the action $\bar{G}$ on $\mathbf{P}(U)$. In particular, the moduli space $\mathcal{F}_g$ is birationally equivalent to the orbit space $G(n-2, U)/\bar{G}$.*

In the case $g = 6$, the generic K3 surface is a complete intersection of X, a linear subspace of dimension 6 and a quadratic hypersurface in $\mathbf{P}(U) \cong \mathbf{P}^9$. We have a similar result on the uniqueness of this expression of the K3 surface (see (4.1)). In the proof of these results, special vector bundles, instead of line bundles in the case $g \leq 5$, play an essential role. For instance, the generic K3 surface (S, L) of genus 10 has a unique (up to isomorphism) stable rank two vector bundle with $c_1(E) = c_1(L)$ and $c_2(E) = 6$ on it and the embedding of S into $X = G/P$ is uniquely determined by this vector bundle E.

The following is the table of the birational type of $\mathcal{F}_g$ for $g \leq 10$:

(0.4)

genus	2	3	4
birational type	$\mathbf{P}(S^6U^3)/\mathrm{PGL}(3)$	$\mathbf{P}(S^4U^4)/\mathrm{PGL}(4)$	$\mathbf{P}(U^{30})/\mathrm{SO}(5)$

5	6	7
$G(3, S^2U^6)/\mathrm{PGL}(6)$	$(U^{13} \oplus U^9)/\mathrm{PGL}(2)$	$G(8, U^{16})/\mathrm{PSO}(10)$

8	9	10
$G(6, \wedge^2 V^6)/\mathrm{PGL}(6)$	$G(4, U^{14})/\mathrm{PSp}(3)$	$G(3, \mathfrak{g})/G_2$

where U^d is a d-dimensional irreducible representation of the universal covering group.

Corollary 0.5. *$\mathcal{F}_g$ is unirational for every $g \leq 10$.*

By [5], there exists a Fano 3-fold V with the property Pic$V \cong \mathbf{Z}(-K_V)$ and $(-K_V)^3 = 22$. The moduli space of these Fano 3-folds are unirational by their description in [5]. The generic K3 surface of genus 12 is an anticanonical divisor of V and hence $\mathcal{F}_{12}$ is also unirational.

Problem 0.6. Describe the birational types, *e.g.*, the Kodaira dimensions, of the 19-dimensional varieties $\mathcal{F}_g$ for $g \gg 0$. Are they of general type?

If (S, L) is a K3 surface of genus g, then every smooth member of $|L|$ is a curve of genus g. Conversely if C is a smooth curve of genus $g \geq 2$ on a K3 surface, then $\mathcal{O}_S(C)$ is pseudo-ample and $(S, \mathcal{O}_S(C))$ is a K3 surface of the same genus as C. In the case $g \leq 9$, the generic curve lies on a K3 surface, that is , the natural rational map

$$\phi_g : \mathcal{P}_g = \bigcup_{(S,L)\in\mathcal{F}_g} |L| \dashrightarrow \mathcal{M}_g = (\text{the moduli space of curves of genus } g)$$

is generically surjective (§6). The inequality $\dim \mathcal{M}_g \leq \dim \mathcal{P}_g = 19 + g$ holds if and only if $g \leq 11$ and ψ_{11} is generically surjective ([10]). But in spite of $\dim \mathcal{M}_{10} = 27 < \dim \mathcal{P}_{10} = 29$, we have

Theorem 0.7. *The generic curve of genus* 10 *cannot lie on a K3 surface.*

Proof. Let $\mathcal{F}'_{10}$ (resp. $\mathcal{M}'_{10}$) be the subset of $\mathcal{F}_{10}$ (resp. $\mathcal{M}_{10}$) consisting of K3 surfaces (resp. curves) of genus 10 obtained as a complete intersection in the homogeneous space $X^5_{18} \subset \mathbf{P}(\mathfrak{g})$. $\mathcal{M}'_{10}$ has a dominant morphism from a Zariski open subset U of $G(4, \mathfrak{g})/\tilde{G}$. Since the automorphism of a curve of genus ≥ 2 is finite, the stabilizer group is finite for every 4-dimensional subspace of $\mathfrak{g}$ which gives a smooth curve of genus 10. Hence we have $\dim \mathcal{M}'_{10} \leq \dim U = \dim G(4, \mathfrak{g}) - \dim G = 26 < \dim \mathcal{M}_{10}$. On the other hand $\mathcal{F}'_{10}$ contains a dense open subset of $\mathcal{F}_{10}$ by Theorem 0.2. Hence the image of ψ_{10} is contained in the closure of $\mathcal{M}'_{10} = \psi_{10}(\mathcal{F}'_{10})$ and ψ_{10} is not generically surjective. *q.e.d.*

Remark 0.8. Every curve of genus 10 has g^4_{12}, a 4-dimensional linear system of degree 12. If C is a general linear section of the homogeneous space $X_{18} \subset \mathbf{P}^{13}$, then every g^4_{12} of C embeds C into a quadric hypersurface in $\mathbf{P}^4$. But if C is the generic curve of genus 10, then the image $C_{12} \subset \mathbf{P}^4$ embedded by any g^4_{12} is not contained in any quadratic hypersurface. This fact gives an alternate proof of the theorem.

In the case $7 \leq g \leq 10$, a Fano 3-fold $V_{2g-2} \subset \mathbf{P}^{g+1}$ is obtained as a complete intersection of the homogeneous space X^n_{2g-2} and a linear subspace of codimension $n-3$ in $\mathbf{P}(U) = \mathbf{P}^{n+g-2}$. By the Lefschetz theorem, the Fano 3-fold $V = V_{2g-2}$ has the property $\mathrm{Pic}V \cong \mathbf{Z}(-K_V)$. The existence of such V has been known classically but was shown by totally different construction ([6]). Theorem 0.2 holds for Fano 3-folds, too.

Theorem 0.9. *Let V_{2g-2} and V'_{2g-2} ($7 \leq g \leq 10$) be two Fano 3-folds which are complete intersections of the homogeneous space $X^n_{2g-2} \subset \mathbf{P}^{n+g-2}$ and linear subspaces of codimension $n-3$. If V_{2g-2} and V'_{2g-2} are isomorphic to each other, then they are equivalent under the action of $\tilde{G}$.*

We note that, by [1], the families of Fano 3-folds in the theorem is locally complete in the sense of [7].

The original version of this article was written during the author's stay at the Max-Planck Institüt für Mathematik in Bonn 1982 and at the Mathematics Institute in University of Warwick 1982-3. He expresses his hearty thanks to both institutions for their hospitality. He also thanks Mrs. Kozaki for nice typing into LaTeX.

Conventions. Varieties and vector spaces are considered over the complex number field $\mathbf{C}$. For a vector space or a vector bundle E, its dual is denoted by $E^\vee$. For a vector space V, $G(r, V)$ (resp. $G(V, r)$) is the Grassmann variety of r-dimensional subspaces (resp. quotient spaces) of V. $G(1, V)$ and $G(V, 1)$ are denoted by $\mathbf{P}_*(V)$ and $\mathbf{P}(V)$, respectively.

§1. Preliminary

We study some properties of the Cayley algebra $\mathcal{C}$ over $\mathbf{C}$. $\mathcal{C}$ is an algebra over $\mathbf{C}$ with a unit 1 and generated by 7 elements e_i, $i \in \mathbf{Z}/7\mathbf{Z}$. The multiplication is given by

$$(1.1)\qquad \begin{array}{l} {e_i}^2 = -1 \text{ and } e_i e_{i+a} = -e_{i+a} e_i = e_{i+3a} \\ \text{for every } i \in \mathbf{Z}/7\mathbf{Z} \text{ and } a = 1, 2, 4. \end{array}$$

The algebra $\mathcal{C}$ is not associative but alternative, *i.e.*, $x(xy) = x^2 y$ and $(xy)y = xy^2$ hold for every $x, y \in \mathcal{C}$. Let $\mathcal{C}_0$ be the 7-dimensional subspace of $\mathcal{C}$ generated by $e_i, i \in \mathbf{Z}/7\mathbf{Z}$ and U the subspace of $\mathcal{C}_0$ spanned by $\alpha = e_3 + \sqrt{-1}e_5$ and $\beta = e_6 - \sqrt{-1}e_7$. It is easily checked that $\alpha^2 = \beta^2 = \alpha\beta = \beta\alpha = 0$, *i.e.*, U is totally isotropic with respect to the multiplication of $\mathcal{C}$. Moreover, U is maximally totally isotropic with respect to the multiplication of $\mathcal{C}$, *i.e.*, if $xU = 0$ or $Ux = 0$, then x belongs to U. Let q be the quadratic form $q(x) = x^2$ on $\mathcal{C}_0$ and b the associated symmetric bilinear form. $b(x, y)$ is equal to $xy + yx$ for every x and $y \in \mathcal{C}_0$. Let V be the subspace of $\mathcal{C}_0$ of vectors orthogonal to U with respect to q (or b). Since U is totally isotropic with respect to q, V contains U and the quotient V/U carries the quadratic form $\bar{q}$.

