LECTURES AT THE UNIVERSITY OF HONG KONG – A GEOMETRIC CONSTRUCTION OF ALGEBRAS

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The difference between this note and what I distributed during lectures are the followings:

(1) Several errors are corrected.
(2) §5 is re-written. I explain the convolution on $\bigoplus_n H_*(X^{[n]})$ first.
(3) §8 is re-written so that the result holds for general quiver varieties, not necessarily fixed point sets of Hilbert schemes.
(4) a subsection on crystal (§9.1) is included.

Lecture Announcement

The theme of lectures is an interplay of representation theory and geometry. I will explain a geometric construction of affine Lie algebras (or their q-analogue) and their representations. The spaces, which I will use, are moduli spaces of vector bundles over complex surfaces or their variant. These spaces have been studied intensively from a geometric point of view, but their relation to the representation theory is a new and hot topic.

Our method is an application of more general technique which has been used successfully in the representation theory during the last several decades. It is the construction of algebras by the *convolution product*, defined on homology groups (or their variants) of manifolds. For example, Weyl groups and affine Hecke algebras were constructed by convolutions on homology groups and equivariant K-homology groups of flag varieties (Springer, Borho-MacPherson, Lusztig, Ginzburg, Kazhdan-Lusztig, etc). Also upper triangular parts of quantum enveloping algebras and their canonical bases were constructed by convolutions using perverse sheaves on moduli spaces of representations of quivers (Lusztig).

More precise plan of the lectures is the following: I will first prepare geometric staff, i.e., the convolution product on homology groups. Then I will introduce Hilbert schemes of points on complex surfaces, and connect their homology groups with infinite dimensional Heisenberg algebras. Then I will introduce *quiver varieties*, and study their homology groups. As an application, we get a geometric construction of Kashiwara's crystal base. If I still have time, I will explain a construction of quantum affine algebras via the convolution on the equivariant K-groups of quiver varieties.

References

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1. Convolution product

1.1. **General definition.** Let X, Y be finite sets. Let $\mathcal{F}(X)$, $\mathcal{F}(Y)$ the vector space of $\mathbb{C}$-valued functions on X, Y. If a $\mathbb{C}$-valued fuction $K(x,y)$ on $X \times Y$ is given, we can define an operator $\mathcal{F}(Y) \to \mathcal{F}(X)$ by

$$\mathcal{F}(Y) \ni f(y) \longmapsto (K * f)(x) \overset{\text{def.}}{=} \sum_{y \in Y} K(x,y) f(y) \in \mathcal{F}(X).$$

This is called the *convolution product* of K and f.

Suppose X, Y, Z are finite sets. For given functions $K(x,y)$ on $X \times Y$ and $K'(y,z)$ on $Y \times Z$, we consider the composition of operators given by the convolution products:

$$K * (K' * f)(x) = \sum_{y \in Y} K(x,y) \left(\sum_{z \in Z} K'(y,z) f(z) \right).$$

This is equal to

$$\sum_{z \in Z} \left(\sum_{y \in Y} K(x,y) K'(y,z) \right) f(z).$$

Hence if we define $(K * K')(x,z) \overset{\text{def.}}{=} \sum_{y \in Y} K(x,y) K'(y,z)$, then the composite of operators is given again by the convolution product.

If we take $X = Y$, then the vector space of $\mathbb{C}$-valued functions on $X \times X$, which we denote by $\mathcal{F}(X \times X)$ is an algebra by the above convolution product $K * K'$. It is clearly associative:

$$(K * K') * K'' = K * (K' * K'').$$

And the unit is given by the characteristic function of the diagonal Δ_X:

$$K * \Delta_X = \Delta_X * K = K, \qquad \text{where } \Delta_X(x,x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

The algebra $\mathcal{F}(X \times X)$ has a natural representation. Namely $\mathcal{F}(X)$ under the convolution product !

Example 1.1 (Trivial). Suppose $\#X = n$. Then $\mathcal{F}(X \times X)$ is the matrix algebra of $n \times n$ matrices. $\mathcal{F}(X)$ is the vector representation.

This example means that we need to consider a subalgebra of $\mathcal{F}(X \times X)$ in order to get an interersting algebra.

1.2. **Iwahori-Hecke algebra (due to Iwahori).** The Iwahori-Hecke algebra $\mathcal{H}_q$ is a q-analogue of the group ring of the Weyl group W associated with a complex simple Lie algebra $\mathfrak{g}$. Here q is an indeterminate (parameter). We consider the simplest case $\mathfrak{g} = \mathfrak{sl}_2$. In this case, the Weyl group W is $\mathbb{Z}/2\mathbb{Z}$. The Iwahori-Hecke algebra $\mathcal{H}_q$ is the $\mathbb{C}[q, q^{-1}]$-algebra with generator T and the defining relation

$$(T - q)(T + 1) = 0.$$

Note that the relation reduces to $T^2 = 1$ if $q = 1$.

Let $k = \mathbb{F}_q$ be the finite field of q elements. We consider the projective line $\mathbb{P}^1(k)$ of k, the space of 1-dimensional subspaces of k^2. We consider a natural action of $\mathrm{SL}_2(k)$ on $\mathbb{P}^1(k)$, and the diagonal action on the product $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$. Let $\mathcal{F}(\mathbb{P}^1(k) \times \mathbb{P}^1(k))^{\mathrm{SL}_2(k)}$ be the vector space of $\mathbb{C}$-valued functions on $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$ which is invariant under the $\mathrm{SL}_2(k)$-action.

By the following elementary result, $\mathcal{F}(\mathbb{P}^1(k) \times \mathbb{P}^1(k))^{\mathrm{SL}_2(k)}$ is an associative algebra (with unit) by the convolution.

Lemma 1.2. *Suppose a group G acts on X. Let $\mathcal{F}(X \times X)^G$ be the vector space of functions on $X \times X$ invariant under the diagonal action of G. Then it is a subalgebra of $\mathcal{F}(X \times X)$ with respect to multiplication given by the convolution.*

The vector space $\mathcal{F}(X \times X)^G$ has a base given by characteristic functions of G-orbits in $X \times X$. In our case $X = \mathbb{P}^1(k)$, $G = \mathrm{SL}_2(k)$, it is easy to see that the diagonal action has two orbits: the diagonal Δ and the complement of the diagonal $U \overset{\text{def.}}{=} \mathbb{P}^1(k) \times \mathbb{P}^1(k) \setminus \Delta$. Let us denote the characteristic functions by the same notation: Δ and U. In order to identify the algebra $\mathcal{F}(\mathbb{P}^1(k) \times \mathbb{P}^1(k))^{\mathrm{SL}_2(k)}$, it is enough to compute the convolution products $\Delta * \Delta$, $\Delta * U$, $U * \Delta$, $U * U$. But the first three are trivial. Δ is unit, so $\Delta * \Delta = \Delta$, $\Delta * U = U$, $U * \Delta = U$. Let us compute the last one:

$$(U * U)(x, z) = \sum_{y \in \mathbb{P}^1(k)} U(x, y)U(y, z) = \#\{y \in \mathbb{P}^1(k) \mid y \neq x, y \neq z\}$$

$$= \begin{cases} q - 1 & \text{if } x \neq y, \\ q & \text{if } x = y, \end{cases}$$

where we have used $\#\mathbb{P}^1(k) = q + 1$. Thus we finally get

$$U * U = (q - 1)U + q\Delta,$$

Or, equivalently

$$(U - q\Delta) * (U + \Delta) = 0.$$

After the substitution $U \to T$, $\Delta \to 1$, this is the defining relation of the Iwahori-Hecke algebra for $\mathfrak{sl}_2$.

This example can be generalized to the case of arbitary Iwahori-Hecke algebra associated with a complex simple Lie algebra $\mathfrak{g}$, by considering the flag variety G/B instead of $\mathbb{P}^1(k)$.

1.3. **The quantum universal enveloping algebra $\mathbf{U}_v(\mathfrak{sl}_2)$ (due to Beilinson-Lusztig-MacPherson).** Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. It is the complex Lie algebra generated by

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with the defining relation

$$[h, e] = 2e, \quad [h, f] = 2f, \quad [e, f] = h.$$

The universal enveloping algebra $\mathbf{U}(\mathfrak{sl}_2)$ is the associative algebra with generators e, f, h and the same defining relation, where $[x, y]$ is understood as $xy - yx$. Geometrically $\mathfrak{sl}_2(\mathbb{C})$ is the space of left invariant vector fields on the Lie group $\mathrm{SL}_2(\mathbb{C})$, and $\mathbf{U}(\mathfrak{sl}_2)$ is the ring of invariant differential operators on $\mathrm{SL}_2(\mathbb{C})$. (A vector field is a 1st order differential operator.)

Let us define a q-analogue of $\mathbf{U}(\mathfrak{sl}_2)$, called the quantum enveloping algebra of Drinfeld-Jimbo, attached to $\mathfrak{sl}_2$. Let v be an indeterminate. (We will use the finite field $\mathbb{F}_q$ again, and the parameter v will be given by $\sqrt{q}$.)

Let us introduce v-intergers:

$$[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}.$$

Let $\mathbf{U}_v(\mathfrak{sl}_2)$ be the associative $\mathbb{C}(v)$-algebra with generators e, f, $k^{\pm}$ and the defining relations

$$kk^{-1} = k^{-1}k = 1,$$
$$kek^{-1} = v^2 e, \qquad kfk^{-1} = v^{-2} f, \tag{1.3}$$
$$ef - fe = \frac{k - k^{-1}}{v - v^{-1}}.$$

Heuristically we can think $k = v^h$. If we make $v \to 1$, then we recover the defining relations of $\mathfrak{sl}_2$.

The representation theory (finite dimensional representations) of $\mathbf{U}_v(\mathfrak{sl}_2)$ is known to be the same as that of $\mathfrak{sl}_2(\mathbb{C})$. In particular, we have the unique irreducible representation of dimension $N+1$ for each $N \in \mathbb{Z}_{\geq 0}$. It is realized on the space of polynomials in x with degree $\leq N$ as

$$kx^d \overset{\text{def.}}{=} v^{N-2d} x^d, \quad ex^d \overset{\text{def.}}{=} \begin{cases} [N-d+1]_v \, x^{d-1} & \text{if } d > 0, \\ 0 & \text{if } d = 0, \end{cases},$$
$$fx^d \overset{\text{def.}}{=} \begin{cases} [d+1]_v \, x^{d+1} & \text{if } d < N, \\ 0 & \text{if } d = N. \end{cases}$$

(The defining relation (1.3) follows from the identity $[N-d]_v[d+1]_v - [d]_v[N-d+1]_v = [N-2d]_v$.)

We give a geometric realization of $\mathbf{U}_v(\mathfrak{sl}_2)$, which is nothing to do with the ring of differential operators on $\mathrm{SL}_2(\mathbb{C})$. In fact, the Lie group $\mathrm{SL}_2(\mathbb{C})$ is absent in the following construction.

Let $k = \mathbb{F}_q$ be the finite field of q elements. Set $v = \sqrt{q}$. Fix a positive integer N. Let $\mathcal{G}$ be the Grassmann variety of all subspaces of k^N. It is a disjoint union of $\mathcal{G}_d$ with $0 \leq d \leq N$, where $\mathcal{G}_d$ is the Grassmann variety of d-dimensional subspace of k^N. We consider the action of $\mathrm{GL}_N(k)$ on $\mathcal{G}$ and the diagonal action on $\mathcal{G} \times \mathcal{G}$. Then the vector space $\mathcal{F}(\mathcal{G} \times \mathcal{G})^{\mathrm{GL}_N(k)}$ of $\mathbb{C}$-valued $\mathrm{GL}_N(k)$-invariant functions on $\mathcal{G} \times \mathcal{G}$ is an associative algebra by the convolution.

It has a base given by the characteristic functions of orbits.

Lemma 1.4. *The $\mathrm{GL}_n(k)$-orbits in $\mathcal{G} \times \mathcal{G}$ are parametrized by 2×2-matrices*

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with entries $a_{ij} \in \mathbb{Z}_{\geq 0}$ satisfying $a_{11} + a_{12} + a_{21} + a_{22} = N$. The corresponding orbit is the set of pairs (V, V') of subspaces of k^N with

$$\dim(V \cap V') = a_{11}, \quad \dim(V/V \cap V') = a_{12},$$
$$\dim(V'/V \cap V') = a_{21}, \quad \dim(k^N/V + V') = a_{22}.$$

Let Δ_d denote the diagonal in $\mathcal{G}_d \times \mathcal{G}_d$. The corresponding matrix is $\begin{bmatrix} d & 0 \\ 0 & N-d \end{bmatrix}$. Let e_d be the orbit

$$\{(V, V') \mid V \subset V', \dim V = d-1, \dim V' = d\}.$$

The corresponding matrix is $\begin{bmatrix} d-1 & 0 \\ 1 & N-d \end{bmatrix}$. Exchanging the role of V and V', we also define f_d:

$$f_d = \{(V, V') \mid V \supset V', \dim V = d+1, \dim V' = d\} \longleftrightarrow \begin{bmatrix} d & 0 \\ 1 & N-d-1 \end{bmatrix}.$$

Let us denote the characteristic functions by the same symbol as orbits. Let $\widetilde{e} \stackrel{\text{def.}}{=} \sum_{d=0}^N e_d$, $\widetilde{f} \stackrel{\text{def.}}{=} \sum_{d=0}^N f_d$. We compute the convolution products of these functions. The followings are obvious:

$$\Delta_d * \Delta_{d'} = \delta_{dd'} \Delta_d, \quad \Delta_d * \widetilde{e} = \widetilde{e} * \Delta_{d+1}, \quad \Delta_d * \widetilde{f} = \widetilde{f} * \Delta_{d-1}.$$

Let us compute the commutator $[\widetilde{e}, \widetilde{f}] = \widetilde{e} * \widetilde{f} - \widetilde{f} * \widetilde{e}$. We have

$$(\widetilde{e} * \widetilde{f})(V, V') = \#\{V'' \subset k^N \mid V \subset V'' \supset V', \dim V'' = \dim V + 1 = \dim V' + 1\},$$
$$(\widetilde{f} * \widetilde{e})(V, V') = \#\{V''' \subset k^N \mid V \supset V''' \subset V', \dim V''' = \dim V - 1 = \dim V' - 1\}.$$

In particular, the both are 0 unless $\dim V = \dim V'$ and $\dim V \cap V' = \dim V - 1$ (or equivalently $\dim V + V' = \dim V + 1$). Moreover, if $V \neq V'$, then we must have $V'' = V + V'$, $V''' = V \cap V'$. This means that the both functions take values 1 on this pair (V, V'). The only remaining case is $V = V'$. We have

$$(\widetilde{e} * \widetilde{f})(V, V) = \#\{V'' \subset k^N \mid V \subset V'', \dim V'' = \dim V + 1\} = \#\mathbb{P}(k^N/V),$$
$$(\widetilde{f} * \widetilde{e})(V, V) = \#\{V''' \subset k^N \mid V \supset V''', \dim V''' = \dim V - 1\} = \#\mathbb{P}(V^*),$$

where $\mathbb{P}(\ )$ is the projective space of 1-dimensional subspace of a given vector space. We have

$$\#\mathbb{P}(k^N/V) = 1 + q + q^2 + \cdots + q^{N - \dim V - 1}, \quad \#\mathbb{P}(V^*) = 1 + q + q^2 + \cdots + q^{\dim V - 1}.$$

