4.1 Consider $\mathbb{H} := \bigoplus_n H^*_{\mathbb{T}}(M(r,n)) \cong \bigoplus_n H^{\mathbb{T}}_{4rn-*}(M(r,n))$

Two natural operators :

a) $\mathcal{E}$: universal sheaf on $M(r,n)$

$H^1(\mathcal{E}(-l_\infty))$ (= $\mathcal{V}$ in the quiver description)

is a rank n vector bdle over $M(r,n)$

mult. of $c_i(\mathcal{V}) \curvearrowright \mathbb{H}$

b) $M(r,n,n+1) = \{(E_1, E_2, \varphi) \mid c_2(E_1) = n,\ c_2(E_2) = n+1,\ E_1 \supset E_2$ isom. on l_∞ framing compat $\}/\cong$

$$\begin{array}{ccc} & M(r,n,n+1) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ M(r,n) & & M(r,n+1) \end{array}$$

Prop (1) $M(r,n,n+1)$ smooth of $\dim = 2rn + 2r + 2$

(2) p_2 is proper

Ex $r = 1$ $\quad M(r,n+1) = \mathrm{Hilb}^{n+1}\mathbb{A}^2 \ni I_{Z_2}$

$\quad M(r,n) = \mathrm{Hilb}^n \mathbb{A}^2 \ni I_{Z_1}$

$Z_1 \subset Z_2 \quad Z_2$ is obtained from Z_1 by adding one point generically.

Now $H^*_{\mathbb{T}}(M(r,n)) \xrightarrow[p_{2*}p_1^*(\cdot)]{} H^*_{\mathbb{T}}(M(r,n+1))$

For the opposite direction, we consider

$M(r,n,n+1)_0 \subset M(r,n,n+1) \quad \mathrm{Supp}\, E_1/E_2 = \{0\}$

$\Rightarrow p_1|_{M(r,n,n+1)_0}$ is proper

Th (Maulik-Okounkov, Schiffmann-Vasserot)
These operators gives a structure of a representation of the W-algebra $W(\mathfrak{gl}_r)$.

I do <u>not</u> make the statement in a precise form.

Today I only study the case $r=1$. And furthermore I set $\varepsilon_1+\varepsilon_2=0$, which means that I restrict $T^2 \supset \mathbb{C}^* \cong \{(t,t^{-1})\}$.

Even in these assumptions, we can still see interesting representation theory.

4.2 study of fixed points

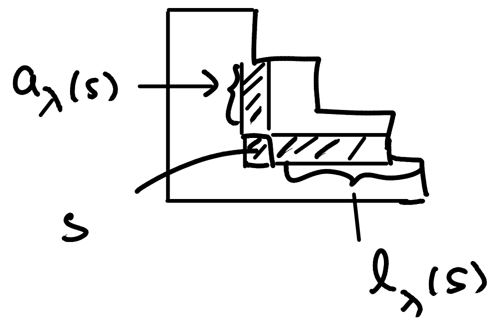
Write $X^{[n]}$ instead of $\mathrm{Hilb}^n \mathbb{A}^2$ hereafter.

I will identify a Young diagram with a partition.

<u>Prop</u> Let $\lambda \in X^{[n]}$ be a fixed pt

$$\mathrm{ch}_{\mathbb{C}^*} T_\lambda X^{[n]} = \sum_{s \in \lambda} t^{h_\lambda(s)} + t^{-h_\lambda(s)},$$

where $h_\lambda(s)$ is the hook length of the box $s \in \lambda$.



$$h_\lambda(s) = a_\lambda(s) + l_\lambda(s) + 1$$

<u>Rem</u> $(X^{[n]})^{\mathbb{C}^*} = (X^{[n]})^{T^2}$

$$\therefore e(T_\lambda X^{[n]}) = (-1)^n \varepsilon^{2n} h(\lambda)^2 \qquad h(\lambda) := \prod_{s \in \lambda} h_\lambda(s)$$

<u>Rem</u> $h(\lambda)$ appears in the representation theory of $\mathfrak{S}_n$

$\dfrac{n!}{h(\lambda)}$ = dim irr. rep. corresponding to the partition λ.