Lemma 1.2. *$x'(xy) = b(x,y)x' - b(x',y)x + y(x'x)$ for every x, x' and $y \in \mathcal{C}_0$.*

Proof. By the alternativity of $\mathcal{C}$, we have $u(vw) + v(uw) = (uv + vu)w$. Hence, if u and v belongs to $\mathcal{C}_0$, then we have $u(vw) + v(uw) = b(u,v)w$. So we have

$$\begin{aligned} x'(xy) &= x'(b(x,y) - yx) = b(x,y)x' - x'(yx) \\ &= b(x,y)x' - (b(x',y)x - y(x'x)) \\ &= b(x,y)x' - b(x',y)x + y(x'x). \end{aligned}$$

q.e.d.

If $x \in U$ and $y \in V$, then $U(xy) = 0$ by the above lemma and hence xy belongs to U. Hence the right multiplication homomorphism $R(y)$, $x \mapsto xy$, by $y \in V$ maps U into itself. Since $R(x)$ is zero on U if and only if $x \in U$, R gives an injective homomorphism $\bar{R} : V/U \to \mathrm{End}(U)$.

Proposition 1.3. (1) *$\bar{R}(\bar{x})^2 = \bar{q}(\bar{x}) \cdot \mathrm{id}$ for every $\bar{x} \in V/U$, and*

(2) *$\bar{R}$ is an isomorphism onto $sl(U)$, the vector space consisting of trace zero endomorphisms of U.*

Proof. (1) follows immediately from the alternativity of $\mathcal{C}$. It is easy to check the following fact: if r is an endomorphism of a 2-dimensional vector space and if r^2 is a constant multiplication, then either r itself is a constant multiplication or the trace of r is equal to zero. Hence by (1), $\bar{R}(\bar{x})$ is a constant multiplication or belongs to $sl(U)$, for every $\bar{x} \in V/U$. Therefore, $\bar{R}(V/U)$ is contained in the 1-dimensional vector space consisting of constant multiplications of U or contained in the 3-dimensional vector space $sl(U)$. Since the quadratic form q is nondegenerate on V/U, the former is impossible and $\bar{R}(V/U)$ coincides with $sl(U)$. *q.e.d.*

Let G be the automorphism group of the Cayley algebra $\mathcal{C}$. It is known that G is a simple algebraic group of type G_2. The automorphisms which map U onto itself form a maximal parabolic subgroup P of G. The subspace spanned by e_1, e_2 and e_4 (resp. by $e_3 - \sqrt{-1}\, e_5$ and $e_6 + \sqrt{-1}\, e_7$) can be identified with $sl(U)$ (resp. $U^\vee$) by $\bar{R}$ (resp. b). $\mathcal{C}$ is isomorphic to $\mathbf{C} \oplus U \oplus sl(U) \oplus U^\vee$ and if $f \in \mathrm{GL}(U)$, then $1 \oplus f \oplus \mathrm{ad}(f) \oplus {}^t f$ is an automorphism of the Cayley algebra $\mathcal{C}$. Hence the maximal parabolic subgroup P contains $\mathrm{GL}(U)$ and $X = G/P$ can be identified with the set of 2-dimensional subspaces of $\mathcal{C}_0$ which are equivalent to U under the action of $G = \mathrm{Aut}\, \mathcal{C}$.

Let $\mathcal{U}$ be the maximally totally isotropic universal subbundle of $\mathcal{C}_0 \otimes \mathcal{O}_X$: the fibre $\mathcal{U}_x \subset \mathcal{C}_0$ at x is the 2-dimensional subspace corresponding to $x \in X$. Let $\mathcal{V}$ be the subsheaf of $\mathcal{C}_0 \otimes \mathcal{O}_X$ consisting of the germs of sections which are orthogonal to $\mathcal{U}$ with respect to the bilinear form $b \otimes 1$ on $\mathcal{C}_0 \otimes \mathcal{O}_X$. $\mathcal{V}$ is a rank 5 subbundle of $\mathcal{C}_0 \otimes \mathcal{O}_X$ and contains $\mathcal{U}$ as a subbundle. The quotient bundle

$(\mathcal{C}_0 \otimes \mathcal{O}_X)/\mathcal{V}$ is isomorphic to $\mathcal{U}^\vee$ by $b \otimes 1$ and $\mathcal{V}/\mathcal{U}$ has a quadratic form $\overline{q \otimes 1}$ induced by $q \otimes 1$ on $\mathcal{C}_0 \otimes \mathcal{O}_X$. By Proposition 1.3, we have

Proposition 1.4. *The right multiplication induces an isomorphism $\bar{R}$ from $\mathcal{V}/\mathcal{U}$ onto the vector bundle $sl(\mathcal{U})$ of trace zero endomorphisms of $\mathcal{U}$ and $\bar{R}(\bar{x})^2$ is equal to $(\overline{q \otimes 1})(\bar{x}) \cdot \mathrm{id}$ for every $\bar{x} \in \mathcal{V}/\mathcal{U}$.*

Next we shall compute the anticanonical class of X and the degree of $\mathcal{O}_X(1)$, the ample generator of Pic$X$, and show some vanishings of the cohomology groups of homogeneous vector bundles $\mathcal{U}(i)$ and $(S^2\mathcal{U})(i)$ etc.

Let G be a simply connected semi-simple algebraic group and P a maximal parabolic subgroup of G. Fixing a Borel subgroup B in P, the Lie algebra $\mathfrak{g}$ of G is the direct sum of $\mathfrak{b}$ and 1-dimensional eigenspaces $\mathfrak{g}^\beta$, where β runs over all negative roots. If we choose a suitable root basis Δ, then there exists a positive root $\alpha \in \Delta$ such that $\mathfrak{p}$ is equal to the direct sum of $\bigoplus \mathfrak{g}^\gamma$ and $\mathfrak{b}$, where γ runs over all positive roots which are linear combinations of the roots in $\Delta \setminus \{\alpha\}$ with nonnegative coefficients. A positive root β is said to be *complementary* if $\mathfrak{g}^\beta \cap \mathfrak{p} = 0$ or equivalently if β cannot be expressed as a linear combination of the roots in $\Delta \setminus \{\alpha\}$ with nonnegative coefficients.

Proposition 1.5. (Borel-Hirzebruch [2]) *Let G, P, Δ and α be as above and L the positive generator of* Pic(G/P). *Then we have*

(1) *the quotient $\mathfrak{g}/\mathfrak{p}$ is isomorphic to $\bigoplus_{\beta \in R_P} \mathfrak{g}^\beta$, where R_P is the set of positive complementary roots. In particular,* $\dim(G/P)$ *is equal to the cardinality n of R_P,*

(2) $(L^n) = n! \prod_{\beta \in R_P} \frac{(\beta, w)}{(\beta, \rho)}$, *where w is the fundamental weight corresponding to α (or L) and ρ is a half of the sum of all positive roots, and*

(3) *the sum of all $\beta \in R_P$ is r times ρ for some positive integer r and $c_1(G/P)$ (or the anticanonical class of G/P) is equal to r times $c_1(L)$.*

A homogeneous vector bundle on G/P is obtained from a representation of P and hence from that of reductive part G_0 of P. Note that the weight spaces of G and G_0 are naturally identified.