Thus we have

$$v^{1-N}(\widetilde{e} * \widetilde{f})(V, V) - v^{1-N}(\widetilde{f} * \widetilde{e})(V, V) = v^{1-N}\left(v^{2\dim V} + v^{2\dim V + 2} + \cdots + v^{2N - 2\dim V - 2}\right)$$
$$= v^{2\dim V - N + 1} + v^{2\dim V - N - 1} + \cdots + v^{N - 2\dim V - 1} = [N - 2\dim V]_v.$$

We define

$$e \stackrel{\text{def.}}{=} \sum_d v^{d-N} e_d, \qquad f \stackrel{\text{def.}}{=} \sum_d v^{-d} f_d,$$

and also define

$$k = \sum_d v^{N-2d} \Delta_d.$$

Then the above computation means that e, f, k satisfy the defining relation (1.3) with parameter $v = \sqrt{q}$. Thus we have an algebra homomorphism

$$\Phi\colon \mathbf{U}_{v=\sqrt{q}}(\mathfrak{sl}_2) \to \mathcal{F}(\mathcal{G} \times \mathcal{G})^{\mathrm{GL}_N(k)}.$$

This cannot be an isomorphism since $\dim \mathcal{F}(\mathcal{G} \times \mathcal{G})^{\mathrm{GL}_N(k)} < \infty$, while $\dim \mathbf{U}_{v=\sqrt{q}}(\mathfrak{sl}_2) = \infty$. However one can show that

Proposition 1.5. *The homomorphism Φ is surjective. So $\mathcal{F}(\mathcal{G} \times \mathcal{G})^{\mathrm{GL}_N(k)}$ is a quotient of $\mathbf{U}_{v=\sqrt{q}}(\mathfrak{sl}_2)$ divided by a two-sided ideal I_N.*

Consider the constant function c_d on $\mathcal{G}_d$. Then we have

$$\begin{aligned}
k * c_d &= v^{N-2d} c_d, \\
(e * c_d)(V) &= v^{d-N} \#\{V' \mid V \subset V', \dim V + 1 = \dim V' = d\} \\
&= \begin{cases} 0 & \text{if } \dim V \neq d-1, \\ v^{d-N} \#\mathbb{P}(k^N/V) & \text{otherwise}, \end{cases} \\
&= [N-d+1]_v \, c_{d-1}(V) \\
(f * c_d)(V) &= v^{-d} \#\{V' \mid V \supset V', \dim V - 1 = \dim V' = d\} \\
&= \begin{cases} 0 & \text{if } \dim V \neq d+1, \\ v^{-d} \#\mathbb{P}(V^*) & \text{otherwise}, \end{cases} \\
&= [d+1]_v \, c_{d+1}(V).
\end{aligned}$$

These equations mean that the representation $\mathcal{F}(\mathcal{G})^{\mathrm{GL}_N(k)}$ of $\mathcal{F}(\mathcal{G} \times \mathcal{G})^{\mathrm{GL}_N(k)}$ is isomorphic to the $(N+1)$-dimensional irreducible representation of $\mathbf{U}_v(\mathfrak{sl}_2)$ via the homomorphism Φ.

This example can be generalized to the quantized universal enveloping algebra $\mathbf{U}_v(\mathfrak{sl}_n)$ by considering the n-step partial flag varieties

$$\{0 = V_0 \subset V_1 \subset \cdots \subset V_n = k^N\}$$

instead of the Grassmann variety. Here the dimensions of V_i is not fixed as above. So the above variety is a disjoint union of varieties of various dimensions.

However, a generalization of this example to $\mathbf{U}_v(\mathfrak{g})$ for arbitary $\mathfrak{g}$ is still open, even for $\mathfrak{g}$ of classical type.

References

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2.1. **Convolution on cohomology.** We can replace finite sets by the Euclidean space $\mathbb{R}^m$, the summation over the finite set by the integration in the definition of the convolution product. Namely, if $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ with coordinates $(x_1, \dots, x_m)$, $(y_1, \dots, y_n)$, then a given function $K(x_1, \dots, x_m, y_1, \dots, y_n)$ on $X \times Y$ defines an operator from the space of functions on Y to the space of functions X by the formula

$$(K * f)(x_1, \dots, x_m) \overset{\text{def.}}{=} \int_{\mathbb{R}^n} K(x_1, \dots, x_m, y_1, \dots, y_n)\, f(y_1, \dots, y_n)\, dy_1 \dots dy_n.$$

The Fourier transform is an example of an operator given by the convolution. Strictly speaking, we must impose some restrictions on functions to have convergence of the integration.

We can further replace $\mathbb{R}^m$, $\mathbb{R}^n$ by oriented C^∞-manifolds and functions by differential forms on the manifolds. Let M and N be oriented C^∞-manifolds. Let $A^*(M)$, $A^*(N)$ be the vector space of all (complex valued) differential form on M and N. If a differential form K on $M \times N$ is given, we want to define an operator $A^*(N) \to A^*(M)$ by

$$A^*(N) \ni \alpha \longmapsto K * \alpha \overset{\text{def.}}{=} \int_N K \wedge \alpha \in A^*(M). \tag{2.1}$$

More precisely, $K \wedge \alpha$ is the exterior product of K and the pullback of α to $M \times N$, and $\int_N$ is the integration of $K \wedge \alpha$ over each $\{x\} \times N$. If M and N are compact, the integration is well-defined. We assume this condition for a moment. However it will be too restrictive for our later purpose.

The composite of this convolution is again a convolution of this type: If K (resp. K') is a differential form on $M \times N$ (resp. $N \times O$) with the above condition, then we have

$$K * (K' * \alpha) = \int_O \left(\int_N K \wedge K' \right) \wedge \alpha.$$

The space $A^*(M)$ of differential forms is too big. We work on the de Rham cohomology group $H^*(M)$, which is by definition, the space of closed forms modulo the space of exact forms:

$$H^k(M) \overset{\text{def.}}{=} \frac{\{\alpha \in A^k(M) \mid d\alpha = 0\}}{\{d\beta \in A^k(M) \mid \beta \in A^{k-1}(M)\}}.$$

(We always consider the cohomology group with complex coefficients. So we do not write $H^k(M, \mathbb{C})$. Moreover, all results which we will use on cohomology can be found in a standard textbook, e.g., [2].)

If K is a closed p-form on $M \times N$, then we have

$$d_M \left(\int_N K \wedge \alpha \right) = \int_N d_{M \times N} (K \wedge \alpha) = \pm \int_N K \wedge d_N \alpha,$$

where we put the suffix to the exterior differential operator in order to emphasize the manifold where the relevant differential form is defined. In particular, the convolution product maps closed (resp. exact) forms to closed (resp. exact) forms. Therefore, we have a well-defined operator

$$K * \cdot \colon H^*(N) \to H^*(M). \tag{2.2}$$

Let us consider the degree more precisely. If α is a k-form, then $K \wedge \alpha$ is $(k+p)$-form, so $\int_N K \wedge \alpha$ is $(k + p - \dim N)$-form.

Moreover, if K is written as $K = d_{M \times N} F$, then the operator on the de Rham cohomology group is 0 as

$$\int_N K \wedge \alpha = \int_N (d_{M \times N} F) \wedge \alpha = \int_N d_{M \times N} (F \wedge \alpha) = 0,$$

where we have used the Stokes theorem in the last equality. This means that the operator (2.2) depends only on the class in

$$[K] \in H^*(M \times N).$$

Take $M = N$. Then the cohomology group $H^*(M \times M)$ has a structure of an associative algebra by the convolution.

Example 2.3. Suppose that M is a compact oriented C^∞-manifold as above. By the Künneth isomorphism $H^*(M\times M) \cong H^*(M)\otimes H^*(M)$, together with the Poincaré duality $\left(H^k(M)\right)^* \cong H^{\dim M - k}(M)$, the algebra $H^*(M \times M)$ is isomorphic to the matrix algebra $\operatorname{End}(H^*(M))$.

2.2. **Borel-Moore homology.** As illustrated by above example, the condition that M is compact is restrictive, and we do not get an interesting algebra by the convolution product on cohomology groups.

If we carefully see the definition (2.1), we find that it is enough to impose the following:

the restriction of the projection $M \times N \to M$ to the support of K is proper.

Recall that a continuous map between topological spaces is *proper*, if the inverse image of a compact set is again compact. Then the above integration is convergent. Thus the operator is well-defined.

In our later examples, we have the following situation: Let Z be a fixed closed subset $Z \subset M \times N$ such that

the restriction of the projection $M \times N \to M$ to Z is proper.

Then we consider a variant of the de Rham cohomology group

$$\frac{\{K \mid d_{M\times N}K = 0, \text{the support of } K \text{ is contained in a small neighbourhood of } Z\}}{\{d_{M\times N}F \mid \text{the support of } F \text{ is contained in a small neighbourhood of } Z\}}.$$

Then the operator $H^*(N) \to H^*(M)$ is well-defined. Namely the integration is convergent, and the result is independent of the choice of the representative in the above coset.

The above definition is a little bit naive. A rigorous defintion is given by the relative cohomology group

$$H^*(M \times N, M \times N \setminus Z),$$

which is, by definition, the cohomology groups of the following complex:

$$\cdots \to A^k(M \times N) \oplus A^{k-1}(M \times N \setminus Z) \xrightarrow{\begin{bmatrix} d & 0 \\ j^* & -d \end{bmatrix}} A^{k+1}(M \times N) \oplus A^k(M \times N \setminus Z) \to \cdots,$$

where $j\colon M \times N \setminus Z \to M \times N$ is the inclusion. For most of our purpose, the above naive definition is sufficient.

For the study of the convolution product, it is more natural to consider the above cohomology group than the usual cohomology group. The above is (a variant of) the so-called Borel-Moore homology group. We give the definition and list its properties.

When X is a topological space which can be embedded as a closed subset in an oriented C^∞-manifold M, we define

$$H_k(X) \overset{\text{def.}}{=} H^{\dim M - k}(M, M \setminus X).$$

The relative cohomology group is defined as above. (**NB**: We will never use the ordinary homology group. So there is no confusion in the notation.)

We must check that the right hand side is independent of the choice of M. Let us study it in an example.

$$H_k(\mathbb{R}^n) = H^{n-k}(\mathbb{R}^n) = \begin{cases} 0 & \text{if } k \neq n, \\ \mathbb{C} & \text{if } k = n. \end{cases}$$

In particular, our Borel-Moore homology is different from the usual homology. Consider the embedding $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ as a linear subspace. So we might define as

$$H_k(\mathbb{R}^n) = H^{n+1-k}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \mathbb{R}^n).$$

Let us check that this gives us the same answer. By the Künneth theorem, the above is equal to

$$\bigoplus_{p+q=n+1-k} H^p(\mathbb{R}^n) \otimes H^q(\mathbb{R}, \mathbb{R} \setminus \{0\}).$$

So the assertion follows from

Lemma 2.4.

$$H^q(\mathbb{R}, \mathbb{R} \setminus \{0\}) = \begin{cases} 0 & \text{if } q \neq 1, \\ \mathbb{C} & \text{if } q = 1. \end{cases}$$

Since the proof is so simple. We give it.

Proof. Obviously the cohomology group vanishes unless $q = 0, 1$. Consider the case $q = 0$ first. Let $j\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the inclusion. By the above definition of the relative cohomology group, a class is represented by a closed form α such that $j^*\alpha = 0$. A closed 0-form on $\mathbb{R}$ is nothing but a constant function. And $j^*\alpha = 0$ means that the constant must be 0.

Next consider the case $q = 1$. A 1-form α on $\mathbb{R}$ is written as

$$\alpha = f(x)dx.$$

By the definition of the relative cohomology group, $H^1(\mathbb{R}, \mathbb{R} \setminus \{0\})$ is represented by a pair $(f(x)dx, g(x))$ of 1-form on $\mathbb{R}$ and a function on $\mathbb{R} \setminus \{0\}$ such that $dg = j^*f(x)dx$, i.e., $g'(x) = f(x)$ for $x \in \mathbb{R}\setminus\{0\}$. If there exists a function $F(x)$ on $\mathbb{R}$ such that $F'(x) = f(x)$, $j^*F(x) = g(x)$, then the class $(f(x)dx, g(x))$ is zero. We define a map

$$H^1(\mathbb{R}, \mathbb{R} \setminus \{0\}) \ni (f(x)dx, g(x)) \longmapsto \int_{-\varepsilon}^{\varepsilon} f(x)dx - (g(\varepsilon) - g(-\varepsilon)) \in \mathbb{C},$$

where ε is a positive real number. It is independent of the choice of the representative of the class. Namely, if $F'(x) = f(x)$, $j^*F(x) = g(x)$ for some $F(x)$, then the above is 0. Moreover, since $g'(x) = f(x)$ outside $\{0\}$, the above is independent of ε.

Obviously the map is linear and surjective. Let us show that is is injective. Define

$$F(x) = \int_{-\varepsilon}^{x} f(t)dt + g(-\varepsilon).$$

It defines a function on $\mathbb{R}$ and satisfies $d\beta = f(x)dx$. It satisfies $F(-\varepsilon) = g(-\varepsilon)$. If $(f(x)dx, g(x))$ is contained in the kernel of the above homomorphism, then it means that $F(\varepsilon) = g(\varepsilon)$. Then $(f(x)dx, g(x)) = dF$, so it is 0 as a cohomology class. □

Note that we can take a representative $(f(x)dx, g(x))$ so that its support is contained in a given arbitrary small neighbourhood of 0. In this sense, we recover the naive definition.

Our canonical isomorphism

$$H^{n-k}(\mathbb{R}^n) \to H^{n+1-k}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\})$$

is given by

$$[\alpha] \longmapsto [\alpha \wedge f(x_{n+1})dx_{n+1}, \alpha \wedge g(x_{n+1})] = [\alpha] \wedge [f(x_{n+1})dx_{n+1}, g(x_{n+1})],$$

where $(f(x)dx, g)$ is a class such that

$$\int_{-\varepsilon}^{\varepsilon} f(x)dx - (g(\varepsilon) - g(-\varepsilon)) = 1.$$

This $[f(x_{n+1})dx_{n+1}, g(x_{n+1})]$ is an example of the *Thom class.*

Theorem 2.5. *If E is an oriented C^∞ vector bundle over a C^∞-manifold M of rank r, then there exists a unique class $\Phi \in H^r(E, E \setminus M)$ such that*

$$\int_{E_x} \Phi = 1$$

for each fiber E_x of E. Here M is embedded in E as the 0-section.

This class is called the *Thom class* of E. And as above, the support of Φ is contained in arbitrary small neighbourhood of M.

If S is an oriented closed submanifold of M, then its tubular neighbourhood is diffeomorphic to the normal bundle $N_{S/M}$. We can consider the Thom class of $N_{S/M}$ as a class of $H^{\operatorname{codim} S}(M, M \setminus S)$. If X is a closed subset of S, then the homomorphism

$$H^{\dim S - k}(S, S \setminus X) \ni \alpha \mapsto \alpha \wedge \Phi \in H^{\dim M - k}(M, M \setminus X)$$

is an isomorphism. This means that two definitions of the Borel-Moore homology group $H_k(X)$, one using S and the other using M, are canonically isomorphic. Based on this result, one can prove

Proposition 2.6. *The Borel-Moore homology group $H_k(X) = H^{\dim M - k}(M, M \setminus X)$ is independent of the choice of the umbient manifold M.*

We list up properties of the Borel-Moore homology, which we will use later.

(**Fundamental class of manifolds**) Suppose M is a connected oriented C^∞ manifold. Then

$$H_k(M) = H^{\dim M - k}(M).$$

If $k = \dim M$, then a constant function on M with value 1 is a generator of $H^0(M)$. We call the corresponding element in $H_{\dim M}(M)$ the *fundamental class* of M, and denote it by $[M]$. Note that it is always nonzero. If M is not necessarily connected, its fundamental class is defined as a sum of the fundamental classes of connected components.