[Macdonald I.(7.6) & §5.Ex.2]

It is natural to consider

$$s_\lambda := \frac{1}{\varepsilon^n h(\lambda)}[\lambda] \in H_*^{\mathbb{C}^*}(X^{[n]}) \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C}(\varepsilon)$$

$$i_\lambda : \{\lambda\} \to X^{[n]} \qquad [\lambda] = i_{\lambda *}[\lambda]$$

$$(-1)^n \int_{X^{[n]}} s_\lambda \cup s_\lambda = (-1)^n \frac{i_\lambda^*(s_\lambda \cup s_\lambda)}{e(T_\lambda X^{[n]})} = \frac{\cancel{(-1)^n}\, i_\lambda^* s_\lambda \cup i_\lambda^* s_\lambda}{\cancel{(-1)^n}\, \varepsilon^{2n} h(\lambda)^2} = 1$$

$$\left(i_\lambda^* s_\lambda = \frac{1}{\varepsilon^n h(\lambda)} i_\lambda^* i_{\lambda *}[\lambda] = (-1)^n \varepsilon^n h(\lambda)[\lambda] \right)$$

So $\{s_\lambda\}$: o.n.b. for $(-1)^n \int_{X^{[n]}} \cdot \cup \cdot$

Let us define an isomorphism

$$\bigoplus_n H_*^{\mathbb{C}^*}(X^{[n]}) \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C}(\varepsilon) \cong \mathbb{C}(\varepsilon) \otimes \underset{\text{symmetric polynomials}}{\Lambda}$$

$$s_\lambda \longmapsto s_\lambda : \text{Schur function}$$

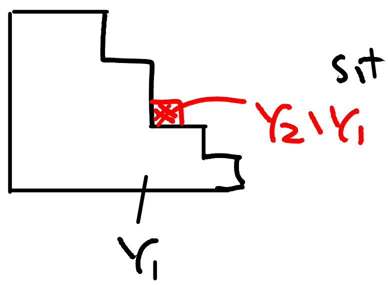
$(-1)^n \int_{X^{[n]}} \cdot \cup \cdot \longleftrightarrow$ standard inner product on Λ

Let us study the operator given by $M(1,n,n+1) = X^{[n,n+1]}$

$$X^{[n,n+1]} = \{ (I_1, I_2) \in X^{[n]} \times X^{[n+1]} \mid I_1 \supset I_2 \}$$

$\circlearrowleft \mathbb{C}^*$ $\qquad$ $2n+2$ dim

A fixed pt is a pair of Young diagrams (Y_1, Y_2) s.t. Y_2 is obtained from Y_1 by adding a box

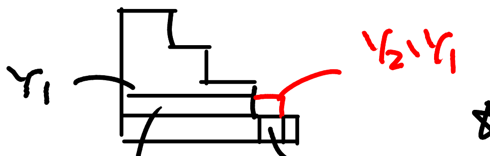


<u>Prop</u> $\mathrm{ch}\, T_{(Y_1,Y_2)} X^{[n,n+1]}$

$$= t + t^{-1} + \sum_{s \in Y_1} t^{-l_{Y_2}(s) - a_{Y_1}(s) - 1} + t^{l_{Y_1}(s) + a_{Y_2}(s) + 1}$$

$\mathbb{C}^2 \circlearrowleft$

<u>Cor.</u> $e(T_{(Y_1,Y_2)} X^{[n,n+1]}) = (-1)^{n+1} \varepsilon^{2(n+1)} h(Y_1) h(Y_2)$



$$a_{Y_1}(s) = a_{Y_2}(s) - 1$$

$$l_{Y_1}(s) = l_{Y_2}(s) - 1$$

$$\prod_{s\in\lambda_1}\left(l_{\lambda_2}(s) + a_{\lambda_1}(s) + 1\right) = \prod_{s\notin \star 1} h_{\lambda_1}(s) \times \prod_{s\in \star 1} \overbrace{\left(h_{\lambda_1}(s)+1\right)}^{h_{\lambda_2}(s)}$$