Theorem 1.6. (Bott [3]) *Let E be a homogeneous vector bundle over G/P induced by an irreducible representation of the reductive part of P. Let γ be the highest weight of the representation and ρ a half of the sum of all positive roots of G. Then we have*

(1) *if $(\gamma + \rho, \beta) = 0$ for a positive root β, then $H^i(G/P, E)$ vanishes for every i, and*

(2) *let i_0 be the number of positive roots β with $(\gamma + \rho, \beta)$ negative (i_0 is called the index of E). Then $H^i(G/P, E) = 0$ for all i except for i_0 and $H^{i_0}(G/P, E)$ is an irreducible G-module.*

Returning to our first situation, our variety X is the quotient of the exceptional Lie group G of type G_2 by a maximal parabolic subgroup P. The root system G_2 has two root basis α_1 and α_2 with different lengths and the root α corresponding to P in the above manner is the longer one, say α_2. The line bundle $L = \mathcal{O}_X(1)$ and the vector bundle $\mathcal{U}^\vee(1)$ on X come from the representation with the highest weights $w_1 = 3\alpha_1 + 2\alpha_2$ and $w_2 = 2\alpha_1 + \alpha_2$, respectively, which are the fundamental weights of G. Since $\mathcal{U}$ is of rank 2 and $\wedge^2\mathcal{U} \cong \mathcal{O}_X(1)$, $\mathcal{U}^\vee$ is isomorphic to $\mathcal{U}(1)$. ρ is equal to $w_1 + w_2$ and the inner products of ρ, w_1, w_2 and the 6 positive roots are as follows:

	α_1	$3\alpha_1+\alpha_2$	$2\alpha_1+\alpha_2$	$3\alpha_1+2\alpha_2$	$\alpha_1+\alpha_2$	α_2
ρ	1	6	5	9	4	3
w_1	0	3	3	6	3	3
w_2	1	3	2	3	1	0

By (1.5), X has dimension 5, $c_1(X) = 3c_1(L)$ and has degree

$$(L^5) = 5!\frac{3\cdot 3\cdot 6\cdot 3\cdot 3}{6\cdot 5\cdot 9\cdot 4\cdot 3} = 18$$

in $\mathbf{P}^{13}$. The homogeneous vector bundles $(S^m\mathcal{U})(n)$ comes from the irreducible representation with the highest weight $mw_1 + (n-m)w_2$. Applying (1.6), we have

Proposition 1.7. *The cohomology groups of* $\mathcal{U}(n), (S^2\mathcal{U})(n)$ *and* $(S^3\mathcal{U})(n)$ *are zero except for the following cases:*

(1) $H^0(X,\mathcal{U}(n))$ *for* $n \geq 1, H^0(X,(S^2\mathcal{U})(n))$ *for* $n \geq 2$ *and* $H^0(X,(S^3\mathcal{U})(n))$ *for* $n \geq 3$,

(2) $H^1(X,(S^3\mathcal{U})(1))$ *and* $H^4(X,(S^3\mathcal{U})(-1))$, *and*

(3) $H^5(X,\mathcal{U}(n)), H^5(X,(S^2\mathcal{U})(n))$ *and* $H^5(X,(S^3\mathcal{U})(n))$ *for* $n \leq -3$.

Let S be a smooth K3 surface which is a complete intersection of 3 members of $|\mathcal{O}_X(1)|$. By using the Koszul complex

(1.8) $0 \longrightarrow \mathcal{O}_X(-3) \longrightarrow \mathcal{O}_X(-2)^{\oplus 3} \longrightarrow \mathcal{O}_X(-1)^{\oplus 3} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_S \longrightarrow 0,$

we have

Lemma 1.9. *If* E *is a vector bundle on* X *and if* $H^{i+j}(X, E(-j)) = 0$ *for every* $0 \leq j \leq 3$, *then* $H^i(S, E|_S) = 0$.

Since $\mathcal{U}$ is of rank 2, $sl(\mathcal{U})$ is isomorphic to $S^2\mathcal{U} \otimes (\det \mathcal{U})^{-1} \cong (S^2\mathcal{U})(1)$. By Proposition 1.7 and Lemma 1.9, we have

Proposition 1.10. *Let S be as above. Then $H^i(S, sl(\mathcal{U})|_S)$ vanishes for every i, $H^1(S, (sl\mathcal{U})(n)|_S)$ vanishes for every n and $\mathcal{U}|_S$ or $(S^3\mathcal{U})(1)|_S$ has no nonzero global sections.*

§2. Proof of Theorem 0.2 in the case $g = 10$

Let A be a 3-dimensional subspace of $H^0(X, L)$ and S_A the intersection of $X = X_{18}$ and the linear subspace $\mathbf{P}(H^0(X, L)/A)$ of $\mathbf{P}(H^0(X, L))$. Let L_A and U_A be the restrictions of L and $\mathcal{U}$ to S_A, respectively. Let Ξ be the subset of the Grassmann variety $G(3, H^0(X, L))$ consisting of A's such that S_A are smooth K3 surfaces and that the vector bundles U_A are stable with respect to the ample line bundles L_A.

Proposition 2.1. *Ξ is a nonempty open subset of $G(3, H^0(X, L))$.*

Proof. U_A is a rank 2 bundle and $\det U_A \cong L_A^{-1}$. By Moishezon's theorem [9], $\mathrm{Pic}\, S_A$ is generated by L_A if A is general. Since $H^0(S_A, U_A) = 0$ by Proposition 1.10, U_A is stable if A is general. Since the stableness is an open condition [8], we have our proposition. *q.e.d.*

In this section we shall prove the following:

(2.2) *If two 3-dimensional subspaces A and B belong to Ξ and if the polarized K3 surfaces (S_A, L_A) and (S_B, L_B) are isomorphic to each other, then S_A and S_B, and hence A and B, are equivalent under the action of G.*

Let $\varphi : S_A \xrightarrow{\sim} S_B$ be an isomorphism such that $\varphi^* L_B \cong L_A$.

Step I. There is an isomorphism $\beta : U_A \xrightarrow{\sim} \varphi^* U_B$.

Proof. Since $c_1(U_A) = -c_1(L_A)$ and $c_1(U_B) = -c_1(L_B)$, the first Chern classes of U_A and $\varphi^* U_B$ are same. Since (S_B, U_B) is a deformation of (S_A, U_A), U_B and U_A have the same second Chern number. Hence the two vector bundles $\mathcal{H}om_{\mathcal{O}_S}(U_A, \varphi^* U_B)$ and $\mathcal{E}nd_{\mathcal{O}_S}(U_A)$ have the same first Chern class and the same second Chern number. Therefore, by the Riemann-Roch theorem and Proposition 1.10, we have

$$\begin{aligned} \chi(\mathcal{H}om_{\mathcal{O}_S}(U_A, \varphi^* U_B)) &= \chi(\mathcal{E}nd_{\mathcal{O}_S}(U_A)) \\ &= \chi(\mathcal{O}_{S_A}) + \chi(sl(U_A)) = 2. \end{aligned}$$

By the Serre duality, we have

$$\begin{aligned} \dim \mathrm{Hom}_{\mathcal{O}_S}(U_A, \varphi^* U_B) &+ \dim \mathrm{Hom}_{\mathcal{O}_S}(\varphi^* U_B, U_A) \\ &\geq \chi(\mathcal{H}om_{\mathcal{O}_S}(U_A, \varphi^* U_B)) = 2. \end{aligned}$$

Hence there is a nonzero homomorphism from U_A to $\varphi^* U_B$ or vice versa. Since U_A and $\varphi^* U_B$ are stable vector bundles and have the same slope, the nonzero homomorphism is an isomorphism. *q.e.d.*

Step II. There is an isomorphism $\gamma : \mathcal{C}_0 \xrightarrow{\sim} \mathcal{C}_0$ (as C-vector spaces) such that the following diagram is commutative:

$$\begin{array}{ccc} U_A & \xrightarrow{\beta} & \varphi^* U_B \\ \cap & & \cap \\ \mathcal{C}_0 \otimes \mathcal{O}_{S_A} & \xrightarrow{\gamma\otimes 1} & \mathcal{C}_0 \otimes \mathcal{O}_{S_A} = \varphi^*(\mathcal{C}_0 \otimes \mathcal{O}_{S_B}) \end{array}$$

Proof. Let γ_0 be the dual map of

$$\mathrm{Hom}(\beta, \mathcal{O}_{S_A}) : \mathrm{Hom}_{\mathcal{O}_S}(\varphi^* U_B, \mathcal{O}_{S_A}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_S}(U_A, \mathcal{O}_{S_A}).$$

Claim: The inclusion $U_A \subset \mathcal{C}_0 \otimes \mathcal{O}_{S_A}$ induces an isomorphism $\mathrm{Hom}(\mathcal{C}_0, \mathbf{C}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S}(U_A, \mathcal{O}_{S_A})$.

Let $\mathcal{K}$ be the dual of the quotient bundle $(\mathcal{C}_0 \otimes \mathcal{O}_X)/\mathcal{U}$ on X. The natural map from $\mathrm{Hom}(\mathcal{C}_0, \mathbf{C})$ to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{U}, \mathcal{O}_X)$ is an isomorphism because both are irreducible G-modules. Hence both $H^0(X, \mathcal{K})$ and $H^1(X, \mathcal{K})$ are zero. By the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{C}_0^\vee \otimes \mathcal{O}_X \xrightarrow{\alpha} \mathcal{U}^\vee \longrightarrow 0$$

and Proposition 1.7, we have $H^i(X, \mathcal{K}(-i)) = H^{i+1}(X, \mathcal{K}(-i)) = 0$ for $i = 1,2$ and 3. Hence by Lemma 1.9, both $H^0(S, \mathcal{K}|_S)$ and $H^1(S, \mathcal{K}|_S)$ are zero and we have our claim.