If S is an oriented submanifold of M, then the fundamental class $[S]$ is identified with the Thom class of the normal bundle under the two realization of the Borel-Moore homology:

$$\begin{array}{ccccccc} H^0(S) & \ni 1 & \longleftrightarrow & \Phi \in & H^{\operatorname{codim} S}(M, M \setminus S) \\ \| & \updownarrow & & \updownarrow & \| \\ H_{\dim S}(S) & \ni [S] & = & [S] \in & H_{\dim S}(S) \end{array}$$

(**Pull-back with support**) Suppose that M and N are oriented C^∞ manifolds with $\dim M = m$, $\dim N = n$, and $f \colon M \to N$ is a smooth map. If $X \subset M$, $Y \subset N$ are closed subsets with $f^{-1}(Y) \subset X$, then we have a homomorphism

$$f^* \colon H_k(Y) \to H_{k-n+m}(X)$$

as a composite

$$H^{n-k}(N, N \setminus Y) \xrightarrow{f^*} H^{n-k}(M, M \setminus f^{-1}(Y)) \to H^{m-(k-n+m)}(M, M \setminus X).$$

This map depends on manifolds M, N, f. A continuous map $\bar{f} \colon X \to Y$ does *not* necessarily induce a homomorphism $\bar{f}^* \colon H_k(Y) \to H_{k-n+m}(X)$.

In particular, we consider the following situation:

- X is an open subset of Y,
- Y is a closed subset of an oriented C^∞-manifold N.

Then we take $M = N \setminus (Y \setminus X)$, which is an open submanifold of N containing X as a closed subset. Then we have a homomorphism

$$H_k(Y) \to H_k(X).$$

(**Pushforward**) (See also Remark 2.11 below) Suppose $f\colon X \to Y$ is a *proper* continuous map. Then we have a homomorphism

$$f_*\colon H_k(X) \to H_k(Y).$$

This is defined as follows. Suppose that X (resp. Y) is embedded in $(0,1)^m$ (resp. $\mathbb{R}^n$) as a closed subset. Then the composition

$$X \xrightarrow{f \times i} Y \times (0,1)^m \to Y \times [0,1]^m \to \mathbb{R}^n \times \mathbb{R}^m$$

is a closed embedding. The properness of f is used to show that the image is closed. Thus

$$H_k(X) = H^{m+n-k}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus X).$$

We have a map

$$H^{m+n-k}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus X) \to H^{m+n-k}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus Y \times [0,1]^m),$$

i.e.,

$$H_k(X) \to H_k(Y \times [0,1]^m).$$

By the Künneth theorem, we have $H_k(Y \times [0,1]^m) = \bigoplus_{p+q=k} H_p(Y) \otimes H_q([0,1]^m)$. But it is easy to see

$$H_q([0,1]^m) \cong \begin{cases} 0 & \text{if } q \neq 0, \\ \mathbb{C} & \text{if } q = 0. \end{cases}$$

The isomorphism for $q = 0$ is given by

$$H_0([0,1]^m) = H^m(\mathbb{R}^m, \mathbb{R}^m \setminus [0,1]^m) \ni [\alpha, \beta] \mapsto \int_{\mathbb{R}^m} \alpha \in \mathbb{C},$$

where we suppose α, β have support contained in a neighbourhood of $[0,1]^m$ as before.

Thus we have a homomorphism

$$f_*\colon H_k(X) \to H_k(Y).$$

Exercise 2.7. Show that f_* is independent of various choices. Show that $(g \circ f)_* = g_* \circ f_*$.

If X is compact, then the projection $P\colon X \to$ point is proper. Thus we have a map $P_*\colon H_0(X) \to H_0(\text{point})$. But $H_0(\text{point})$ is isomorphic to $\mathbb{C}$, where the constant function on point with value 1 corresponds to 1 in $\mathbb{C}$. This map is identified with

$$H_0(X) = H^{\dim M}(M, M \setminus X) \ni [\alpha, \beta] \mapsto \int_M \alpha \in \mathbb{C},$$

where we take the representative $[\alpha, \beta]$ so that its support is contained in a small neighbourhood of X.

Exercise 2.8. Check the above assertion from the definition.

(**Long exact sequence**) Let U be an open set of X, and $Y = X \setminus U$ be the complement. Let $i\colon Y \to X$, $j\colon U \to X$ be inclusions. We have a long exact sequence

$$\cdots \to H_k(Y) \xrightarrow{i_*} H_k(X) \xrightarrow{j^*} H_k(U) \xrightarrow{\delta^*} H_{k-1}(Y) \to \cdots,$$

where δ^* is the boundary homomorphism.

(**Intersection with support**) Suppose X, Y are closed subsets of an oriented C^∞ manifold M with $\dim M = m$. Then we can define a cap product (in M)

$$\cap\colon H_k(X) \otimes H_l(Y) \to H_{k+l-m}(X \cap Y)$$

from a cup product in the relative cohomology:

$$\cup\colon H^k(M, M \setminus X) \otimes H^l(M, M \setminus Y) \to H^{k+l}(M, M \setminus (X \cap Y)).$$

Note that this depends on the umbient space M.

Exercise 2.9. Suppose that X and Y are oriented submanifolds of M. Assume that they intersect transversally. Namely, $T_xX + T_xY = T_xM$ for all $x \in X \cap Y$. Then $X \cap Y$ is an oriented manifold with dimension $\dim X + \dim Y - \dim M$, where the orientation is induced from that of X and Y. We have the following formula:

$$[X] \cap [Y] = [X \cap Y]$$

in $H_{\dim X + \dim Y - \dim M}(X \cap Y)$.

(**Self-intersection and Euler class**) We suppose X, Y are oriented submanifolds of M. We want to compute the intersection product $[X] \cap [Y]$ without assuming the intersection is transverse. The most extreme case is when $X = Y$. In this case $[X] \cap [X]$ is called *self-intersection*. Let $\Phi \in H^{\operatorname{codim} X}(M, M \setminus X)$ be the Thom class of the normal bundle. Let $\vartheta \colon H^*(M, M \setminus X) \to H^*(M)$ be the natural homomorphism, and let $i \colon X \to M$ be the inclusion. Then it is easy to check that $[X] \cap [X]$ is identified with $i^*\vartheta\Phi$ under the isomorphism $H^{\operatorname{codim} X}(X) \cong H_{\dim X - \operatorname{codim} X}(X)$. In general, the pullback of the Thom class of an oriented vector bundle E is called the *Euler class* of E. Thus $i^*\vartheta\Phi$ is the Euler class of the normal bundle.

If i' is a small pertubation of $i \colon X \to M$, then i and i' is homotopic, so the class $i^*\vartheta\Phi$ is equal to $i'^*\vartheta\Phi$. Using the above argument backwords, we find

$$i'^*\vartheta\Phi = j_*\left([i'X] \cap [X]\right),$$

where $j \colon i'X \cap X \to X$ is the inclusion. We can choose i' so that $i'X$ and X is transversal. Then the right hand side is $j_*([i'X \cap X])$. Combining all these discussions, we get

$$[X] \cap [X] = [i'X \cap X].$$

(**Fundamental class of subvarieties**) Let M be a complex manifold, and $X \subset M$ be a closed subvariety (not necessarily irreducible) with $\dim_{\mathbb{C}} X = n$.

Proposition 2.10. *We have $H_k(X) = 0$ for $k > 2n$ and $H_{2n}(X)$ has a base corresponding to irreducible components of X of dimension n.*

Proof. If X is nonsingular, this is obvious from $H_k(X) = H^{2n-k}(X)$. We prove the assertion for general case by induction on $\dim_{\mathbb{C}} X$. The case $\dim_{\mathbb{C}} X = 0$ is obvious. We have a closed subvariety $Z \subset X$ with $\dim Z < n$ such that and $X \setminus Z$ is nonsingular, of pure dimension n. We consider the long exact sequence

$$\cdots \to H_k(Z) \xrightarrow{i_*} H_k(X) \xrightarrow{j^*} H_k(X \setminus Z) \xrightarrow{\delta^*} H_{k-1}(Z) \to \cdots.$$

By the induction hypothesis, $H_k(Z) = 0$ if $k > 2(n-1)$. And we have $H_k(X \setminus Z) = 0$ if $k > 2n$, and $H_{2n}(X \setminus Z)$ has a base given by fundamental classes of its connected components. Since the connected components of $X \setminus Z$ are the irreducible components of X with dimension n, we get the assertion. □

If X is irreducible, we denote by $[X]$ the class in $H_{2n}(X)$ given by the above lemma, and call it the *fundamental class*. If X is not irreducible, its fundamental class is the sum of fundamental classes of irreducible components of dimension n.

Remark 2.11. (See also Professor Borel's note [1].) It is known that our Borel-Moore homology group $H_k(X)$ is isomorphic to homology group of infinite singular chains with locally finite support. More precisely, a formal *infinite* singular chains $\sum_i a_i\sigma_i$, where σ_i is a simplex, $a_i \in \mathbb{C}$, is called *locally finite*, if for any compact subset $D \subset X$ there are only finitely many nonzero a_i such that $D \cap \operatorname{Supp} \sigma_i \neq \emptyset$. One can define the boudary operator exactly as in the usual *finite* singular chains. It preserves the locally finiteness condition, so one can define the associated homology group. It is canonically isomorphic to our $H_k(X)$.

Moreover, it is clear that a *proper* continuous map $f\colon X\to Y$ induces a homomorphism $f_*\colon H_k(X)\to H_k(Y)$ exactly as in the case of usual homology groups, since the locally finiteness condition is preserved under the proper map f.

2.3. **Lagrangian construction of the Weyl group (due to Ginzburg).** Let M_1, M_2, M_3 be oriented C^∞ manifolds with $\dim M_i = m_i$. Let $Z_{12}\subset M_1\times M_2$, $Z_{23}\subset M_2\times M_3$ be closed subsets satisfying

> the restrictions of the projections $M_1\times M_2\to M_1$, $M_2\times M_3\to M_2$ to Z_{12}, Z_{23} are proper.

Let $p_{12}\colon M_1\times M_2\times M_3\to M_1\times M_2$, etc, be the projection. Then we can define the convolution product by

$$H_k(Z_{12})\otimes H_l(Z_{23})\ni K\otimes K' \longmapsto p_{13*}\left(p_{12}^*K\cap p_{23}^*K'\right)\in H_{k+l-m_2}(p_{13}(Z_{12}\times M_3\cap M_1\times Z_{23})).$$

More precisely, we take the cup product of $p_{12}^*K\in H_{k+m_3}(Z_{12}\times M_3)$ and $p_{23}^*K'\in H_{l+m_1}(M_1\times Z_{23})$ in $M_1\times M_2\times M_3$. Then the restriction of p_{13} to $Z_{12}\times M_3\cap M_1\times Z_{23}$ is proper by the above condition. Thus the pushforward is well-defined. Note that $p_{13}(Z_{12}\times M_3\cap M_1\times Z_{23})$ is a closed subset of $M_1\times M_3$.

Let $M = T^*\mathbb{P}^1(\mathbb{C})$, the cotangent bundle of the complex projective line. It is the set of pairs

$$T^*\mathbb{P}^1(\mathbb{C}) = \{(V,\xi)\in\mathbb{P}^1(\mathbb{C})\times\operatorname{End}(\mathbb{C}^2)\mid \xi(V)=0, \xi(\mathbb{C}^2)\subset V\}.$$

Note that ξ is nilpotent by the condition. We define the Steinberg variety

$$Z \overset{\text{def.}}{=} \{(V_1,V_2,\xi)\in\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})\times\operatorname{End}(\mathbb{C}^2)\mid (V_1,\xi),(V_2,\xi)\in T^*\mathbb{P}^1(\mathbb{C})\}.$$

It is a closed subvariety in $T^*\mathbb{P}^1(\mathbb{C})\times T^*\mathbb{P}^1(\mathbb{C})$. If $\xi\neq 0$, then $V_1 = V_2 = \operatorname{Ker}\xi$. Thus it is contained in the diagonal of $T^*\mathbb{P}^1(\mathbb{C})\times T^*\mathbb{P}^1(\mathbb{C})$. Thus Z is a union of two 2-dimensional complex submanifolds

$$\Delta_{T^*\mathbb{P}^1(\mathbb{C})}\cup\left(\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})\right),$$

where $\mathbb{P}^1(\mathbb{C})$ is contained in $T^*\mathbb{P}^1(\mathbb{C})$ as $\xi=0$ (0-section). Thus

$$H_4(Z) = \mathbb{C}[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]\oplus\mathbb{C}[\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})].$$

By the definition, the map $Z\to T^*\mathbb{P}^1(\mathbb{C})$ is proper. Hence we have the convolution product on $H_4(Z)$:

$$H_4(Z)\times H_4(Z)\ni(K,K')\longmapsto p_{13*}(p_{12}^*K\cap p_{23}^*K')\in H_4(Z),$$

where we should notice $p_{13}(Z\times M\cap M\times Z) = Z$.

Theorem 2.12. *$H_4(Z)$ is isomorphic to the group ring $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$ of the Weyl group $\mathbb{Z}/2\mathbb{Z}$ of $\mathfrak{sl}_2$.*

Proof. Let us compute the convolution product

$$[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]*[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}],\quad [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]*[\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})],$$
$$[\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})]*[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}],\quad [\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})]*[\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})].$$

The first threes are easy. The intersections are transversal, and we easily get

$$[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]*[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}] = [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}],$$
$$[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]*[\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})] = [\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})]*[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}] = [\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})].$$

Namely $[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]$ is the unit. This holds in general.

Let us consider the last one. We have

$$[\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})] * [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})] = p_{13*}\left([\mathbb{P}^1(\mathbb{C})] \times \left([\mathbb{P}^1(\mathbb{C})] \cap [\mathbb{P}^1(\mathbb{C})]\right) \times [\mathbb{P}^1(\mathbb{C})]\right)$$
$$= P_*\left([\mathbb{P}^1(\mathbb{C})] \cap [\mathbb{P}^1(\mathbb{C})]\right) [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})],$$

where $[\mathbb{P}^1(\mathbb{C})] \cap [\mathbb{P}^1(\mathbb{C})]$ is the intersection product in $M_2 = T^*(\mathbb{P}^1(\mathbb{C}))$, and $P\colon \mathbb{P}^1 \to \text{point}$ is the projection to the single point. So $P_*([\mathbb{P}^1(\mathbb{C})] \cap [\mathbb{P}^1(\mathbb{C})])$ is an element in $H_0(\text{point})$. But it is considered as a real number by the isomorphism $H_0(\text{point}) \cong \mathbb{C}$.

Exercise 2.13. Compute the self-intersection $[\mathbb{P}^1(\mathbb{C})]$ in $T^*\mathbb{P}^1(\mathbb{C})$:

$$[\mathbb{P}^1(\mathbb{C})] \cap [\mathbb{P}^1(\mathbb{C})] = -2[\text{point}],$$

where point is the fundamental class of a point in $[\mathbb{P}^1(\mathbb{C})]$. (It is independent of the choice of points.)