$$= h(\lambda_1) \times \prod_{s\in\star 1} \frac{h_{\lambda_2}(s)}{h_{\lambda_1}(s)}$$

$$\prod_{s\in\lambda_1}\left(l_{\lambda_1}(s) + 1 + a_{\lambda_2}(s)\right) = \prod_{s\notin \star 2} h_{\lambda_1}(s) \times \prod_{s\in \star 2} h_{\lambda_1}(s)+1$$

$$= h(\lambda_1) \times \prod_{s\in\star 2} \frac{h_{\lambda_2}(s)}{h_{\lambda_1}(s)}$$

$$\therefore \prod_{s\in\lambda_1}\left(l_{\lambda_2}(s) + a_{\lambda_1}(s) + 1\right) \prod_{s\in\lambda_1}\left(l_{\lambda_1}(s) + 1 + a_{\lambda_2}(s)\right)$$

$$= h(\lambda_1)^2 \prod_{s\in \star 1 \sqcup \star 2} \frac{h_{\lambda_2}(s)}{h_{\lambda_1}(s)} = h(\lambda_1) h(\lambda_2) \quad //$$

<u>Prop.</u> (up to sign) $[X^{[n,n+1]}] : H_*^{\mathbb{C}^*}(X^{[n]}) \to H_*^{\mathbb{C}^*}(X^{[n+1]})$ corresponds to the multiplication by e_1 (1st elementary symmetric func.)

$\because$

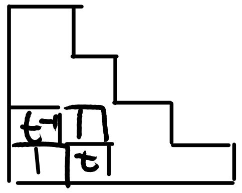
$$X^{[n]} \xleftarrow{p_1} X^{[n,n+1]} \xrightarrow{p_2} X^{[n+1]}$$

$$[S_{\lambda_1}] = \frac{[\lambda_1]}{h(\lambda_1)} \xrightarrow{p_1^*} h(\lambda_1)[\lambda_1] \xrightarrow{\cap X^{[n,n+1]}} \sum_{\lambda_2 > \lambda_1} \frac{h(\lambda_1)[\lambda_1]}{h(\lambda_1)h(\lambda_2)}$$

$$= \sum_{\lambda_2 \supset \lambda_1} s_{\lambda_2}$$

This coincides with the Pieri formula for Schur function //

Next we study $G(\mathcal{V})$ $\mathcal{V}$: tautological bdle

$\mathrm{ch}\,\mathcal{V}|_\lambda =$  $x^i y^j \mapsto t^{i-j} x^i y^j$

$$\therefore\ G(\mathcal{V})|_\lambda = \sum_{(i,j)=s\in\lambda} (i-j) = \sum_{s\in\lambda} c(s)$$

c: content of s

$$= n(\lambda^t) - n(\lambda)$$ (Macdonald I.1, Ex.3

where $n(\lambda) = \sum (i-1)\lambda_i$

So it becomes a combinatorial question!

Q. What is the operator G on Λ: symmetric func, given by
$$G s_\lambda = (n(\lambda^t) - n(\lambda))\, s_\lambda\ ?$$

A. G = Goulden operator
$\rightsquigarrow$ Virasoro algebra

cf. Frenkel-Wang math.QA/0006087

$$G := \frac{1}{2}\sum_{m,n=1}^{\infty} (\alpha_{-m}\alpha_{-n}\alpha_{m+n} + \alpha_{-m-n}\alpha_m\alpha_n)$$

up to the normalization

Goulden operator

NB. Mac. I.7.Ex 7 $n(\lambda^t) - n(\lambda) = \dfrac{\chi^\lambda_\rho}{f^\lambda} h_\rho$

where $\rho = (21^{n-2})$

χ^λ_ρ: character χ^λ at the conjugacy class ρ
$f^\lambda = \chi^\lambda(1) = \dim \lambda$
$h_\rho = n!/z_\rho = n(n-1)/2$