By the claim and by applying the claim to $\varphi^* U_B \subset \varphi^*(\mathcal{C}_0 \otimes \mathcal{O}_{S_B})$, we have a homomorphism $\gamma : \mathcal{C}_0 \longrightarrow \mathcal{C}_0$ such that the following diagram

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{\gamma} & \mathcal{C}_0 \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Hom}_{\mathcal{O}_S}(U_A, \mathcal{O}_{S_A})^\vee & \xrightarrow{\gamma_0} & \mathrm{Hom}_{\mathcal{O}_S}(\varphi^* U_B, \mathcal{O}_{S_A})^\vee \end{array}$$

is commutative. Since β is an isomorphism, γ_0 and γ are isomorphisms and γ enjoys our requirement. *q.e.d.*

Step III. There is an isomorphism $\gamma : \mathcal{C}_0 \xrightarrow{\sim} \mathcal{C}_0$ (as C-vector spaces) such that $(\gamma \otimes 1)(U_A) = \varphi^* U_B \subset \mathcal{C}_0 \otimes \mathcal{O}_X$ and $x^2 = \gamma(x)^2$ for every $x \in \mathcal{C}_0$.

Proof. Take an isomorphism γ which satisfies the requirement of Step II. Put $q(x) = x^2$ and $q'(x) = \gamma(x)^2$. Then q and q' are quadratic forms on $\mathcal{C}_0$ and both $q \otimes 1$ and $q' \otimes 1$ are identically zero on U_A. Hence replacing γ by some multiple by a nonzero constant if necessary, we have our assertion by the following:

Claim: The quadratic forms Q on $\mathcal{C}_0$ such that $(Q \otimes 1)|_{U_A} = 0$ form at most one dimensional vector space.

Let $\mathcal{N}$ be the kernel of the homomorphism $S^2\alpha : S^2\mathcal{C}_0 \otimes \mathcal{O}_X \longrightarrow S^2\mathcal{U}^\vee$. Since $S^2\mathcal{C}_0$ is a sum of two irreducible G-modules of dimension 1 and 27 and since $H^0(S^2\alpha)$ is a homomorphism of G-modules, we have dim $H^0(X,\mathcal{N}) =$ dim $\mathrm{Ker} H^0(S^2\alpha) = 1$. By the exact sequence

$$H^{i-1}(X, S^2\mathcal{U}^\vee(-n)) \longrightarrow H^i(X,\mathcal{N}(-n)) \longrightarrow H^i(X, S^2\mathcal{C}_0 \otimes \mathcal{O}_X(-n))$$

and Proposition 1.7, $H^i(X,\mathcal{N}(-i))$ is zero for every $i = 1,2$ and 3. Hence by the Koszul complex (1.8), the restriction map $H^0(X,\mathcal{N}) \longrightarrow H^0(S,\mathcal{N}|_S)$ is surjective and we have dim $H^0(S,\mathcal{N}|_S) \leq$ dim $H^0(X,\mathcal{N}) = 1$, which shows our claim. *q.e.d.*

Step IV. There is an isomorphism $\gamma : \mathcal{C}_0 \xrightarrow{\sim} \mathcal{C}_0$ such that $(\gamma \otimes 1)(U_A) = \varphi^* U_B, x^2 = \gamma(x)^2$ for every $x \in \mathcal{C}_0$ and $(\gamma \otimes 1)(xy) = ((\gamma \otimes 1)(x))((\gamma \otimes 1)(y))$ for every $x \in U_A$ and $y \in V_A$.

Proof. Take an isomorphism γ which satisfies the requirements of Step III. Then $\gamma \otimes 1$ maps V_A onto $\varphi^* V_B \subset \mathcal{C}_0 \otimes \mathcal{O}_X$ and induces an isomorphism $\Gamma : V_A/U_A \longrightarrow \varphi^*(V_B/U_B)$ which is compatible with the quadratic forms on V_A/U_A and V_B/U_B. Let $r_A : V_A/U_A \longrightarrow sl(U_A)$ be the restriction of $\bar{R} : \mathcal{V}/\mathcal{U} \longrightarrow sl(\mathcal{U})$ to S_A. Consider the following diagram:

$$\begin{array}{ccc} V_A/U_A & \xrightarrow{r_A} & sl(U_A) \\ \Gamma\downarrow & & \downarrow \mathrm{ad}(\gamma\otimes 1) \\ \varphi^*(V_B/U_B) & \xrightarrow{\varphi^* r_B} & \varphi^* sl(U_B) \end{array}$$

The vector bundles $sl(U_A)$ and $sl(U_B)$ have the quadratic forms $f \mapsto (\mathrm{tr} f^2)/2$ and all the homomorphisms in the above diagram are isomorphisms compatible with the quadratic forms by Proposition 1.4. If g is an automorphism of $sl(U_A)$ and preserves the quadratic form, then g or $-g$ comes from an automorphism of U_A because $H^1(S, \mathbf{Z}/2\mathbf{Z}) = 0$. Since every endomorphism of U_A is a constant multiplication, g is equal to $\pm$id. Therefore, the above diagram is commutative up to sign. Hence, for γ or $-\gamma$, the above diagram is commutative. Since $xy = r_A(\bar{y})(x)$ for every $x \in U_A$ and $y \in V_A$, γ or $-\gamma$ satisfies our requirements, where $\bar{y} \in V_A/U_A$ is the image of $y \in V_A$. *q.e.d.*

We shall show that, for the isomorphism γ in Step IV, $\tilde{\gamma} = 1 \oplus \gamma : \mathcal{C}_0 \longrightarrow \mathcal{C}^0$ satisfies $\tilde{\gamma}(xy) = \tilde{\gamma}(x)\tilde{\gamma}(y)$ for every $x, y \in \mathcal{C}_0$. If $x, y \in \mathcal{C}_0$, then $xy + yx$ is equal to $b(x,y)$, where $b(x,y)$ is the inner product associated to the quadratic form q. Hence the *real* part of xy is equal to $b(x,y)/2$, that is, $xy - b(x,y)/2$ belongs to $\mathcal{C}_0$. Since γ preserves the quadratic form q, $\tilde{\gamma}(x,y)$ and $\tilde{\gamma}(x)\tilde{\gamma}(y)$ have the same *real* part, that is, their difference belongs to $\mathcal{C}_0$. Put $\delta(x,y) = \tilde{\gamma}(x,y) - \tilde{\gamma}(x)\tilde{\gamma}(y)$for every $x, y \in \mathcal{C}_0$. $\delta : \mathcal{C}_0 \otimes \mathcal{C}_0 \longrightarrow \mathcal{C}_0$ is skew-symmetric and $\delta \otimes 1$ is identically zero on $U_A \otimes V_A \subset \mathcal{C}_0 \otimes \mathcal{C}_0 \otimes \mathcal{O}_{S_A}$.

Step V. $\delta \otimes 1$ is identically zero on $V_A \otimes V_A \subset \mathcal{C}_0 \otimes \mathcal{C}_0 \otimes \mathcal{O}_{S_A}$.

Proof. Since $\delta \otimes 1$ is skew-symmetric and identically zero on $U_A \otimes V_A$, $\delta \otimes 1$ induces a skew-symmetric form $\bar{\delta}$ on V_A/U_A. Since V_A/U_A is isomorphic to $sl(U_A), \wedge^2(V_A/U_A)^\vee$ is also isomorphic to $sl(U_A)$ and has no nonzero global sections. Hence $\bar{\delta}$ is zero and $\delta \otimes 1$ is identically zero on $V_A \otimes V_A$. *q.e.d.*

Step VI. Every homomorphism f from V_A to U_A is zero.

Proof. Since V_A/U_A is isomorphic to $sl(U_A)$, there are no nonzero homomorphisms from V_A/U_A to $\mathcal{O}_{S_A}$. Hence V_A/U_A cannot be a subsheaf of $\mathcal{C}_0 \otimes \mathcal{O}_{S_A}$. Therefore, the exact sequence $0 \longrightarrow U_A \longrightarrow V_A \longrightarrow V_A/U_A \longrightarrow 0$ does not split. Hence the restriction $f|_{U_A} : U_A \longrightarrow U_A$ of f to U_A is not an isomorphism. Since every endomorphism of U_A is a constant multiplication, $f|_{U_A}$ is zero and f induces a homomorphism $\bar{f} : V_A/U_A \longrightarrow U_A$. Since $V_A/U_A \cong sl(U_A)$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_S}(V_A/U_A, U_A) &\cong H^0(S_A, sl(U_A) \otimes U_A) \\ &\cong H^0(S_A, U_A \oplus (S^3 U_A) \otimes L_A). \end{aligned}$$

Hence by Proposition 1.10, $\bar{f}$ is zero and f is also zero. *q.e.d.*

Step VII. δ is zero.