By this exercise, $T \overset{\text{def.}}{=} [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})] + [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]$ satisfies $T^2 = [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]$. Thus we get the assertion. □

Remark 2.14. The result of this section and that in §1.2 is deeply connected. The result of §1.2 can be reformulated by using SL_2-invariant mixed perverse sheaves on $\mathbb{P}^1$. Functions appeared in §1.2 are traces of the Frobenius homomorphism on stalks of perverse sheaves on rational points. Forgetting the mixed structure, one can formulate the result on equivariant D-modules on the complex manifold $\mathbb{P}^1(\mathbb{C})$ (the Riemann-Hilbert correspondence). It gives the group ring $\mathbb{Z}[W]$, the specialization of the Hecke algebra $\mathcal{H}_q$ at $q = 1$. There is a natural passage from D-modules to cycles in cotangent bundles, i.e., characteristic cycles.

Exercise 2.15 (See [4]). By considering the cotangent bundle of the Grassmann variety, construct $\mathbf{U}(\mathfrak{sl}_2)$.

References

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3. Symmetric groups and symmetric functions

The results of this section is well-known. See [2, 1] for example.

3.1. Let $\mathfrak{S}_n$ be the symmetric group of n letters. Let $R(\mathfrak{S}_n)$ be the complexified representation ring of $\mathfrak{S}_n$. It is defined to be the complex vector space generated by the isomorphism classes $[\rho]$ of all representations of $\mathfrak{S}_n$, modulo the subspace generated by all $[\rho\oplus\rho']-[\rho]-[\rho']$. It is isomorphic to the space of class functions:

$$\mathfrak{F}(\mathfrak{S}_n)^{\mathfrak{S}_n} \overset{\text{def.}}{=} \left\{ f\colon \mathfrak{S}_n \to \mathbb{C} \mid f(\sigma x \sigma^{-1}) = f(x) \text{ for all } x, \sigma \in \mathfrak{S}_n \right\}.$$

The isomorphism is given by the characters of representations. (Of course, this holds for any finite group G, not necessarily $\mathfrak{S}_n$.)

We define an inner product on $R(\mathfrak{S}_n)$ as usual. Namely, if we identify $R(\mathfrak{S}_n)$ with $\mathfrak{F}(\mathfrak{S}_n)^{\mathfrak{S}_n}$, then it is given by

$$(f,g) \overset{\text{def.}}{=} \frac{1}{\#\mathfrak{S}_n} \sum_{x\in\mathfrak{S}_n} f(x)\overline{g(x)}.$$

Forgetting the multiplicative structure on $R(\mathfrak{S}_n)$, we define a new multiplication on the direct sum $\bigoplus_{n=0}^{\infty} R(\mathfrak{S}_n)$ as follows. (We set $\mathfrak{S}_0 = \{1\}$.) Using the inclusion $\mathfrak{S}_m\times\mathfrak{S}_n \subset \mathfrak{S}_{m+n}$, we consider the induction functor

$$\operatorname{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}} \colon R(\mathfrak{S}_m)\otimes R(\mathfrak{S}_n) \to R(\mathfrak{S}_{m+n}).$$

This can be considered as a multiplication on $\bigoplus_n R(\mathfrak{S}_n)$. It is easy to check that this multiplication is commutative. We also have a comultiplication $\Delta\colon \bigoplus_n R(\mathfrak{S}_n) \to \bigoplus_n R(\mathfrak{S}_n) \otimes \bigoplus_n R(\mathfrak{S}_n)$, given by the restriction functor

$$\operatorname{Res}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}} \colon R(\mathfrak{S}_{m+n}) \to R(\mathfrak{S}_m)\otimes R(\mathfrak{S}_n).$$

It is also easy to check that it is cocommutative.

These operators are examples of covolution products. For $x\in\mathfrak{S}_m\times\mathfrak{S}_n$, $y\in\mathfrak{S}_{m+n}$ we define

$$Z(x,y) \overset{\text{def.}}{=} \begin{cases} \#Z_{\mathfrak{S}_{m+n}}(y) & \text{if } x \text{ and } y \text{ are conjugate in } \mathfrak{S}_{m+n}, \\ 0 & \text{otherwise,} \end{cases}$$

then the induction operator is identified with

$$f(x) \longmapsto \frac{1}{\#\mathfrak{S}_m\#\mathfrak{S}_n}(f*Z)(y) = \frac{1}{\#\mathfrak{S}_m\#\mathfrak{S}_n} \sum_{x\in\mathfrak{S}_m\times\mathfrak{S}_n} f(x)Z(x,y),$$

while the restriction opeator is identified with

$$g(y) \longmapsto \frac{1}{\#\mathfrak{S}_{m+n}}(Z*g)(x) = \frac{1}{\#\mathfrak{S}_{m+n}} \sum_{y\in\mathfrak{S}_{m+n}} Z(x,y)g(y).$$

Frobenius reciprocity $(\operatorname{Res}\rho,\rho') = (\rho,\operatorname{Ind}\rho')$ is expressed as an obvious identity

$$\#\mathfrak{S}_{m+n}(f*Z,g) = \#\mathfrak{S}_m\#\mathfrak{S}_n(f,Z*g).$$

Exercise 3.1. (1) Check the above identifications of ind/res operators.

(2) Check that $\bigoplus_n R(\mathfrak{S}_n)$ is a graded commutative/cocommutative Hopf algebra. The counit $\varepsilon\colon \bigoplus_n R(\mathfrak{S}_n)\to\mathbb{C}$ is the linear map which vanishes on $R(\mathfrak{S}_n)$ for $n\ge 1$, and $\varepsilon(1)=1$. ($R(\mathfrak{S}_0)\cong\mathbb{C}$.) The antipode $S\colon \bigoplus_n R(\mathfrak{S}_n)\to\bigoplus_n R(\mathfrak{S}_n)$ is just $-\operatorname{id}$ on $R(\mathfrak{S}_n)$ for $n\ge 1$ and id on $R(\mathfrak{S}_0)$. (cf. [3])

We would like to have a concrete description of this Hopf algebebra $\bigoplus_n R(\mathfrak{S}_n)$. For this, we relate it to the representation ring of general linear groups, and the ring of symmetric function Λ. Let V be a complex vector space. For a representation $\rho\colon \mathfrak{S}_n \to W$, we define a representation of $\mathrm{GL}(V)$ by

$$V^{\otimes n} \otimes_{\mathbb{C}[\mathfrak{S}_n]} W = \frac{V^{\otimes n} \otimes_{\mathbb{C}} W}{\mathrm{Span}\{v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \otimes x - v_1 \otimes \cdots \otimes v_n \otimes \sigma(x) \mid \sigma \in \mathfrak{S}_n\}},$$

where $\mathrm{GL}(V)$ acts through the diagonal action on $V^{\otimes n}$. ($\mathbb{C}[\mathfrak{S}_n]$ is the complexified group ring of $\mathfrak{S}_n$.) Then we consider the trace of an element $\mathrm{diag}(x_1, \dots, x_N) \in \mathrm{GL}(V)$ (where $N = \dim V$) on this representation space. It is a symmetric function on $x_1, \dots, x_N$. If we make the dimension of V large, i.e., letting $N \to \infty$, it defines an element in the projective limit, i.e., the ring of symmetric functions. For example, if $\rho\colon \mathfrak{S}_n \to W$ is the trivial representation of $\mathfrak{S}_n$, then $V^{\otimes n} \otimes_{\mathbb{C}[\mathfrak{S}_n]} W$ is the symmetric power $S^n V$, so the corresponding symmetric function is the nth complete symmetric function h_n:

$$h_n = \text{the sum of all monomials of degree } n.$$

If $\rho\colon \mathfrak{S}_n \to W$ is the sign representation of $\mathfrak{S}_n$, then $V^{\otimes n} \otimes_{\mathbb{C}[\mathfrak{S}_n]} W$ is the exterior power $\bigwedge^n V$, so the corresponding symmetric function is the nth elementary symmetric function e_n:

$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \dots x_{i_n}.$$

This map

$$\bigoplus_n R(\mathfrak{S}_n) \to \Lambda \tag{3.2}$$

is an algebra homomorphism since

$$V^{\otimes m+n} \otimes_{\mathbb{C}[\mathfrak{S}_{m+n}]} \left(\mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} W \boxtimes W'\right) = \left(V^{\otimes m} \otimes_{\mathbb{C}[\mathfrak{S}_m]} W\right) \otimes \left(V^{\otimes n} \otimes_{\mathbb{C}[\mathfrak{S}_n]} W'\right).$$

Since it is known that Λ is a polynomial ring in h_n's, so the map is surjective.

It is well-known that a conjugacy classe of $\mathfrak{S}_n$ corresponds to a partition of n: An element σ of $\mathfrak{S}_n$ is a product of cycles, and the lengths of the cycles arranged in decreasing order can be viewed as a partition of n. Let λ be a partition of n and C_λ be the corresponding conjugacy class. We identify it with its characteristic function, and consider it as an element of $\mathfrak{F}(\mathfrak{S}_n)^{\mathfrak{S}_n} = R(\mathfrak{S}_n)$. It is known that it is mapped under (3.2) to

$$\frac{\#C_\lambda}{\#\mathfrak{S}_n} p_\lambda, \qquad \text{where } p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots \text{ and } p_n = x_1^n + x_2^n + x_3^n + \cdots.$$

It is also well-known that Λ is a polynomial ring in p_n's

$$\Lambda = \mathbb{C}[p_1, p_2, \cdots].$$

Since $\{C_\lambda\}$ is a base of $\mathfrak{F}(\mathfrak{S}_n)^{\mathfrak{S}_n}$, the map (3.2) is injective.

Theorem 3.3. *The map* (3.2) *is an algebra isomorphism.*

Since we know that C_λ is mapped to $\frac{\#C_\lambda}{\#\mathfrak{S}_n} p_\lambda$, it is easy to describe the comultiplication and the inner proudct in terms of symmetric functions. They are given by

$$\Delta p_n = 1 \otimes p_n + p_n \otimes 1, \tag{3.4}$$

$$(p_\lambda, p_\mu) = \delta_{\lambda\mu} \frac{\#C_\lambda}{\#\mathfrak{S}_{|\lambda|}}, \qquad \text{where } |\lambda| = n \text{ if } \lambda \text{ is a partition of } n. \tag{3.5}$$

Exercise 3.6. (This will be used later.) Using $\Delta p_n = 1 \otimes p_n + p_n \otimes 1$, $(p_m, p_n) = m\delta_{mn}$ and $(\Delta f, g \otimes h) = (f, gh)$, show (3.5)

3.2. **The infinite dimensional Heisenberg algebra.** Next we relate Λ to a certain Lie algebra.

Let us consider Λ as a polynomial ring $\mathbb{C}[p_1, p_2, \cdots]$ of infinitely many variables. Let us define an operators $P[n]$ acting on Λ by

$$P[n] \overset{\text{def.}}{=} \begin{cases} \text{multiplication of } p_{-n} & \text{if } n < 0, \\ 0 & \text{if } n = 0, \\ n\dfrac{\partial}{\partial p_n} & \text{if } n > 0. \end{cases}$$

It is straightforward to check that $P[n]$ is the hermitian adjoint of $P[-n]$:

$$(P[n]f, g) = (f, P[-n]g).$$

It is also clear that $P[n]$'s satisfy the following relation

$$[P[m], P[n]] = m\delta_{m+n,0}\,\mathrm{id}\,.$$

Let $\mathfrak{s}$ be the Lie algebra with generators $P[n]$ ($n \in \mathbb{Z}$), K, d satisfying the defining relation

$$[P[m], P[n]] = m\delta_{m+n,0}K, \qquad [P[n], K] = 0, \qquad [d, P[n]] = nP[n].$$

This (almost commutative) Lie algebra is called the *infinite dimensional Heisenberg algebra.* The space Λ is a representation of $\mathfrak{s}$, where $P[n]$ is mapped to the operator denoted by the same symbol, and K is mapped to the identity. It is called the *Fock space.*

Digression: The *usual* Heisenberg algebra has generators $P[n]$ ($n = \pm 1$), K and the same defining relation as above.

The Lie algebra $\mathfrak{s}$ plays a fundamental role in the representation theory of the affine Lie algebra $\widehat{\mathfrak{g}}$ associated with a complex simple Lie algebra $\mathfrak{g}$, which is

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with the Lie algebra structure

$$[X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n} + m\delta_{m+n,0}(X, Y)K, \quad [\widehat{\mathfrak{g}}, K] = 0,$$
$$[d, X \otimes z^m] = mX \otimes z^m,$$

where (X, Y) is the Killing form of $\mathfrak{g}$.

In physics, $\mathfrak{s}$ is a fundamental object in string theory. We have the following dictionary:

$$\Lambda \longleftrightarrow \text{the Fock space}$$
$$\Lambda \ni 1 \longleftrightarrow \text{the vaccum vector}$$
$$P[n]\ (n < 0) \longleftrightarrow \text{creation operator}$$
$$P[n]\ (n > 0) \longleftrightarrow \text{annihilation operator}$$

Remark 3.7. We have the following nice expression of the generating function of dimensions of $R(\mathfrak{S}_n)$, in other words, the character of the Fock space:

$$\sum_{n=0}^{\infty} q^n \dim R(\mathfrak{S}_n) = \sum_{n=0}^{\infty} q^n \#\{\text{partitions of } n\} = \prod_{d=1}^{\infty} \frac{1}{1 - q^d}.$$

This is essentially Dedekind's η-function. It is an example of a modular form. But the modular invariance is completely misterious from the view point of symmetric groups.

References

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In this section, we give a geometric object whose homology group has exactly the same structure of the Hopf algebra studied in the previous section. Since the symmetric group $\mathfrak{S}_n$ played the crusial role there, one might guess that such a geometric object should be a kind of a quotient space by $\mathfrak{S}_n$. We have such a space, namely a symmetric product $S^nX = X^n/\mathfrak{S}_n$ of a C^∞-manifold X. We have a candidate for induction and restriction, namely the graph of the map

$$S^mX \times S^nX \ni (C_1, C_2) \mapsto C_1 + C_2 \in S^{m+n}X.$$

We can view it as a submanifold of $S^mX \times S^nX \times S^{m+n}X$ and can define functors by convolution. (Although S^nX is not a manifold, it is an orbifold, or a V-manifold in Satake's terminology. The rational homology group of an orbifold satisfies the Poincaré duality.) However, it turns out that the homology group of the symmetric product is far simpler than the representation ring $R(\mathfrak{S}_n)$. One of the reason of simplicity is that lots of information are lost when taking the quotient by $\mathfrak{S}_n$.

A right object is 'the Hilbert scheme of points on a complex surface', which is a resolution of singularities of the symmetric product of the surface.

Exercise 4.1. Study the Hopf algebra structure of $\bigoplus_{n=0}^{\infty} H_*(S^nX)$.

Remark 4.2. One can define the structure of a Hopf algebra on $\bigoplus_n K^{\mathfrak{S}_n}(X^n)$, where $K^{\mathfrak{S}_n}(X^n)$ is the equivariant K-theory of the n-th Cartesian product X^n. If $X = \mathbb{C}^2$, it is known that $K^{\mathfrak{S}_n}(X^n)$ is isomorphic to $R(\mathfrak{S}_n)$. However, $K^{\mathfrak{S}_n}(X^n)$ and the Hopf algebra structure can be defined for any topological space X. Thus one might have interesting examples of Hopf algebras. This observation is due to Grojnowski ([4] in the next section). An interesting example was found by Wang [10].

4.1. **Definition.** In this subsection, we define the Hilbert scheme of points on the complex plane, and study its geometric properties. We do not use a general construction due to Grothendieck, and give an elementary treatment which works only our special case.

First we do not restrct ourselves to the case when dimension is 2. Let X be the N-dimensional complex affine space $\mathbb{C}^N$. We define the Hilbert scheme of points by

$$X^{[n]} \overset{\text{def.}}{=} \{I \mid I \text{ is an ideal of } \mathbb{C}[x_1, \dots, x_N] \text{ with } \dim \mathbb{C}[x_1, \dots, x_N]/I = n\}.$$

So far, we consider $X^{[n]}$ just a set.