Proof. Let T be the cokernel of the natural injection $\wedge^2 V_A \longrightarrow \wedge^2 \mathcal{C}_0 \otimes \mathcal{O}_{S_A}$. Since $\delta \otimes 1$ belongs to $\mathrm{Hom}_{\mathcal{O}_S}(T, \mathcal{C}_0 \otimes \mathcal{O}_{S_A})$, it suffices to show that $\mathrm{Hom}_{\mathcal{O}_S}(T, \mathcal{O}_{S_A})$ is zero. There is an exact sequence

$$0 \longrightarrow V_A \otimes E_A \longrightarrow T \longrightarrow \bigwedge^2 E_A \longrightarrow 0,$$

where E_A is the quotient bundle $(\mathcal{C}_0 \otimes \mathcal{O}_{S_A})/V_A$ and isomorphic to $U_A^\vee$ by the bilinear form b associated to q. By Step VI, we have $\mathrm{Hom}_{\mathcal{O}_S}(V_A \otimes E_A, \mathcal{O}_{S_A}) \cong \mathrm{Hom}_{\mathcal{O}_S}(V_A, U_A) = 0$. Since $\wedge^2 E_A$ is an ample line bundle, $\mathrm{Hom}_{\mathcal{O}_S}(\wedge^2 E_A, \mathcal{O}_{S_A})$ is zero. Therefore, by the above exact sequence, $\mathrm{Hom}_{\mathcal{O}_S}(T, \mathcal{O}_{S_A})$ is zero. *q.e.d.*

By Step VII, $1 \oplus \gamma$ is an automorphism of the Cayley algebra $\mathcal{C}$. The automorphism of $X_{18} = G/P$ induced by $1 \oplus \gamma$ maps S_A onto S_B. Hence we have (2.2) and, in particular, Theorem 0.2.

§3. Generic K3 surfaces of genus 7,8, and 9

The proof of Theorem 0.2 in the case $g = 7, 8$, and 9 is very similar to and rather easier than the case $g = 10$ dealt in the previous sections. The $(24 - 2g)$-dimensional homogeneous spaces $X = X_{2g-2} \subset \mathbf{P}^{22-g}$ ($g = 7$,8 and 9) are also generalized Grassmann variety as in the case $g = 10$. In the case $g = 8$, $X_{14} \subset \mathbf{P}^{14}$ is the Grassmann variety $G(V, 2)$ of 2-dimensional quotient spaces of a 6-dimensional vector space V embedded into $\mathbf{P}(\wedge^2 V)$ by the Plücker coordinates. In the case $g = 9$, $X \subset \mathbf{P}^{13}$ is the Grassmann variety of 3-dimensional totally isotropic quotient spaces of a 6-dimensional vector space V with a nondegenerate skew-symmetric tensor $\sigma \in \wedge^2 V$, where a quotient $f : V \to V'$ is totally isotropic with respect to σ if $(f \otimes f)(\sigma)$ is zero in $V' \otimes V'$. The embedding $X_{16} \subset \mathbf{P}^{13}$ is the linear hull of the composite of the natural embedding $X \subset G(V, 3)$ and the Plücker embedding $G(V, 3) \subset \mathbf{P}(\wedge^3 V)$. In the case $g = 7$, $X \subset \mathbf{P}^{15}$ is a 10-dimensional spinor variety. Let V be a 10-dimensional vector space with a non-degenerate second symmetric tensor. The subset of $G(V, 5)$ consisting of 5-dimensional totally isotropic quotient spaces of V has exactly two connected components, one of which is our spinor variety X. The pull-back of the tautological line bundle $\mathcal{O}_{\mathbf{P}}(1)$ by the composite $X \hookrightarrow G(V, 5) \hookrightarrow \mathbf{P}(\wedge^5 V)$ is twice the positive generator L of $\operatorname{Pic} X$ and the vector space $H^0(X, L)$ is a half spinor representation of $\operatorname{Spin}(V)$, the universal covering groups of $\operatorname{SO}(V)$. In each case, X is a compact hermitian symmetric space and the anticanonical class of X is equal to $\dim X - 2$ times the positive generator L of $\operatorname{Pic} X$ (Proposition 1.5 and [2] §16). Moreover, by Proposition 1.5 and an easy computation, we have that the embedded variety $X \hookrightarrow \mathbf{P}(H^0(X, L))$ has degree $2g - 2$. Hence every smooth complete intersection of X and a linear subspace of codimension $n - 2$ (resp. $n - 3$, $n - 1$) is the projective (resp. canonical, anticanonical) model of a K3 surface (resp. curve, Fano 3-fold) of genus g.

Each homogeneous space $X = X_{2g-2}$ has a natural homogeneous vector bundle $\mathcal{E}$ on it. In the case $g = 8$, we have the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow V \otimes \mathcal{O}_X \xrightarrow{\alpha} \mathcal{E} \longrightarrow 0, \tag{3.1}$$

where $\mathcal{E}$ (resp. $\mathcal{F}$) is the universal quotient (resp. sub-) bundle and is of rank 2 (resp. 4). In the case $g = 7$ (resp. 9), we have the exact sequence

$$0 \longrightarrow \mathcal{E}^\vee \longrightarrow V \otimes \mathcal{O}_X \xrightarrow{\alpha} \mathcal{E} \longrightarrow 0, \tag{3.2}$$

where $\mathcal{E}$ is the universal maximally totally isotropic quotient bundle with respect to $\sigma \otimes 1 \in V \otimes V \otimes \mathcal{O}_X$ and is of rank 5 (resp. 3).

Theorem 0.2 is a consequence of the openness of the stability condition and the following:

Theorem 3.3. *Let S and S' be two K3 surfaces which are complete intersections of $X_{2g-2} \subset \mathbf{P}^{22-g}$ ($g = 7, 8$ and 9) and linear subspaces R and R', respectively. Then we have*

1. *if R is general, then the vector bundle $\mathcal{E}|_S$ is stable with repsect to $\mathcal{O}_S(1)$, the restriction of $L = \mathcal{O}_X(1)$ to S, and*
2. *if $\mathcal{E}|_S$ and $\mathcal{E}|_{S'}$ are stable with respect to $\mathcal{O}_S(1)$ and $\mathcal{O}_{S'}(1)$ and if $S \subset R$ and $S' \subset R'$ are projectively equivalent, then R and R' are equivalent under the action of G on X.*

For the proof we need the following property of the vector bundle $E = \mathcal{E}|_S$.

Proposition 3.4. *Let S be a complete intersection of $X = X_{2g-2} \subset \mathbf{P}^{22-g}$ and a g-dimensional linear subspace and E the restriction of $\mathcal{E}$ to S. Then we have*

(1) $H^i(S, sl(E)) = 0$ *for every* i,

(2) *the homomorphism* $H^0(\alpha) : V \to H^0(S, E)$ *is an isomorphism,*

(3) *in the case* $g = 7$ *(resp. 9), the kernel of the homomorphism* $H^0(S^2\alpha) : S^2V \to H^0(S, S^2E)$ *(resp.* $H^0(\wedge^2\alpha) : \wedge^2V \to H^0(S, \wedge^2E)$*) is 1-dimensional and generated by* $\sigma \otimes 1$, *and*

(4) *in the case* $g = 7$ *(resp. 8, resp. 9),* $E(-1), (\wedge^2E)(-1)$, $(\wedge^3E)(-2)$ *or* $(\wedge^4E)(-2)$ *(resp.* $E(-1)$, *resp.* $E(-1)$ *or* $(\wedge^2E)(-1)$*) has no nonzero global sections.*

We prove the proposition in the case $g = 7$. The other cases are similar. According to [4], we take $\alpha_i = e_i - e_{i+1}$, $1 \le i \le 4$, and $\alpha_5 = e_4 + e_5$ as a root basis of SO(10). The positive roots are $e_i \pm e_j, i < j$ and the conjugacy class of the maximal parabolic subgroup P corresponds to α_5 (or α_4). The homogeneous vector bundles $\mathcal{O}_X(1), \wedge^i\mathcal{E}, sl(\mathcal{E})$ and $S^2\mathcal{E}$ are induced by the irreducible representations of the reductive part of P with the highest weights $\frac{1}{2}(e_1 + \cdots + e_5)$, $e_1 + \cdots + e_i$, $e_1 - e_5$ and $2e_1$, respectively. The half ρ of the sum of positive roots is equal to $4e_1 + 3e_2 + 2e_3 + e_4$. Applying Bott's theorem, we have

Lemma 3.5. ($g = 7$) *The cohomology groups of* $\mathcal{E}(n), (\wedge^2\mathcal{E})(n), (sl\,\mathcal{E})(n)$ *and* $(S^2\mathcal{E})(n)$ *vanish except for the following cases:*

(1) $H^0(X, \mathcal{E}(n)), H^0(X, (\wedge^2\mathcal{E})(n))$, $H^0(X, (S^2\mathcal{E})(n))$ *for* $n \ge 0$ *and* $H^0(X, (sl\,\mathcal{E})(n))$ *for* $n \ge 1$,

(2) $H^9(X, (\wedge^2\mathcal{E})(-8))$, *and*

(3) $H^{10}(X, \mathcal{E}(n)), H^{10}(X, (sl\,\mathcal{E})(n))$ *for* $n \le -9$ *and* $H^{10}(X, (\wedge^2\mathcal{E})(m))$, $H^{10}(X, (S^2\mathcal{E})(m))$ *for* $m \le -10$.

Remark 3.6. In the above case $g = 7$, the 10 roots $e_i + e_j$, $1 \le i < j \le 5$, are complementary to P. Their sum is equal to $4(e_1 + \cdots + e_5)$ and this is 8 times the fundamental weight w. By Proposition 1.5, the self intersection number of $\mathcal{O}_X(1)$ is equal to

$$10! \prod_{\beta \in R_P} \frac{(\beta, w)}{(\beta, \rho)} = 10! \prod_{0 \le i < j \le 4} (i + j)^{-1} = 12.$$

Hence X is a 10-dimensional variety of degree 12 in $\mathbf{P}^{15}$ and the anticanonical class is 8 times the hyperplane section.

Proof of Proposition 3.4 (in the case $g = 7$): S is a complete intersection of 8 members of $|\mathcal{O}_X(1)|$. Hence, if $\mathcal{A}$ is a vector bundle on X and $H^{i+a}(X, \mathcal{A}(-a))$ vanishes for every $0 \le a \le 8$, then so does $H^i(S, \mathcal{A}|_S)$.

(1 and 4) (1) and the vanishings of $H^0(S, E(-1))$ and $H^0(S, (\wedge^2 E)(-1))$ follow immediately from Lemma 3.5. Since $\wedge^5 \mathcal{E}$ is isomorphic to $\mathcal{O}_X(2)$, $\wedge^k \mathcal{E}$ is isomorphic to $(\wedge^{5-k}\mathcal{E})^\vee \otimes \mathcal{O}_X(2)$. Hence by the Serre duality and Lemma 3.5, we have

$$\begin{aligned} H^i(X, (\wedge^3 \mathcal{E})(-2-i)) &\cong H^{10-i}(X, (\wedge^3\mathcal{E})^\vee(2+i-8))^\vee \\ &\cong H^{10-i}(X, (\wedge^2\mathcal{E})(i-8))^\vee = 0 \end{aligned}$$

and

$$\begin{aligned} H^i(X, (\wedge^4 \mathcal{E})(-2-i)) &\cong H^{10-i}(X, (\wedge^4\mathcal{E})^\vee(2+i-8))^\vee \\ &\cong H^{10-i}(X, \mathcal{E}(i-8)))^\vee = 0, \end{aligned}$$

for every $0 \le i \le 8$. Therefore, $(\wedge^3 E)(-2)$ or $(\wedge^4 E)(-2)$ has no nonzero global sections.

(2) By the Serre duality, $H^i(X, E^\vee(-i))$ and $H^{i+1}(X, E^\vee(-i))$ are isomorphic to $H^{10-i}(X, \mathcal{E}(i-8))^\vee$ and $H^{9-i}(X, \mathcal{E}(i-8))^\vee$, respectively and both are zero for every $0 \le i \le 8$, by Lemma 3.5. Hence both $H^0(S, E^\vee)$ and $H^1(S, E^\vee)$ vanish. Therefore, by the exact sequence (3.2), we have (2).

(3) Let $\mathcal{K}$ be the kernel of the homomorphism $S^2\alpha : S^2V \otimes \mathcal{O}_X \to S^2\mathcal{E}$. We have the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow S^2V \otimes \mathcal{O}_X \longrightarrow S^2\mathcal{E} \longrightarrow 0.$$

The G-module S^2V is isomorphic to the direct sum of an irreducible G-module of dimension 54 and a trivial G-module generated by σ. Hence the G-module $H^0(X, \mathcal{K}) = \mathrm{Ker}\ H^0(S^2\alpha)$ is 1-dimensional and generated by σ. By Lemma 3.5 and the Kodaira vanishing theorem, $H^{i-1}(X, (S^2\mathcal{E})(-i))$ and $H^i(X, \mathcal{O}_X(-i))$ are zero. Hence by the above exact sequence, $H^i(X, \mathcal{K}(-i))$ vanishes for every $1 \le i \le 8$. By using the Koszul complex, we have that the restriction map $H^0(X, \mathcal{K}) \to H^0(S, \mathcal{K}|_S)$ is surjective. Therefore, the kernel of $H^0(S^2\alpha|_S)$ is at most 1-dimensional. It is clear that the kernel contains $\sigma \otimes 1$. Hence we have (3). *q.e.d.*

Proof of Theorem 3.3: Let S (resp. S') be a K3 surface which is a complete intersection of X and a linear subspace P (resp. P') and E (resp. E') the restriction of $\mathcal{E}$ to S (resp. S'). If P is general, then Pic S is generated by $\mathcal{O}_S(1)$ and, by (4) of Proposition 3.4, E is stable. Hence we have i). Assume that S and S' are isomorphic to each other as polarized surfaces and that E and E' are stable. By (1) of Proposition 3.4 and the same argument as Step I in §2, E and E' are isomorphic to each other. By (2) of Proposition 3.4, we have an isomorphism $\beta : V \xrightarrow{\sim} V'$ and a commutative diagram

$$\begin{array}{ccccc} V \otimes \mathcal{O}_S & \xrightarrow{\alpha|_S} & E & \longrightarrow & 0 \\ {\scriptstyle \beta\otimes 1}\downarrow\wr & & \downarrow\wr & & \\ V' \otimes \mathcal{O}_{S'} & \xrightarrow{\alpha|_{S'}} & E' & \longrightarrow & 0. \end{array}$$

Hence, in the case $g = 8$, S and S' are equivalent under the action of $\mathrm{GL}(V)$. In the case $g = 7$ or 9, by (3) of Proposition 3.4, $S^2\beta$ maps σ to $a\sigma$ for a nonzero constant a. Hence, replacing β by $a^{1/2}\beta$, we may assume that $S^2\beta$ preserves σ. Hence S and S' are equivalent under the action of $\mathrm{SO}(V,\sigma)$ or $\mathrm{Sp}(V,\sigma)$. *q.e.d.*

§4. Generic K3 surface of genus 6

A K3 surface of genus 6 is obtained as a complete intersection in the Grassmann variety $G(2, V^5)$ of 2-dimensional subspaces in a fixed 5-dimensional vector space V^5. $G(2, V^5)$ is embedded into $\mathbf{P}^9$ by Plücker coordinates and has degree 5. A smooth complete intersection $X_5 \subset \mathbf{P}^6$ of $G(2, V^5)$ and 3 hyperplanes in $\mathbf{P}^9$ is a Fano 3-fold of index 2 and degree 5. A smooth complete intersection X_5 and a quadratic hypersurface in $\mathbf{P}^6$ is an anticanonical divisor of X_5 and is a K3 surface of genus 6. The isomorphism class of X_5 does not depend on the choice of 3 hyperplanes and X_5 has an action of $\mathrm{PGL}(2)$ (see below).

Theorem 4.1. *Let S and S' be two general smooth complete intersections of X_5 and a quadratic hypersurface in $\mathbf{P}^6$. If $S \subset \mathbf{P}^6$ and $S' \subset \mathbf{P}^6$ are projectively equivalent, then they are equivalent under the action of* $\mathrm{PGL}(2)$ *on X_5.*

All the Fano 3-folds of index 2 and degree 5 are unique up to isomorphism [5]. There are several ways to describe the Fano 3-folds. The following is most convenient for our purpose: Let V be a 2-dimensional vector space and $f \in S^6V$ an invariant polynomial of an octahedral subgroup of $\mathrm{PGL}(V)$. f is equal to $xy(x^4 - y^4)$ for a suitable choice of a basis $\{x, y\}$ of V. Then the closure X_5 of the orbit $\mathrm{PGL}(V)\cdot\bar{f}$ in $\mathbf{P}_*(S^6V) := (S^6V - \{0\})/\mathbf{C}^*$ is a Fano 3-fold of index 2 and degree 5, [11]. $H^0(X_5, \mathcal{O}_X(2))$ is generated by $H^0(X_5, \mathcal{O}_X(1)) = S^6V$, [5], and has dimension $\frac{1}{2}(-K_X)^3 + 3 = 23$. Hence the kernel A of the natural map $S^2H^0(X, \mathcal{O}_X(1)) \longrightarrow H^0(X, \mathcal{O}_X(2))$ is a 5-dimensional $\mathrm{SL}(V)$-invariant subspace. As an $\mathrm{SL}(V)$-module, $S^2H^0(X, \mathcal{O}_X(1))$ is isomorphic to $S^2(S^6V) \cong S^{12}V \oplus S^8V \oplus S^4V \oplus \mathbf{1}$. Hence we have