The Hilbert scheme $X^{[n]}$ is related to the symmetric product S^nX in the following way. If we have distinct n points $p_1, \dots, p_n$ in X, then it defines both a point in S^nX and a point in $X^{[n]}$. In fact, if we set

$$I \overset{\text{def.}}{=} \{f \in \mathbb{C}[x_1, \dots, x_N] \mid f \text{ vanishes at } p_1, \dots, p_n\},$$

it is an ideal with $\dim \mathbb{C}[x_1, \dots, x_N]/I = n$.

However, the difference occurs if some points collide. Consider the case $n = 2$. In this case, there are two types of ideals in $X^{[2]}$. The first type is an ideal given by two distinct points p, q. The other type is an ideal given by

$$I = \{f \mid f(p) = 0,\ df_p(v) = 0\} \tag{4.3}$$

for some point $p \in X$ and nonzero tangent vector $v \in T_pX$. This ideal is a limit of ideals of the first type when q approaches to p. And the information of the direction in which q approaches to p is remembered in I. In the symmetric product, the limit is simply $2p$, and this information is lost. When the number of points is greater than 2, much more complicated ideals will occur.

Exercise 4.4. Show that the Hilbert scheme $X^{[n]}$ coincides with the symmetric product S^nX when the dimension of the base space is 1.

Our definition of $X^{[n]}$ can be modified to the case when the base space is a projective space $\mathbb{P}^N$. Let $Y = \mathbb{P}^N$. We consider the homogeneous coordinate ring of $\mathbb{P}^N$, i.e.,

$$\mathbb{C}[x_0, x_1, \dots, x_N].$$

We define the Hilbert scheme $Y^{[n]}$ of points in Y by

$$\{I \mid I \text{ is a } \textit{homogeneous} \text{ ideal of } \mathbb{C}[x_0, x_1, \dots, x_N] \text{ with } \dim \mathbb{C}[x_0, x_1, \dots, x_N]/I = n\}.$$

Let $\mathbb{C}[x_0, x_1, \dots, x_N]_m$ be the degree m part of $\mathbb{C}[x_0, x_1, \dots, x_N]$, i.e., the vector space of homogeneous polynomials of degree m. Then it is clear that

$$I \cap \mathbb{C}[x_0, x_1, \dots, x_N]_m = \mathbb{C}[x_0, x_1, \dots, x_N]_m$$

if $m \geq n$. Thus I can be considered as an ideal of

$$\frac{\mathbb{C}[x_0, x_1, \dots, x_N]}{\bigoplus_{m:m\geq n} \mathbb{C}[x_0, x_1, \dots, x_N]_m}.$$

This is a finite-dimensional space ! Thus $Y^{[n]}$ is a subset of the Grassmann manifold of codimension n subspaces of the above space. For such a subspace S, the condition that it is a point in $Y^{[n]}$, i.e., it is an ideal, is just

$$x_i S \subset S \quad \text{for } i = 0, \dots, N.$$

This shows that $Y^{[n]}$ is a closed subvariety of the Grassmann manifold. This discussion is easily generalized to the case of a projective variety $X \subset \mathbb{P}^N$ (see [8]).

Exercise 4.5. Let $X = \mathbb{C}^N$, $Y = \mathbb{P}^N$. Show that $X^{[n]}$ is an open subset of $Y^{[n]}$.

We return back to the case $X = \mathbb{C}^N$. We give a *matrix description* of $X^{[n]}$. Let $V = \mathbb{C}[x_1, \dots, x_N]/I$. We define linear operators B_i on V by

$$B_i(f \bmod I) \overset{\text{def.}}{=} x_i f \bmod I.$$

We define a vector $v \in V$ as $v \overset{\text{def.}}{=} 1 \bmod I$. Then it is clear that they satisfy the following properties

(4.6.1) $[B_i, B_j] = 0$,

(4.6.2) v is a cyclic vector, i.e., if a subspace $S \subset V$ contains v and is invariant under B_i's, then it must be the whole space V.

Conversely, if a vector space V and such $(B_1, \dots, B_N, v)$ is given, we can define an ideal I as a kernel of a surjective homomorphism

$$\mathbb{C}[x_1, \dots, x_N] \ni f(x_1, \dots, x_N) \longmapsto f(B_1, \dots, B_N)v \in V.$$

Here $f(B_1, \dots, B_N)$ makes sense since $[B_i, B_j] = 0$. Moreover, the surjectivity follows from the cyclicity of v. Thus I is a point in $X^{[n]}$. This I is not changed under the action of $\mathrm{GL}(V)$ given by

$$(B_1, \dots, B_N, v) \longmapsto (gB_1g^{-1}, \dots, gB_Ng^{-1}, gv).$$

Moreover, it is easy to check that these maps are mutually inverse. We have a set-theoretical bijection

$$X^{[n]} \longleftrightarrow \{(B_1, \dots, B_N, v) \mid (4.6.1), (4.6.2)\} \,/\, \mathrm{GL}(V).$$

When a $\mathrm{GL}(V)$-orbit through $(B_1, \dots, B_N, v)$ is considered as a point in $X^{[n]}$, we denote it by $[(B_1, \dots, B_N, v)]$.

For example, consider the case $n = 2$. Since $[B_i, B_j] = 0$, we can make B_i's simultaneously into upper triangular matrices as

$$B_1 = \begin{bmatrix} x_1 & a_1 \\ 0 & y_1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} x_2 & a_2 \\ 0 & y_2 \end{bmatrix}, \qquad \dots.$$

If $(x_1, x_2, \ldots, x_N) \neq (y_1, y_2, \ldots, y_N)$, then we can simultaneously diagonalize all B_i's. This case corresponds to the ideal given by distinct two points. Suppose $(x_1, x_2, \ldots, x_N) = (y_1, y_2, \ldots, y_N)$. Then the cyclicity implies that $(a_1, a_2, \ldots, a_N) \neq (0, 0, \ldots, 0)$. Now it is not difficult to see that this case corresponds to an ideal of type (4.3) with $v = (a_1, a_2, \ldots, a_N)$.

From now on we assume $N = 2$.

Theorem 4.7. *$X^{[n]}$ is a nonsingular complex manifold of dimension $2n$.*

Proof. Let $\widetilde{X}^{[n]} \overset{\text{def.}}{=} \{(B_1, B_2, v) \mid (4.6.1), (4.6.2)\}$, i.e., $X^{[n]} = \widetilde{X}^{[n]} / \operatorname{GL}(V)$.

Step 1. We first show that $\widetilde{X}^{[n]}$ is a nonsingular complex manifold of dimension $2n + n^2$. Let

$$\mu \colon \operatorname{End}(V) \times \operatorname{End}(V) \times V \to \operatorname{End}(V)$$

be a map defined by

$$\mu(B_1, B_2, v) = [B_1, B_2].$$

Then $\widetilde{X}^{[n]}$ is an open subset of $\mu^{-1}(0)$. The differential of μ at (B_1, B_2, v) is given by

$$d\mu(\delta B_1, \delta B_2, \delta v) = [B_1, \delta B_2] + [\delta B_1, B_2].$$

It is enough to show that the cokernel of $d\mu$ is dimension n for any $(B_1, B_2, v) \in \widetilde{X}^{[n]}$. We identify the dual space of $\operatorname{End}(V)$ with itself by the inner product given by trace. Then the cokernel of $d\mu$ is

$$\begin{aligned} &\{C \in \operatorname{End}(V) \mid \operatorname{tr}(C d\mu(\delta B_1, \delta B_2, \delta v)) = 0 \text{ for all } (\delta B_1, \delta B_2, \delta v)\} \\ = &\{C \in \operatorname{End}(V) \mid [B_1, C] = [B_2, C] = 0\}. \end{aligned}$$

This space is isomorphic to V under the map $C \mapsto Cv$ thanks to the conditions (4.6.1), (4.6.2). This completes the step 1.

Step 2. Next we show that the action of $\operatorname{GL}(V)$ on $\widetilde{X}^{[n]}$ is free. Suppose that $g \in \operatorname{GL}(V)$ stabilizes (B_1, B_2, v), i.e.,

$$gB_1g^{-1} = B_1, \; gB_2g^{-1} = B_2, \; gv = v.$$

Then $S \overset{\text{def.}}{=} \operatorname{Ker}(g - 1)$ is a subspace of V which is invariant under B_1, B_2 and contains v. Hence $S = V$ by the condition (4.6.2). Thus we have $g = 1$.

Step 3. We show that every $\operatorname{GL}(V)$-orbit in $\widetilde{X}^{[n]}$ is closed. In fact, the closure of an orbit is a union of orbits, but any orbit cannot be contained in the closure of another orbit since both have the dimension $\dim \operatorname{GL}(V)$ by Step 2. In particular, the quotient space $\widetilde{X}^{[n]} / \operatorname{GL}(V)$ is Hausdorff.

Step 4. We show that a bijection

$$\operatorname{GL}(V) \times \widetilde{X}^{[n]} \to \Gamma \overset{\text{def.}}{=} \{(x, gx) \in \widetilde{X}^{[n]} \times \widetilde{X}^{[n]} \mid g \in \operatorname{GL}(V)\}$$

is a homeomorphism. Thus we want to show that the inverse of the map is continuous. Suppose

$$\lim_{i \to \infty} ((B_{1,i}, B_{2,i}, v_i), (g_i B_{1,i} g_i^{-1}, g_i B_{2,i} g_i^{-1}, g_i v_i)) = ((B_1, B_2, v), (gB_1g^{-1}, gB_2g^{-1}, gv)).$$

We need to show that g_i converges to g. Set

$$B'_{1,i} = g_i B_{1,i} g_i^{-1}, \quad B'_{2,i} = g_i B_{2,i} g_i^{-1}, \quad v'_i = g_i v_i.$$

We have

$$g_i B_{1,i} = B'_{1,i} g_i, \quad g_i B_{2,i} = B'_{2,i} g_i.$$

We consider $h_i = g_i/\|g_i\|$. Then $\|h_i\| = 1$, so we may assume that h_i converges to an endomorphism $h \in \operatorname{End}(V)$ with $\|h\| = 1$ if we replace h_i by a subsequence. Therefore, we have

$$hB_1 = gB_1g^{-1}\,h, \quad hB_2 = gB_2g^{-1}\,h.$$

Suppose $\|g_i\| \to \infty$. Then

$$hv = \lim_{i\to\infty} h_i v_i = \lim_{i\to\infty} \frac{1}{\|g_i\|} v_i' = 0.$$

This means the kernel of h contains v and invariant under B_1, B_2. Thus $h = 0$ by (4.6.2). This contradicts with $\|h\| = 1$. Therefore $\|g_i\|$ is bounded, and may assume g_i converges to $g' \in \operatorname{End}(V)$. As above, we have

$$g'B_1 = gB_1g^{-1}\,g', \quad g'B_2 = gB_2g^{-1}\,g', \quad g'v = gv.$$

By Step 2 (we do not need the invertibility of g'), we have $g = g'$. This completes the proof of Step 4.

Step 5. The rest of the proof is a standard argument (see e.g., [9, Theorem 2.9.10]). So we explain it only briefly.

Take $(B_1, B_2, v) \in \widetilde{X}^{[n]}$. Consider the deformation complex at (B_1, B_2, v):

$$\operatorname{End}(V) \xrightarrow{\iota} \operatorname{End}(V) \times \operatorname{End}(V) \times V \xrightarrow{d\mu} \operatorname{End}(V),$$

where ι is the differential of the $\operatorname{GL}(V)$-action, i.e.,

$$\iota(\xi) = ([\xi, B_1], [\xi, B_2], \xi v).$$

By the arugment in Step 1, we know that ι is injective. We can take a submanifold S of $\widetilde{X}^{[n]}$ passing through x such that

(1) its tangent space T_xS is complementary to $\operatorname{Im}\iota$,
(2) $\operatorname{GL}(V) \cdot S$ is an open subset of $\widetilde{X}^{[n]}$ and the map $\operatorname{GL}(V) \times S \to \operatorname{GL}(V) \cdot S$ is an isomorphism of complex manifolds.

We can give a structure of a complex manifold to the quotient space $\widetilde{X}^{[n]}/\operatorname{GL}(V) = X^{[n]}$ so that the natural map $\{1\} \times S \to \operatorname{GL}(V) \cdot S \to X^{[n]}$ is an isomorphism onto an open set of $X^{[n]}$. □

Exercise 4.8. Show that the above complex structure on $X^{[n]}$ is isomorphic to the complex structure induced by

$$X^{[n]} \subset Y^{[n]} \subset (\text{Grassmann manifold}),$$

where $Y = \mathbb{P}^2$ and the embedding $Y^{[n]} \subset$ (Grassmann manifold) is the one discussed above.

Remark 4.9. (1) Theorem 4.7 is originally due to Fogarty [6]. Our proof here is completely different, and somehow similar to the construction of a moduli space in the gauge theory.

(2) We will see later this description of $X^{[n]}$ is a symplectic quotient (in the category of complex manifolds). In particular, $X^{[n]}$ is a symplectic manifold. Originally this result was proved by Beauville [2].

4.2. **The Hilbert-Chow morphism and the punctual Hilbert scheme.** Let S^nX be the nth symmetric product of $X = \mathbb{C}^2$. It is an orbifold, locally isomorphic to an open set of the Euclidean space divided by an action of a finite group. In particular, it has a natural topology and complex structure. It is an affine algebraic variety, whose coordinate ring is $\mathbb{C}[\lambda_1, \mu_1, \dots, \lambda_n, \mu_n]^{\mathfrak{S}_n}$. It is known that the ring is generated by $\sum_i \lambda_i^p \mu_i^q$ for various p, q.

The symmetric product has a natural stratification indexed by partitions of n:

$$S^nX = \bigsqcup_\lambda S^n_\lambda X, \qquad \text{where } S^n_\lambda X = \left\{ \sum_i \lambda_i x_i \in S^nX \,\middle|\, x_i \neq x_j (i \neq j) \right\}.$$

For example, if $\lambda = (1^n) = (1, \dots, 1)$, then $S^n_{(1^n)}X$ is the open set consisting of distinct n points. It is a nonsingular locus of S^nX, i.e., S^nX has singularities along the complement $S^nX \setminus S^n_{(1^n)}X$. The other extreme is $\lambda = (n)$. Then $S^n_{(n)}X$ is the set of points with multiplicity n. Hence $S^n_{(n)}X$ is isomorphic to X.

Let $[(B_1, B_2, v)] \in X^{[n]}$. Since $[B_1, B_2] = 0$, we can make B_1 and B_2 simultaneously into upper triangular matrices as

$$B_1 = \begin{bmatrix} \lambda_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \mu_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mu_n \end{bmatrix}.$$

We define a map $\pi\colon X^{[n]} \to S^nX$ by

$$\pi([B_1, B_2, v]) = (\lambda_1, \mu_1) + \dots + (\lambda_n, \mu_n).$$

From the above remark on the coordinate ring of S^nX, it is clear that this is a morphism between complex analytic varieties. It is called the *Hilbert-Chow morphism.* If $[(B_1, B_2, v)]$ corresponds to an ideal given by distinct n points, then it is easy to see that the corresponding matrices B_1, B_2 are simultaneously diagonalizable, and the eigenvalues are the given points. This shows that π is an isomorphism on an open set consisting of ideals given by distinct points, i.e., $\pi^{-1}(S^n_{(1^n)}X)$.

The other extreme is the inverse image of a point in $S^n_{(n)}X$. We define

$$X_0^{[n]} \overset{\text{def.}}{=} \pi^{-1}(n0), \qquad X_*^{[n]} \overset{\text{def.}}{=} \pi^{-1}(S^n_{(n)}X)$$

where 0 is the origin of $X = \mathbb{C}^2$. The formar $X_0^{[n]}$ is called the *punctual Hilbert scheme.* These are closed subvarieties of $X^{[n]}$ and we have $X_*^{[n]} = X_0^{[n]} \times X$. If $n = 1$, $X_0^{[1]} = \{0\}$. If $n = 2$, $X_0^{[2]} \cong \mathbb{P}^1$ by the description explained in §4.1. The inverse image of the other points can be easily described. If $C \in S^n_\lambda X$, then

$$\pi^{-1}(C) \cong X_0^{[\lambda_1]} \times X_0^{[\lambda_2]} \times \cdots, \tag{4.10}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a partition of n.

It is known that

Theorem 4.11. *If $n \neq 0$, $X_0^{[n]}$ is an $(n-1)$-dimensional irreducible subvariety. (If $n = 0$, we understand $X_0^{[n]} = \{point\}$.)*

This result is due to Briançon [4] (see also Iarrobino [7]). Later we will show a weaker statement:

(4.12) There is only one $(n-1)$-dimensional irreducible component in $X_0^{[n]}$.

In fact, as was remarked in [5], the stronger statement follows from the weaker statement together with a result of Gaffney-Lazarsfeld.

Theorem 4.13. *$\pi\colon X^{[n]} \to S^nX$ is a resolution of singularities. Namely, π is a proper surjective morphism such that*

(1) *$X^{[n]}$ is nonsingular,*
(2) *π is an isomorphism on $\pi^{-1}(S^n_{(1^n)}X)$.*
(3) *$\pi^{-1}(S^n_{(1^n)}X)$ is a dense subset in $X^{[n]}$.*

Moreover, $X^{[n]}$ is irreducible.

Proof. We do not prove that π is proper yet. For example, it becomes obvious if we consider the Hilbert scheme $(\mathbb{P}^2)^{[n]}$ of points on projective plane. Then $(\mathbb{P}^2)^{[n]}$ is a projective variety since it is a subvariety of the Grassmann manifold. In particular, it is compact. We can define the Hilbert-Chow morphism $\pi\colon (\mathbb{P}^2)^{[n]} \to S^n(\mathbb{P}^2)$. It is clearly proper. Then $X^{[n]} = \pi^{-1}(S^n X)$, where $S^n X$ is an open subset of $S^n\mathbb{P}^2$. Thus the properness is clear.

Another way to show the properness is to identify $S^n X$ with the quotient in the geometric invariant theory:

$$S^n X = \mu^{-1}(0) /\!\!/ \operatorname{GL}(V).$$

Then the properness follows from a general theory in the geometric invariant theory. This argument is necessary for *quiver varieties*.

Now we check other conditions. The surjectivity of π is clear. The remaining one is the condition (3). We have proved that $X^{[n]}$ has dimension $2n$. Thus the condition (3) follows if we show that $\dim \pi^{-1}(S^n X \setminus S^n_{(1^n)} X) < 2n$. And this follows easily from the previous theorem and (4.10). Since it is clear that $S^n_{(1^n)} X$ is connected, it also implies that $X^{[n]}$ is connected. □

Remark 4.14. For the proof of this theorem, the full strength of Theorem 4.11 is not necessary. It is enough to prove the weaker statement (4.12). In fact, even weaker statement $\dim X_0^{[n]} \le n-1$ is enough. There is a very simple proof of this statement based on the symplectic geometry on $X^{[n]}$ ([Lecture, 1.13]).

Remark 4.15. Using Theorem 4.11, one can show that

$$\dim \pi^{-1}(C) = n - \#\text{ of nonzero entries in } \lambda = \frac{1}{2}\operatorname{codim} S^n_\lambda X^n, \quad C \in S^n_\lambda X^n.$$

Thus the map π is *semi-small*. This immediately gives us a formula of Betti numbers of $X^{[n]}$. (See [Lecture, Chapter 5].)

4.3. **Bialynicki-Birula decomposition associated with a $\mathbb{C}^*$-action.** We first explain a general result about a $\mathbb{C}^*$-action on a projective manifold. (In fact, the result more generally holds for a Hamiltonian S^1-action on a compact (real) symplectic manifold). Unfortunately, it is less elementary compared with results used in other sections. However one can check the result for flag varieties (Exercise 4.17). And I hope this topic will be explained in Prof. Lu's lectures......

Suppose that the multiplicative group $\mathbb{C}^*$ acts on a projective manifold M algebraically. By a theorem of Borel, the action must have a fixed point. We further assume that fixed points are isolated. So the fixed point set is a finite set $p_1, \dots, p_N$.

We define $(\pm)$-attracting sets:

$$S_i = \left\{ p \,\middle|\, \lim_{t\to 0} t\cdot p = p_i \right\},$$
$$U_i = \left\{ p \,\middle|\, \lim_{t\to \infty} t\cdot p = p_i \right\}.$$

It is known that

(1) S_i and U_i are locally closed submanifolds of M.
(2) Each S_i (resp. U_i) is isomorphic to an affine space whose dimension is equal to the dimension of the positive (resp. negative) weight space in $T_{p_i}M$ with respect to the $\mathbb{C}^*$-action. (In particular, $\dim S_i = \operatorname{codim} U_i$.)
(3) If we order the fixed point set p_i in an appropriate way, then $\bigcup_{i:i\le i_0} S_i$ (resp. $\bigcup_{i:i\le i_0} U_i$) is open (resp. closed) for each i_0.

The statements (1) and (2) are due to Bialynicki-Birula [3]. The ordering in (3) is given so that $\dim S_i \ge \dim S_j$ if $i < j$. There is a symplectic geometric approach due to Atiyah-Bott [1] where S_i and U_i are identified with stable and unstable manifolds for the gradient flow of the moment map of the action of the maximal compact subgroup $S^1 \subset \mathbb{C}^*$. In their approach, the ordering in (3) is given by the value of the moment map, and the statement (3) is clear.

Using the long exact sequence in §2.2 and $H_k(\mathbb{C}^m) = \begin{cases} \mathbb{R} & \text{if } k = 2m, \\ 0 & \text{otherwise,} \end{cases}$ one can show the following:

Theorem 4.16. *The odd homology group $H_{odd}(M)$ vanishes, and the fundamental classes of closures of S_i (or U_i) give a base of $H_{even}(M)$:*

$$H_{even}(M) = \bigoplus \mathbb{R}[\overline{S_i}], \qquad H_{even}(M) = \bigoplus \mathbb{R}[\overline{U_i}].$$

Moreover, one can easily see that

$$\overline{S_i} \cap \overline{U_j} \neq \emptyset \Longrightarrow \dim S_i \geq \operatorname{codim} U_j. \text{ And if '=' holds, then } i = j.$$

It is also clear that $\overline{S_i}$ and $\overline{U_i}$ intersect only at p_i and the intersection is transversal. Thus we can determine the intersection pairing as

$$\langle [\overline{S_i}], [\overline{U_j}] \rangle \stackrel{\text{def.}}{=} P_* \left([\overline{S_i}] \cap [\overline{U_j}] \right) = \delta_{ij},$$

where $P\colon M \to \text{point}$ is the projection, and $H_0(\text{point})$ is identified with $\mathbb{R}$ as before.

Exercise 4.17. Consider the flag variety $M = G/B$. Choosing a generic one parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ so that a point is fixed by T if and only if it is fixed by $\lambda(\mathbb{C}^*)$. Identify the fixed points set with the Weyl group W and $(\pm)$-attracting sets with Schubert cells.

Remark 4.18. One can drop the condition that fixed points are isolated. In fact, for the study of quiver varieties, one need the theory with this generality.

4.4. **Torus action on the Hilbert scheme.** We apply the theory in the previous subsection to the Hilbert scheme $X^{[n]}$ of points on $X = \mathbb{C}^2$. It is not a projective (only quasi-projective), so we cannot directly apply the theory. However, there are two ways to remedy the lack of projectivity. One is to consider the Hilbert scheme of points $(\mathbb{P}^2)^{[n]}$ on the projective plane. It has a $\mathbb{C}^*$-action and contains our $X^{[n]}$ as an invariant open subset. Thus we apply the theory to $(\mathbb{P}^2)^{[n]}$, and check that $(\pm)$-attracting sets are contained in the open subset $X^{[n]}$. Another is to use the symplectic approach to the theory. Our $X^{[n]}$ has a natural Kähler metric (in fact, it is hyper-Kähler) and the corresponding moment map is proper. Then the gradient flow technique works. This is explained in [Lecture, Chapter 5]. Anyhow, this modification is a technical matter, so we do not discuss it in full detail.

The 2-dimensional torus $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on the affine plane $X = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{C}\}$ by

$$(x_1, x_2) \mapsto (t_1 x_1, t_2 x_2) \qquad \text{for } (t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*.$$

It induces a torus action on the Hilbert scheme $X^{[n]}$. In the matrix description, it is given by

$$[(B_1, B_2, v)] \mapsto [(t_1 B_1, t_2 B_2, v)].$$

The following is obvious:

Lemma 4.19. *A fixed point of the torus action is an ideal generated by monomials in x_1, x_2.*

An ideal I generated by monomials corresponds to a Young diagram as follows. We write a monomial $x_1^i x_2^j$ at the coordinate (i, j). A monomial which is *not* contained in I is surrounded by a box. Since I is an ideal, the number of boxes in each row or each column is nonincreasing. (See Figure 1.) The corresponding matrices (B_1, B_2, v) are given as follows. The vector v is the left bottom box, B_1 maps a box to the right box, B_2 maps a box to the upper box.

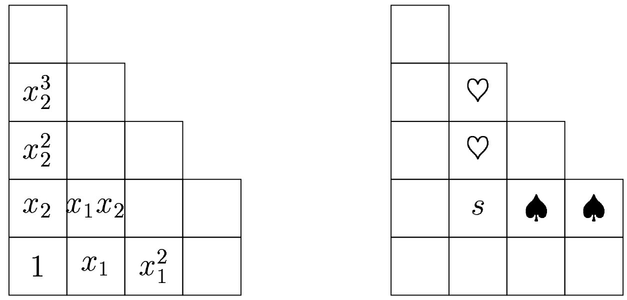

FIGURE 1. Young diagram

In particular, it is clear that the number of fixed points in $X^{[n]}$ is equal to the number of partitions of n. This gives us immediately that

$$\text{Euler number of } X^{[n]} = \#\{\text{partitions of } n\} = \dim R(\mathfrak{S}_n).$$

We need a more precise information on homology groups. Namely we study the weights of the action on the tangent space at a fixed point.

Lemma 4.20. *Let $I \in X^{[n]}$ be a fixed point of the torus action, corresponding to a Young diagram as above. Then the character of the tangent space $T_I X^{[n]}$ as a T-module is given by*

$$\sum_s (t_1^{l(s)+1} t_2^{-a(s)} + t_1^{-l(s)} t_2^{a(s)+1}),$$

where s is a box in the Young diagram, and $a(s)$ (resp. $l(s)$) is the number of boxes with ♡ (resp. ♠).

This lemma can be proved by a direct computation using the description of the tangent space given in the proof of Theorem 4.7 (see [Lecture, Chapter 5] for detail).

We choose a one-parameter subgroup

$$f(t) = (t^N, t) \in T.$$

If N is sufficiently large, then we have

$$I \text{ is fixed by the } T\text{-action} \iff I \text{ is fixed by the } f(\mathbb{C}^*)\text{-action}.$$

Let I_λ be the fixed point set corresponding to the partition λ of n. We consider the $(\pm)$-attracting set with respect to $f(\mathbb{C}^*)$, i.e.,

$$S_\lambda \overset{\text{def.}}{=} \left\{ I \in X^{[n]} \,\middle|\, \lim_{t\to 0} f(t)\cdot I \in I_\lambda \right\},$$
$$U_\lambda \overset{\text{def.}}{=} \left\{ I \in X^{[n]} \,\middle|\, \lim_{t\to \infty} f(t)\cdot I \in I_\lambda \right\}.$$

Lemma 4.21. (1) $\bigsqcup_\lambda S_\lambda = X^{[n]}$.

(2) $\bigsqcup_\lambda U_\lambda = X_0^{[n]}$.

(3) $\dim S_\lambda = n + l(\lambda)$, *where $l(\lambda)$ is the number of rows in the Young diagram corresponding to λ.*

Proof. (1) The Hilbert-Chow morphism $\pi\colon X^{[n]} \to S^n X$ is equivariant. Consider the $\mathbb{C}^*$-action in $S^n X$. Then $f(t)\cdot x \in S^n X$ converges to the origin $n0$ when $t \to 0$ for arbitray x. By a general theory, it is known that the limit of $f(t)\cdot I$ exists if and only if it stays in a compact subset. Thus any point $I \in X^{[n]}$ has a limit $\lim_{t\to 0} f(t)\cdot I$. This means the first statement.

(2) When $t \to \infty$, the flow $f(t)\cdot I$ may not have a limit. Consider the problem in $S^n X$. Then, the limit exists if and only if the point is the origin $n0$. Since π is proper, $\lim_{t\to\infty} f(t)\cdot I$ exists if and only if $I \in X_0^{[n]}$. Thus we get (2).

(3) We know that $\dim S_\lambda$ is equal to the dimension of the positive weight space of the tangent space at I_λ. The character is given by

$$\sum_s \left(t^{N(l(s)+1)-a(s)} + t^{-Nl(s)+a(s)+1}\right).$$

Recall that we choose N sufficiently large compared with n. Thus $N(l(s)+1) - a(s)$ is always positive. On the other hand, $-Nl(s) + a(s) + 1$ is positive if and only if $l(s) = 0$, i.e., s is the right end of a row. Therefore we get the result. □

Now by the discussion in the previous subsection, we get the following:

Theorem 4.22. (1) *$H_{odd}(X^{[n]}) = 0$, and*

$$\dim H_{2k}(X^{[n]}) = \#(\textit{Young diagrams with } n \textit{ boxes and } k-n \textit{ rows}).$$

In particular, $\dim H_{4n}(X^{[n]}) = 1$, so $X^{[n]}$ is irreducible. And $\dim H_{2n+2}(X^{[n]}) = 1$. Lower degree homology groups vanish.

(2) *$H_{odd}(X_0^{[n]}) = 0$ and*

$$\dim H_{2k}(X_0^{[n]}) = \#(\textit{Young diagrams with } n \textit{ boxes and } n-k \textit{ rows}).$$

In particular, $\dim H_{2n-2}(X_0^{[n]}) = 1$, and higher homology groups vanish. So irreducible components of $X_0^{[n]}$ are at most $(n-1)$-dimensional and there is only one $(n-1)$-dimensional irreducible component.

(3) *The pairing*

$$\langle\ ,\ \rangle\colon H_*(X^{[n]}) \otimes H_*(X_0^{[n]}) \ni (c, c') \mapsto P_*(c \cap c') \in H_0(point) = \mathbb{R} \tag{4.23}$$

is nondegenerate.

Remark 4.24. The results in this subsection are due to Ellingsrud and Strømme [5] and our proof is essentially the same as them.