Proposition 4.2. (1) *$H^0(X_5, \mathcal{O}_X(-K_X))$ is isomorphic to $S^{12}V \oplus S^8V \oplus \mathbf{1}$ as* $\mathrm{SL}(V)$*-module, and*

(2) *the vector space A of quadratic forms which vanish on $X_5 \subset \mathbf{P}^6$ is isomorphic to S^4V as* $\mathrm{SL}(V)$*-module.*

There is a non-empty open subset Ξ of $|-K_X|$ and a natural morphism $\Xi/\mathrm{PGL}(V) \longrightarrow \mathcal{F}_6$. Both the target and the source are of dimension 19 and the morphism is birational by the theorem. Hence by the proposition we have

Corollary 4.3. *The generic K3 surface of genus* 6 *can be embedded into* X_5 *as an anticanonical divisor in a unique way up to the action of* $\mathrm{PGL}(V)$. *In particular, the moduli space* $\mathcal{F}_6$ *is birationally equivalent to the orbit space* $(S^{12}V \oplus S^8V)/\mathrm{PGL}(V)$.

First we need to show that $\mathrm{PGL}(V)$ is the full automorphism group of X_5 :

Proposition 4.4. *The automorphism group* $\mathrm{Aut}\, X_5$ *of* X_5 *is connected and the natural homomorphism* $\mathrm{PGL}(V) \longrightarrow \mathrm{Aut}\, X_5$ *is an isomorphism.*

Proof. There is a 2-dimensional family of lines on $X_5 \subset \mathbf{P}^6$ and a 1-dimensional subfamily of lines ℓ of special type, *i.e.*, lines such that $N_{\ell/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$. The union of all lines of special type is a surface and has singularities along a rational curve C. C is the image of the 6-th Veronese embedding of $\mathbf{P}(V) \cong \mathbf{P}^1$ into $\mathbf{P}(S^6V)$. C is invariant under the action of $\mathrm{Aut}\, X_5$. Every automorphism of X_5 induces an automorphism of C. Hence we have the homomorphism $\alpha : \mathrm{Aut}\, X_5 \longrightarrow \mathrm{Aut}\, C \cong \mathrm{PGL}(V)$. Since $\alpha|_{\mathrm{PGL}(V)}$ is an isomorphism, $\mathrm{Aut}\, X_5$ is isomorphic to $\mathrm{PGL}(V) \times \mathrm{Ker}\, \alpha$. Let g be an automorphism of X_5 which commutes with every element of $\mathrm{SL}(V)$. Since S^6V is an irreducible $\mathrm{SL}(V)$-module, g is the identity by Schur's lemma. *q.e.d.*

Next we construct an equivariant embedding of X_5 into the Grassmann variety $G(2, S^4V)$. Let W be the 2-dimensional subspace of S^4V generated by $x^4 + y^4$ and x^2y^2 for some basis $\{x, y\}$ of V and Y the closure of the orbit $\mathrm{PGL}(V)\cdot[W]$ in $G(2, S^4V)$. Consider the morphism $J : G(2, S^4V) \longrightarrow \mathbf{P}_*(S^6V)$ for which

$$J([\mathbf{C}g + \mathbf{C}h]) = \det \begin{pmatrix} g_X & g_Y \\ h_X & h_Y \end{pmatrix},$$

where $\{X, Y\}$ is the dual basis of $\{x, y\}$. Then J is a $\mathrm{PGL}(V)$-equivariant morphism and sends $[W]$ to the point $\bar{f}, f = xy(x^4 - y^4)$. Hence J maps Y onto $X_5 \subset \mathbf{P}_*(S^6V)$. Define two $\mathrm{GL}(V)$-homomorphisms $\varphi : \wedge^2 S^4V \longrightarrow S^2V \otimes (\det V)^3$ and $j : \wedge^2 S^4V \longrightarrow S^6V \otimes \det V$ by

$$\varphi(g \wedge h) = \sum_{i,j,k=\pm 1} ijk(D_iD_jD_kg)(D_{-i}D_{-j}D_{-k}h) \otimes (X \wedge Y)^{-3}$$

and

$$j(g \wedge h) = \det \begin{pmatrix} D_1(g) & D_{-1}(g) \\ D_1(h) & D_{-1}(h) \end{pmatrix} \otimes (X \wedge Y)^{-1},$$

where $D_{\pm 1}$ are the derivations by X and Y. The GL(V)-module $\wedge^2 S^4V$ is decomposed into the direct sum of irreducible GL(V)-submodules $\operatorname{Ker}\varphi$ and $\operatorname{Ker} j$. Since $\varphi(\wedge^2 W) = 0$, the Plücker coordinates of W lies in the linear subspace $P = \mathbf{P}_*(\operatorname{Ker}\varphi)$ of $\mathbf{P}_*(\wedge^2 S^4V)$ and Y is contained in the intersection $G(2, S^4V) \cap P$. The morphism J is the composite of the Plücker embedding $G(2, S^4V) \subset \mathbf{P}_*(\wedge^2 S^4V)$ and the projection $\mathbf{P}_*(j) : \mathbf{P}_*(\wedge^2 S^4V) \cdots \to \mathbf{P}_*(S^6V)$ from the linear subspace $\mathbf{P}_*(\operatorname{Ker} j)$. Since the restriction of $\mathbf{P}_*(j)$ to P is an isomorphism, J gives a PGL(V)-equivariant isomorphism from the projective variety $Y \subset P$ onto $X_5 \subset \mathbf{P}_*(S^6V)$.

Lemma 4.5. *Y coincides with the intersection of $G(2, S^4V)$ and P in $\mathbf{P}_*(\wedge^2 S^4V)$.*

Proof. Let Y' be the intersection of $G(2, S^4V)$ and P and B (resp. B') the vector space consisting of quadratic forms on P which vanish on Y (resp. Y'). Both Y and Y' are intersections of quadratic hypersurfaces. Hence it suffices to show that $B = B'$. Since $G(2, S^4V)$ does not contain P, B' is not zero. On the other hand, since $Y \subset P$ is isomorphic to $X_5 \subset \mathbf{P}^6$, B is an irreducible SL(V)-module by Proposition 4.2. As we saw above, Y is contained PGL(V)-equivariantly in Y' and hence B' is an SL(V)-submodule of B. Hence B' coincides with B. *q.e.d.*

So we have constructed a PGL(V)-equivariant embedding of X_5 into $G(2, S^4V)$ and shown that X_5 coincides with the intersection of its linear hull and $G(2, S^4V)$.

Proof of Theorem 4.1. There is a universal exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow S^4V \otimes \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow 0$$

on $G(2, S^4V)$, where $\mathcal{E}$ (resp. $\mathcal{F}$) is the universal sub- (resp. quotient) bundle and has rank 2 (resp. 3). Let S and S' be two members of the anticanonical linear system $|-K_X|$ on X_5. By the same arguments as in Sections 2 and 3, we have

(i) $H^i(S, sl(\mathcal{E})|_S) = 0$ for every i,

(ii) If S is general, then the vector bundle $\mathcal{E}|_S$ is stable with respect to $\mathcal{O}_S(1)$, and

(iii) If $\mathcal{E}|_S$ and $\mathcal{E}|_{S'}$ are stable with respect to $\mathcal{O}_S(1)$ and $\mathcal{O}_{S'}(1)$, respectively, and if S and S' are isomorphic as polarized surfaces, then there are isomorphisms $\alpha : \mathcal{E}|_S \longrightarrow \mathcal{E}|_{S'}$ and $\beta \in \mathrm{GL}(S^4V)$ such that the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & E|_S & \longrightarrow & S^4V \otimes \mathcal{O}_S \\ & & \alpha\downarrow\wr & & \downarrow\wr\ \beta\otimes 1 \\ 0 & \longrightarrow & E|_{S'} & \longrightarrow & S^4V \otimes \mathcal{O}_{S'} \end{array}$$

is commutative. In particular, the automorphism $\bar{\beta}$ of $G(2, S^4V)$ induced by β maps S onto S' isomorphically.

Since X_5 is the intersection of $G(2, S^4V)$ and the linear span of S (resp. S'), the automorphism $\bar{\beta}$ maps X_5 onto itself. Hence, by Proposition 4.4, S and S' are equivalent under the action of $\mathrm{PGL}(V)$ on X_5. *q.e.d.*

§5. Fano 3-folds of genus 10

In this section we shall prove Theorem 0.9 in the case $g = 10$. The other cases $g = 7, 8$ and 9 are very similar.