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5. Convolution on homology groups of Hilbert schemes

5.1. Let Z be the closed subvariety of $X^{[m]} \times X^{[n]} \times X^{[m+n]}$

$$Z \overset{\text{def.}}{=} \{(I_1, I_2, I_3) \mid \pi(I_1) + \pi(I_2) = \pi(I_3)\},$$

where π is the Hilbert-Chow morphism. If the supports of I_1 and I_2 are disjoint, then I_3 must be the intersection $I_1 \cap I_2$. Thus

$$Z^\circ \overset{\text{def.}}{=} \{(I_1, I_2, I_1 \cap I_2) \mid \operatorname{Supp}(I_1) \cap \operatorname{Supp}(I_2) = \emptyset\},$$

is a connected open subset in Z, whose dimension is $2m + 2n$. Using $\dim X_0^{[n]} \le n - 1$, one can check that the complement $Z \setminus Z^\circ$ is lower dimensional. Therefore, $H_{4m+4n}(Z) = \mathbb{R}[Z]$.

Exercise 5.1. Check $\dim(Z \setminus Z^\circ) < 2m + 2n$.

Let us define the following operators by using the convolution product with respect to $[Z]$:

$$\bullet\colon H_*(X^{[m]}) \otimes H_*(X^{[n]}) \to H_*(X^{[m+n]}),$$
$$\Delta\colon H_*(X^{[m+n]}) \to H_*(X^{[m]}) \otimes H_*(X^{[n]}),$$

and

$$\bullet\colon H_*(X_0^{[m]}) \otimes H_*(X_0^{[n]}) \to H_*(X_0^{[m+n]}),$$
$$\Delta\colon H_*(X_0^{[m+n]}) \to H_*(X_0^{[m]}) \otimes H_*(X_0^{[n]}).$$

Let p_1, p_2 be the projections to the first and the second components in $\left(X^{[m]} \times X^{[n]}\right) \times X^{[m+n]}$. Then these operators are given by

$$c \bullet c' = p_{2*}\left(p_1^*(c \boxtimes c') \cap [Z]\right),$$
$$\Delta c'' = p_{1*}\left(p_2^*(c'') \cap [Z]\right).$$

Remark that the restriction of p_1, p_2 to Z is proper, so the pushforward is defined. These operators are defined by the same formula for both $H_*(X^{[n]})$ and $H_*(X_0^{[n]})$. In order to guarantee that $\bullet$, Δ are operators on $H_*(X_0^{[n]})$, we must check that

$$p_2(p_1^{-1}(X_0^{[m]} \times X_0^{[n]}) \cap Z) \subset X_0^{[m+n]}, \qquad p_1(p_2^{-1}(X_0^{[m+n]}) \cap Z) \subset X_0^{[m]} \times X_0^{[n]}.$$

But these are clear from the definition of Z.

We collect these operators and consider these as multiplication and comultiplication on the direct sums $\bigoplus_n H_*(X^{[n]})$ and $\bigoplus_n H_*(X_0^{[n]})$. The following is obvious.

Theorem 5.2. *Both $\bigoplus_n H_*(X^{[n]})$ and $\bigoplus_n H_*(X_0^{[n]})$ are graded commutative and cocommutative Hopf algebras, where the grading is given by n, not by the degree of homology groups.*

We want to give an explicit description of this Hopf algebra as in the case of $\Lambda \cong \bigoplus_n R(\mathfrak{S}_n)$. For this purpose, we use two subvarieties $X_0^{[n]}, X_*^{[n]} \subset X^{[n]}$ in §4.2.

Lemma 5.3.

$$\Delta[X_*^{[n]}] = [X_*^{[n]}] \otimes 1 + 1 \otimes [X_*^{[n]}], \qquad \Delta[X_0^{[n]}] = [X_0^{[n]}] \otimes 1 + 1 \otimes [X_0^{[n]}],$$

where 1 is the fundamental class of point $= X^{[0]}$.

Proof. Note that $[X_*^{[m+n]}] \in H_{2(m+n+1)}(X^{[m+n]})$ is, in fact, defined in $H_{2(m+n+1)}(X_*^{[m+n]})$, and considered as an element of $H_{2(m+n+1)}(X^{[m+n]})$ by the pushforward i_* of the inclusion $i\colon X_*^{[m+n]} \to X^{[m+n]}$. Then we can refine the convolution product to see

$$\Delta[X_*^{[m+n]}] \in H_{2(m+n+1)}\left(p_1\left(Z \cap p_2^{-1}(X_*^{[m+n]})\right)\right).$$

We have

$$p_1\left(Z \cap p_2^{-1}(X_*^{[m+n]})\right) \subset \{(I_1, I_2) \mid \pi(I_1) = \pi(I_2) = np \text{ for some } p \in X\}.$$

The right hand side is $(n-1)+(m-1)+2 = n+m$ dimensional if $n, m \neq 0$ by Theorem 4.11. Thus the class vanishes. The same proof works for $X_0^{[n]}$. $\square$

The lemma means that the homology classes $[X_*^{[n]}]$ and $[X_0^{[n]}]$ *cannot* be decomposed in nontrivial way. These are very similar to p_n, which is a multiple of the characteristic function of a single cycle $(12\dots n)$. This cycle cannot be decompose to a product of cycles, and this was the reason why we have $\Delta p_n = 1 \otimes p_n + p_n \otimes 1$. By a similar reason, we had the above lemma.

The pairing $\langle \ , \ \rangle$ in (4.23) satisfies

$$(5.4)\qquad \begin{aligned} &\langle c \bullet c', c''\rangle = \langle c\otimes c', \Delta c''\rangle \qquad c \in H_*(X^{[m]}),\ c' \in H_*(X^{[n]}),\ c'' \in H_*(X_0^{[m+n]}),\\ &\langle c, c' \bullet c''\rangle = \langle \Delta c, c'\otimes c''\rangle \qquad c \in H_*(X^{[m+n]}),\ c' \in H_*(X_0^{[m]}),\ c'' \in H_*(X_0^{[n]}). \end{aligned}$$

Theorem 5.5.

$$\langle [X_*^{[n]}], [X_0^{[n]}]\rangle = (-1)^{n-1} n.$$

This result is due to Ellingsrud and Strømme [2]. We give a different proof in §5.3 (cf. [Lecture, Chapter 9]). But first two cases are easy. If $n=1$, $X^{[1]} = X_*^{[1]} = X$, $X_0^{[1]} = \{0\}$. We get 1. If $n=2$, $X^{[2]} = X \times T^*\mathbb{P}^1$, $X_*^{[2]} = X\times \mathbb{P}^1$, $X_0^{[2]} = \{0\}\times\mathbb{P}^1$. Thus we get (-2) by Exercise 2.13.

We normalize the inner product

$$(\ ,\ ) \overset{\text{def.}}{=} (-1)^{n-1}\langle\ ,\ \rangle \quad \text{on } H_*(X^{[n]}) \times H_*(X_0^{[n]}).$$

Then Ellingsrud-Strømme formula and (5.4) imply

$$\left([X_*^{[\lambda_1]}]\bullet [X_*^{[\lambda_2]}]\bullet\cdots,\ [X_0^{[\mu_1]}]\bullet[X_0^{[\mu_2]}]\bullet\cdots\right) = \delta_{\lambda\mu}\frac{\# C_\lambda}{\#\mathfrak{S}_{|\lambda|}}$$

by Exercise 3.6. Here $\lambda = (\lambda_1, \lambda_2, \cdots)$, $\mu = (\mu_1, \mu_2, \cdots)$ are partitions. In particular, the classes $\left\{[X_*^{[\lambda_1]}]\bullet[X_*^{[\lambda_2]}]\bullet\cdots\right\}_\lambda$ and $\left\{[X_0^{[\lambda_1]}]\bullet[X_0^{[\lambda_2]}]\bullet\cdots\right\}_\lambda$ are linearly independent. Thus the two ring homomorphisms

$$(5.6)\qquad \begin{aligned} \Lambda = \mathbb{C}[p_1, p_2, \cdots] \ni p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots &\longmapsto [X_*^{[\lambda_1]}]\bullet[X_*^{[\lambda_2]}]\bullet\cdots \in \bigoplus_n H_*(X^{[n]}),\\ &\longmapsto [X_0^{[\lambda_1]}]\bullet[X_0^{[\lambda_2]}]\bullet\cdots \in \bigoplus_n H_*(X_0^{[n]}). \end{aligned}$$

are injective. However, we know that the both sides have the same dimension. And it is clear that it preserves the comultiplication and the inner product. Therefore

Theorem 5.7. *The homomorphisms $\Lambda \to \bigoplus_n H_*(X^{[n]})$ and $\Lambda\to\bigoplus_n H_*(X_0^{[n]})$ in (5.6) are isomorphisms of Hopf algebras.*

Remark 5.8. Let us remark that it is enough to show that $\langle [X_*^{[n]}], [X_0^{[n]}]\rangle \neq 0$ for the proof of this theorem. And this assertion follows from the nondegeneracy of the pairing (4.23). When I proved the above result first, I used this weaker result. Then I asked Ellingsrud-Strømme to compute the precise value. Theorem 5.5 was their answer.

Exercise 5.9. Recall that we have bases $\{[U_\lambda]\}_\lambda$, $\{[S_\lambda]\}_\lambda$ of $\bigoplus_n H_*(X^{[n]})$ and $\bigoplus_n H_*(X_0^{[n]})$ respectively. Show the followings:

$$[S_\lambda] = \frac{1}{\alpha_1!\alpha_2!\cdots}[X_*^{[\lambda_1]}]\bullet[X_*^{[\lambda_2]}]\bullet\cdots,$$

$$[U_\lambda] = \frac{(-1)^{|\lambda|-1}}{\lambda_1\lambda_2\cdots}[X_0^{[\lambda_1]}]\bullet[X_0^{[\lambda_2]}]\bullet\cdots,$$

where

$$\lambda = (\lambda_1, \lambda_2, \cdots) = (1^{\alpha_1} 2^{\alpha_2} \cdots).$$

5.2. We *cannot* define an inner product on $\bigoplus_n H_*(X^{[n]})$ or $\bigoplus_n H_*(X_0^{[n]})$ simply because the naive intersection pairing vanishes for $H_k(X^{[n]}) = 0$ for $k < 2n+2$, $H_k(X_0^{[n]}) = 0$ for $k > 2n-2$. So we introduce a subvariety of $X^{[n]}$ whose homology group has a natural inner product.

Let $C = \{x_2 = 0\} \subset X$ be the x_1-axis. We define a closed subvariety $L^n C$ of $X^{[n]}$ by

$$\begin{aligned} L^n C &\stackrel{\text{def.}}{=} \bigsqcup_n \{ I \in X^{[n]} \mid \pi(L) \in S^n C \} \\ &= \bigsqcup_n \{ (B_1, B_2, v) \mid B_2 \text{ is nilpotent} \}. \end{aligned}$$

We decompose $L^n C$ according to the decomposition $S^n C = \bigsqcup_\lambda S^n_\lambda C$ as

$$L^n_\lambda C \stackrel{\text{def.}}{=} \pi^{-1}(S^n_\lambda C).$$

In the matrix description, $L^n_\lambda C$ is the set such that the Jordan normal form of B_2 is λ.

Let us compute the dimensions of $L^n_\lambda C$. First note

$$L^n_{(n)} C = \pi^{-1}(S^n_{(n)} C) = \{ I \in X^{[n]} \mid \pi(I) = p \text{ for some } p \in C.$$

It is the product of $X_0^{[n]}$ and C. If $\lambda = (\lambda_1, \lambda_2, \cdots)$, then $L^n_\lambda C$ is an open subset of

$$L^{\lambda_1}_{(\lambda_1)} C \times L^{\lambda_2}_{(\lambda_2)} C \times \cdots .$$

(See Figure 2.) By Theorem 4.11, we have $\dim L^n_{(n)} C = n - 1 + 1 = n$. Therefore we have

$$\dim L^n_\lambda C = \sum_i \lambda_i = n.$$

In particular, $L^n_\lambda C$ is a middle dimensional subvariety in $X^{[n]}$. Again by Theorem 4.11, $L^n_{(n)} C$ is irreducible. Therefore $L^n_\lambda C$ is also irreducible. We have

$$H_{2n}(L^n C) = \bigoplus_\lambda \mathbb{R}[\overline{L^n_\lambda C}].$$

In order to simplify the notation, we denote simply by $[L^n_\lambda C]$, the fundamental class of $\overline{L^n_\lambda C}$ hereafter.

Exercise 5.10. Consider the $\mathbb{C}^*$-action on $X^{[n]}$ induced by the $\mathbb{C}^*$-action on X given by

$$(x_1, x_2) \mapsto (x_1, t x_2) \qquad t \in \mathbb{C}^*.$$

Study the fixed point set. (In this case, it is not isolated.) Study $(-)$-attracting sets of the Bialynicki-Birula decomposition, and identify the closure of them with $\overline{L^n_\lambda C}$.

Remark 5.11. It is known that $L^\lambda C$ is a lagrangian subvariety in the symplectic manifold $X^{[n]}$. This can be shown by using the above exercise. (See [Lecture, Chapter 7].)

By the same formula as §5.1, we can define

$$\begin{aligned} \bullet &\colon H_k(L^m C) \otimes H_l(L^n C) \to H_{k+l}(L^{m+n} C), \\ \Delta &\colon H_p(L^{m+n} C) \to \bigoplus_{k+l=p} H_k(L^m C) \otimes H_l(L^n C). \end{aligned}$$

We are interested the top degree part $\bigoplus_n H_{2n}(L^n C)$. By the above formula, this part is a Hopf subalgebra.

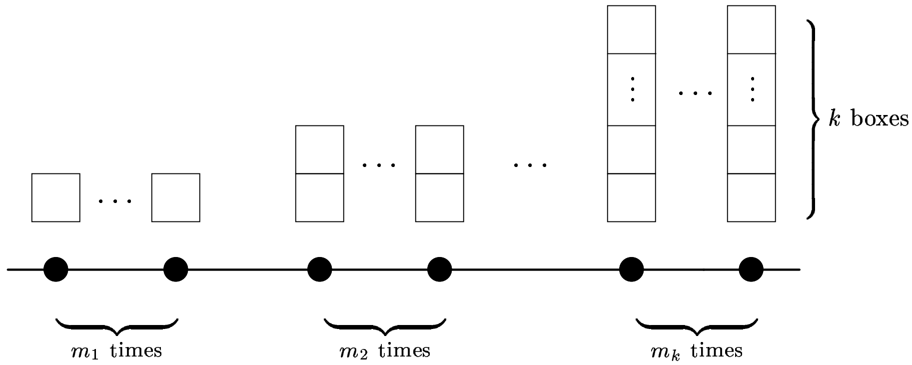

FIGURE 2. a point in $S^\lambda C$ for $\lambda = (1^{m_1} 2^{m_2} \dots n^{m_n})$

As above $\bigoplus_n H_{2n}(L^n C)$ is a commutative and cocommutative graded Hopf algebra. The following formula can be proved exactly as Lemma 5.3:

$$\Delta[L^n_{(n)}C] = [L^n_{(n)}C] \otimes 1 + 1 \otimes [L^n_{(n)}C].$$

Next we want to define an inner product on $\bigoplus_n H_{2n}(L^n C)$. A naive intersection pairing on $\bigoplus_n H_{2n}(L^n C)$ is *not* well-defined since $L^n C$ is noncompact. So we consider an involution $\tau\colon X \to X$ defined by $\tau(x_1, x_2) = (x_2, x_1)$, and the induced involution $\tau\colon X^{[n]} \to X^{[n]}$. Then if I is belonged to the intersection $L^n C \cap \tau(L^n C)$, then its support is contained in $C \cap \tau(C) = \{0\}$. Therefore $L^n C \cap \tau(L^n C)$ is compact. We define an inner product by

$$\langle c, c' \rangle \overset{\text{def.}}{=} P_*(c \cap \tau_*(c')), \qquad c, c' \in H_{2n}(L^n C),$$

where $\cap$ is taken in $X^{[n]}$, P is the projection $L^n C \cap \tau(L^n C) \to \text{point}$, and $H_0(\text{point})$ is identified with $\mathbb{R}$ as before.