Let V and V' be Fano 3-folds which are complete intersections of $X_{18} \subset \mathbf{P}^{13}$ and linear subspaces of codimension 2. By the Lefschetz theorem, both $\mathrm{Pic}\, V$ and $\mathrm{Pic}\, V'$ are generated by hyperplane sections. Let $\mathcal{U}$ be the universal subbundle of $\mathcal{C}_0 \otimes \mathcal{O}_{X_{18}}$ as in Section 1 and F and F' the restrictions of $\mathcal{U}$ to V and V', respectively.

Proposition 5.1. *Let $\varphi : V \xrightarrow{\sim} V'$ be an isomorphism. Then $\varphi^*(F')$ is isomorphic to F.*

Proof. Let S be the generic member of $|-K_V|$ and put $S' = \varphi(S)$. The Picard group of S is generated by the hyperplane section. The restrictions $E = F|_S$ and $E' = F'|_{S'}$ are stable vector bundles as we saw in the proof of Proposition 2.1. Hence F and F' are also stable vector bundles. Put $M = \mathcal{H}om_{\mathcal{O}_V}(F, \varphi^*F')$. By Step I in Section 2, there is an isomorphism $f_0 : E \xrightarrow{\sim} (\varphi|_S)^*E'$. Hence the restriction of M to S is isomorphic to $\mathcal{E}nd_{\mathcal{O}_S}(E)$. By Proposition 1.10, we have

$$H^1(S, M(n)|_S) \cong H^1(S, \mathcal{O}_S(n)) \oplus H^1(S, (sl\, E)(n)) = 0$$

for every integer n. Since $H^1(V, M(n))$ is zero, if n is sufficiently negative, we have by induction on n that $H^1(V, M(n))$ is zero for every n. In particular $H^1(V, M(-1))$ vanishes and hence the restriction map $H^0(V, M) \longrightarrow H^0(S, M|_S)$ is surjective. It follows that there is a nonzero homomorphism $f : F \longrightarrow \varphi^*F'$ such that $f|_S = f_0$. Since f_0 is an isomorphism, the cokernel of f has a support on a finite set. Since the Hilbert polynomials $\chi(F(n))$ and $\chi((\varphi^*F')(n))$ are same, the cokernel of f is zero and f is an isomorphism. *q.e.d.*

By Proposition 5.1 and similar arguments as Step II-VII in Section 2, we have an isomorphism $\beta : F \xrightarrow{\sim} \varphi^*F'$ and an isomorphism $\gamma : \mathcal{C}_0 \longrightarrow \mathcal{C}_0$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\beta} & \varphi^*(F') \\ \cap & & \cap \\ \mathcal{C}_0 \otimes \mathcal{O}_V & \xrightarrow{\gamma \otimes 1} & \mathcal{C}_0 \otimes \mathcal{O}_V = \varphi^*(\mathcal{C}_0 \otimes \mathcal{O}_{V'}) \end{array}$$

is commutative and such that $1 \oplus \gamma$ is an automorphism of the Cayley algebra $\mathcal{C}$. Hence the automorphism of $X_{18} = G/P$ induced by $1 \oplus \gamma$ maps V onto V', which shows Theorem 0.9 in the case $g = 10$.

§6. Curves of genus ≤ 9

In this section we shall show the following:

Theorem 6.1. *The generic curve of genus ≤ 9 lie on a K3 surface.*

In the case $g \leq 6$, the generic curve is realized as a plane curve C of degree $d \leq 6$ with only ordinary double points. Take a general plane curve D of degree $6-d$ and let S be the double covering of the plane with branch locus $C \cup D$. Then the minimal resolution $\tilde{S}$ of S is a K3 surface and contains a curve isomorphic to C.

In the case $6 \leq g \leq 9$, we shall show that the generic curve C of genus g can be embedded into $\mathbf{P}^5$ by the complete linear system of a line bundle L of degree $g+4$ and that there is a K3 surface S which is a complete intersection of 3 quadratic hypersurfaces in $\mathbf{P}^5$ and which contains the image of C.

Let C be a curve of genus $6 \leq g \leq 9$ and D an effective divisor on C of degree $g-6$. Put $L = \omega_C \otimes \mathcal{O}_C(-D)$. Then L is a line bundle of degree $g+4$. If D is general, then $\dim H^0(C, L) = 6$. Since $\deg L^{\otimes 2} > \deg \omega_C$, we have $\dim H^0(C, L^{\otimes 2}) = 2(g+4) + 1 - g = g + 9$.

Proposition 6.2. *If C and D are general, then we have*

(1) *L is very ample and* $\dim H^0(C, L) = 6$,

(2) *the natural map*
$$S^2 H^0(C, L) \longrightarrow H^0(C, L^{\otimes 2})$$
is surjective and its kernel V is of dimension $12 - g$, *and*

(3) *there are 3 quadratic hypersurfaces Q_1, Q_2 and Q_3 in $\mathbf{P}(H^0(C, L))$ which contains the image of C by $\Phi_{|L|}$ and such that the intersection $S = Q_1 \cap Q_2 \cap Q_3$ is a K3 surface.*

Proof. It suffices to show that there exists a pair of C and D which satisfies the conditions (1), (2) and (3). Let R be a smooth rational curve of degree $g-4$ in $\mathbf{P}^5$ whose linear span $< R >$ has dimension $g-4$. Since R is an intersection of quadratic hypersurfaces, the intersection of 3 general quadratic hypersurfaces Q_1, Q_2 and Q_3 which contain R is a smooth K3 surface. Let C_0 be the intersection of S and a general hyperplane H. We show that the pair of the generic member C of the complete linear system $|C_0 + R|$ on S and the divisor $D = R|_C$ satisfies the conditions (1), (2) and (3).

The intersection number $(C_0 \cdot R)$ is equal to $\deg R = g - 4 \geq 2$. Hence the linear system $|C_0 + R|$ has no base points. Therefore C is smooth and D is effective. The genus of C is equal to $(C_0 + R)^2/2 + 1 = g$ and the degree of D is equal to $(C_0 + R.R) = g - 6$. Since ω_C is isomorphic to $\mathcal{O}_C(C) \cong \mathcal{O}_C(C_0 + R)$, the line bundle $L = \omega_C(-D)$ is isomorphic to $\mathcal{O}_C(C_0)$, the restriction of the tautological line bundle of $\mathbf{P}^5$ to C. There is a natural exact sequence

$$0 \longrightarrow \mathcal{O}_S(-R) \longrightarrow \mathcal{O}_S(C_0) \longrightarrow \mathcal{O}_C(C_0) \longrightarrow 0.$$

Since $H^i(S, \mathcal{O}_S(-R)) = 0$ for $i = 0$ and 1, the restriction map $H^0(\mathbf{P}^5, \mathcal{O}_{\mathbf{P}}(1)) \xrightarrow{\sim} H^0(S, \mathcal{O}_S(C_0)) \longrightarrow H^0(C, \mathcal{O}_C(C_0))$ is an isomorphism. Hence the morphism $\Phi_{|L|}$ is nothing but the inclusion map $C \hookrightarrow \mathbf{P}^5$ and (1) and (3) are obvious by our construction of C.

Claim. Let V_0 be the vector space of the quadratic forms on $\mathbf{P}^5$ which are identically zero on $C_0 \cup R$. Then the dimension of V_0 is at most $12 - g$.

Let $F_i = 0$ be the defining equation of the quadratic hypersurface Q_i for $i = 1, 2$ and 3 and $G = 0$ that of the hyperplane H. Let F be any quadratic form on $\mathbf{P}^5$ which is identically zero on $C_0 \cup R$. Since F is identically zero on C_0, F is equal to $a_1F_1 + a_2F_2 + a_3F_3 + GG'$ for some constants a_1, a_2 and a_3 and linear form G'. Since F_1, F_2, F_3 and F are identically zero on R, so is GG'. Hence G' is identically zero on R. Therefore, the vector space V_0 is generated by F_1, F_2, F_3 and GG', G' being all linears from vanishing on $< R >$. Since $\dim < R >= g - 4$, we have $\dim V_0 \le 3 + 5 - (g - 4) = 12 - g$.

Since C is a general deformation of $C_0 \cup R$, we have, by the claim, that the dimension of V is also at most $12 - g$. Since

$$\dim S^2H^0(C, L) - \dim H^0(C, L^{\otimes 2}) = 21 - (g + 9) = 12 - g,$$

$H^0(C, L^{\otimes 2})$ is generated by $H^0(C, L)$ and V has exactly dimension $12 - g$.

q.e.d.

By the theorem and Corollaries 0.3 and 4.3, we have

Corollary 6.3. *The generic curve of genus* $3 \le g \le 9$ *is a complete intersection in a homogeneous space.*

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