From the definitions of $\bullet$, Δ and the pairing, the following is clear:

$$\langle c \bullet c', c'' \rangle = \langle c \otimes c', \Delta c'' \rangle \qquad c \in H_{2m}(L^m C),\ c' \in H_{2n}(L^n C),\ c'' \in H_{2(m+n)}(L^{m+n} C).$$

By using Ellingsrud and Strømme formula (Theorem 5.5), it is not difficult to prove

$$\langle [L^n_{(n)}C], [L^n_{(n)}C] \rangle = (-1)^{n-1} n.$$

In fact, it is equivalent to Ellingsrud and Strømme formula.

We normalize the inner product

$$(\ ,\ ) \overset{\text{def.}}{=} (-1)^{n-1} \langle\ ,\ \rangle \quad \text{on } H_{2n}(L^n C).$$

As in the previous subsection, we have

Theorem 5.12. *The map*

$$\Lambda = \mathbb{C}[p_1, p_2, \cdots] \ni p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \longmapsto [L^{\lambda_1}_{(\lambda_1)}C] \bullet [L^{\lambda_2}_{(\lambda_2)}C] \bullet \cdots \in \bigoplus_n H_*(L^n C)$$

is an isomorphism of Hopf algebras respecting the inner products.

By the discussion in §3.2, $\bigoplus_n H_{2n}(L^n C)$ is the representation of the infinite dimensional Lie algebra $\mathfrak{s}$ (the Fock space). The operators $P[n]$ are given by

$$P[n] \overset{\text{def.}}{=} \begin{cases} \text{multiplication of } [L^{-n}_{(-n)}C] & \text{if } n < 0, \\ 0 & \text{if } n = 0, \\ \text{the hermitian adjoint of } P[-n] & \text{if } n > 0. \end{cases}$$

Let us re-write these operators in terms of the convolution. For convention, we assume $m, n \in \mathbb{Z}_{>0}$ below. Let q_1, q_2 be the projections to the first and second factors of $X^{[m]} \times X^{[m+n]}$. We denote by p_{13} the projection $X^{[m]} \times X^{[n]} \times X^{[m+n]} \to X^{[m]} \times X^{[m+n]}$. The operator $P[-n]$ is given by

$$P[-n]c = q_{2*}\left(q_1^* c \cap [Z']\right),$$

where

$$[Z'] = p_{13*}\left([X^{[m]} \times L^n_{(n)}C \times X^{[m+n]}] \cap [Z]\right).$$

More explicitly, it is the fundamental class of the subvariety

$$Z' = \left\{(I_1, I_2) \in X^{[m]} \times X^{[m+n]} \mid I_1 \supset I_2,\ \pi(I_2) = \pi(I_1) + n\,p \text{ for some } p \in C\right\}$$

The operator $P[n]$ is given by

$$P[n]c = (-1)^{n-1} q_{1*}\left(q_2^* c \cap [Z'']\right),$$

where

$$[Z''] = p_{13*}\left([X^{[m]} \times L^n_{(n)}\tau(C) \times X^{[m+n]}] \cap [Z]\right).$$

Note that q_1 and q_2 are interchanged, and C is replaced by $\tau(C)$ in the formula. These description was the original one used in [Lecture, Chapter 8].

5.3. **Further study of $\bigoplus_n H_{2n}(L^nC)$.** Our next task is to describe the classes $[L^n_\lambda C]$ in terms of symmetric functions. So far, we only identified $[L^n_{(n)}C]$. For this purpose, we need to introduce another family of symmetric functions.

Let λ be a partition such that the number of nonzero entries (denoted by $l(\lambda)$) is less than or equal to N. Let

$$m_\lambda(x_1, \dots, x_N) \overset{\text{def.}}{=} \sum_{\alpha \in \mathfrak{S}_N \cdot \lambda} x_1^{\alpha_1} \cdots x_N^{\alpha_N} = \frac{1}{\#\{\sigma \in \mathfrak{S}_N \mid \sigma \cdot \lambda = \lambda\}} \sum_{\sigma \in \mathfrak{S}_N} x_1^{\lambda_{\sigma(1)}} \cdots x_N^{\lambda_{\sigma(N)}},$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ is a permutation of $\lambda = (\lambda_1, \dots, \lambda_N)$. (We allow $\lambda_k = 0$ in this notation.) If $N \le M$, we have

$$\rho^n_{M,N} m_\lambda(x_1, \dots, x_M) = m_\lambda(x_1, \dots, x_N),$$

where $\rho_{M,N}$ is the map given by setting $x_{N+1} = \cdots = x_M = 0$. Hence m_λ defines an element in Λ, which is also denoted by m_λ. It is called a *monomial symmetric function.* Clearly $\{m_\lambda\}_\lambda$ is a basis for Λ.

The previous symmetric functions are written as

$$e_n = m_{(1^n)}, \quad h_n = \sum_{|\lambda|=n} m_\lambda, \quad p_n = m_{(n)}.$$

A generating function is useful to express relations among symmetric functions. Let z be a formal variable. Let

$$E(z) \overset{\text{def.}}{=} \sum_{n=0}^\infty e_n z^n = \prod_{i=1}^\infty (1 + x_i z),$$

$$H(z) \overset{\text{def.}}{=} \sum_{n=0}^\infty h_n z^n = \prod_{i=1}^\infty \frac{1}{1 - x_i z} = E(-z)^{-1},$$

$$P(z) \overset{\text{def.}}{=} \sum_{n=1}^\infty p_n z^{n-1} = \sum_{i=1}^\infty \sum_{n=1}^\infty x_i^n z^{n-1}.$$

We have

$$P(z) = \sum_{i=1}^\infty \frac{d}{dz} \log \frac{1}{1 - x_i z} = \frac{d}{dz} \log H(z).$$

Therefore,

$$H(z) = \exp \int P(z)dz = \exp\left(\sum_{n=1}^{\infty} \frac{p_n z^n}{n}\right).$$
$$E(z) = H(-z)^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{p_n z^n}{(-1)^{n-1}n}\right).$$

Exercise 5.13. We need the following elementary lemma whose proof is left to the reader:
If we multiply the power sum p_n to the orbit sum m_ν, we get

$$p_n m_\lambda = \sum_\mu a_{\lambda\mu} m_\mu,$$

where the summation is over partitions μ of $|\lambda| + n$ which are obtained as follows: (a) add i to a term in λ, say λ_k (possibly 0), and then (b) arrange it in descending order. The coefficient $a_{\lambda\mu}$ is $\#\{l \mid \mu_l = \lambda_k + n\}$.
For example,

$$p_1 m_{(4,3,2)} = m_{(4,3,2,1)} + 2m_{(4,4,2)} + m_{(5,3,2)} + 2m_{(4,3,3)}$$
$$p_3 m_{(4,3,2)} = 2m_{(4,3,3,2)} + m_{(5,4,3)} + m_{(6,4,2)} + m_{(7,3,2)}.$$

Theorem 5.14. *Under the isomorphism in Theorem 5.12, $[L^n_\lambda C]$ corresponds to m_λ.*

Proof. We show that

$$[L^n_{(n)} C] \bullet [L^m_\lambda C] = \sum_\mu a_{\lambda\mu} [L^{m+n}_\mu C],$$

where λ (resp. μ) is a partition of m (resp. $m+n$), and $a_{\lambda\mu}$ is the coefficient in Exercise 5.13. Once this formula is shown, then the assertion follows by induction on the dominance order of λ.

Since $\left\{[L^{m+n}_\mu C]\right\}_\mu$ is a base of $H_{2(m+n)}(L^{m+n}C)$, we have the above equality for some $a_{\lambda\mu}$. We must check that this coincides with one given in Exercise 5.13. In order to determine the coefficients $a_{\lambda\mu}$, it is enough to study it in a small neighbourhood U of a generic point of $L^{m+n}_\mu C$. Namely, if $j\colon U \to X^{[n]}$ is the inclusion, then

$$j^*\left([L^n_{(n)} C] \bullet [L^m_\lambda C]\right) = a_{\lambda\mu}[L^{m+n}_\mu C \cap U].$$

Take a point $(I_1, I_2, I_3) \in Z \subset X^{[m]} \times X^{[n]} \times X^{[m+n]}$. Suppose that $I_1 \in L^m_\lambda C$, $I_2 \in L^n_{(n)} C$. Thus

$$\pi(I_1) = \sum_i \lambda_i p_i, \quad \pi(I_2) = np.$$

for some p_i, $p \in C$. We may assume that p is equal to p_k for some k, setting $\lambda_k = 0$ if p is not equal to any of p_i's. Since $\pi(I_1) + \pi(I_2) = \pi(I_3)$, we have

$$\pi(I_3) = \sum_i \mu_i p_i, \qquad \mu_i = \begin{cases} \lambda_k + n & \text{if } i = k, \\ \lambda_i & \text{otherwise.} \end{cases}$$

As remarked above, we assume I_3 is a generic point of $L^{m+n}_\mu C$, so p_i are all distinct. Then for given λ, μ, how many possible choices of k with the above equation. It is exactly $a_{\lambda\mu}$! It means that there exists $a_{\lambda\mu}$ disjoint open subsets U_1, U_2, ... , $U_{a_{\lambda\mu}}$ such that the intersection

$$\left(L^n_{(n)} C \times L^m_\lambda C \times \left(L^{m+n}_\mu C \cap U\right)\right) \cap Z$$

is contained in the union $U_1 \cup U_2 \cup \cdots \cup U_{a_{\lambda\mu}}$. Therefore our assertion follows if we show that

- the intersection is transversal in each open set U_i,

- the intersection in each U_i is isomorphic to $L^{m+n}_\mu C \cap U$ under the projection to the third factor.

One can show these by taking coordinate systems around I_2, I_3. The details are explained in [Lecture, p.112]. We omit it. □

As we promised, we give the proof of Theorem 5.5. It is enough to show

$$\langle [L^n_{(n)}C], [L^n_{(1^n)}C] \rangle = 1. \tag{5.15}$$

In fact, by the preceding theorem and the formula in the symmetric polynomial, we have

$$[L^n_{(1^n)}C] = \text{the coefficient of } z^n \text{ of } \exp\left(\sum_{n=1}^{\infty} \frac{[L^n_{(n)}C]z^n}{(-1)^{n-1}n}\right),$$

where the multiplication in the exponential is $\bullet$. Expanding the exponential and using (5.4), (5.3), we find that the nontrivial contribution is

$$\langle [L^n_{(n)}C], \frac{[L^n_{(n)}C]}{(-1)^{n-1}n} \rangle.$$

Therefore (5.15) implies Ellingsrud-Strømme's formula.

Let us show (5.15). By the definition of the inner product, we want to compute the intersection

$$[L^n_{(n)}C] \cap [\tau(L^n_{(1^n)}C)].$$

In fact, $\tau(L^n_{(1^n)}C)$, more precisely its closure $\overline{\tau(L^n_{(1^n)}C)}$ can be explicitly described. It is

$$\left\{ I = \left(x_1, x_2^n + a_{n-1}x_2^{n-1} + \cdots + a_0\right) \,\middle|\, a_0, a_1, \ldots, a_{n-1} \in \mathbb{C} \right\} \subset X^{[n]},$$

or in the matrix description,

$$\{[B_1, B_2, v] \mid B_1 = 0\}.$$

Its set-theoretical intersection with $L^n_{(n)}C$ is a single point

$$I = (x_1, x_2^n)$$

or, in the matrix description,

$$B_1 = 0, \quad B_2 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \\ \text{\Large 0} & & & & 0 \end{bmatrix}, \quad i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

One can show that the intersection is transversal, again using the coordinate system around I. (Probably this case is easier.) Thus we get the assertion.

5.4. **Further works on Hilbert schemes.** Our discussion followed along the line of Grojnowski [4], rather than that of [Lecture]. The subvarieties $L^n_\lambda C$ were defined by him, although their relation to symmetric functions were not discussed by him. The formula expressing $[L^n_{(1^n)}C]$ in terms of $[L^n_{(n)}C]$ is an example of so-called *vertex operator*. The relation between homology groups of Hilbert schemes and Borcherds' vertex algebras is *not* fully understood.

de Cataldo and Migliorini determine the Chow groups of Hilbert schemes of complex surfaces [1]. When the base space is $\mathbb{C}^2$, the Chow groups of Hilbert schemes are isomorphic to homology groups. But in general, they are different. (There is always a map from the Chow group to the homology group for any variety.)

There is a further connection between Hilbert schemes and symmetric functions, due to Haiman [5]. In particular, he proved the so-called Macdonald positivity conjecture, using the Hilbert schemes. It is a conjecture about the transition matrix between Macdonald polynomials and Schur functions. The statement of the conjecture is purely combinatorial, but so far, there is only a geometric proof.

There is an attempt to generalize Hilbert schemes to Lie algebra of other types by Ginzburg [3]. (Note that the symmetric product $S^n X$ is, essentially, the quotient of two copies of the Cartan subalgebra divided by the Weyl group of type A_{n-1}, i.e., $(\mathfrak{h} \oplus \mathfrak{h})/W$.)

It is known that a Hilbert scheme has a hyper-Kähler structure. In particular, it has a family of complex structures parametrized by $\mathbb{P}^1 = S^2$. A different complex structure was identified with the phase space for an n-particle integral system, called the Calogero-Moser system by Wilson [9]. This is a rather surprising link of Hilbert schemes to other area of mathematics !

There are some works on the ring structure of homology groups of Hilbert schemes [6, 7, 8]. For $X = \mathbb{C}^2$, the ring is isomorphic to the center of a *modified group ring* of $\mathfrak{S}_n$.

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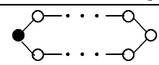
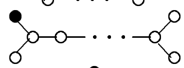
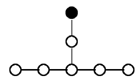
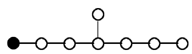
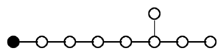
[7] W.P. Li, Z. Qin and W. Wang, *Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces*, preprint, math.AG/0009132; *Generators for the cohomology ring of Hilbert schemes of points on surfaces*, preprint, math.AG/0009167;

[8] E. Vasserot, *Sur l'anneau de cohomologie du schéma de Hilbert de* $\mathbb{C}^2$, preprint, math.AG/0009127.

[9] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, Invent. Math. **133** (1998), 1–41.

Quiver varieties were introduced in [N1]. They arised a natural generalization of moduli spaces of anti-self-dual connections of the so-called ALE spaces, studied in [6]. However, we give a different geometric description in these lectures.

6.1. **Γ-fixed point set.** Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. The classification of such subgroups has been well-known to us, since they are symmtry groups of regular polytopes via the double covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$. The classification table is the following:

type	affine Dynkin graph	group
A_n $(n \geq 0)$		$\left\{\left[\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{smallmatrix}\right] \,\middle|\, \varepsilon^{n+1} = 1\right\}$, the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$
D_n $(n \geq 4)$		the binary dihedral group of order $4(n-1)$
E_6		the binary tetrahedral group
E_7		the binary octahedral group
E_8		the binary icosahedral group

The link between this classification and that of simple simply-laced complex Lie algebra of type ADE will be explained below.

The group Γ acts on the complex plane $X = \mathbb{C}^2$, and also on the Hilbert scheme $X^{[n]}$. We want to consider the fixed point variety

$$\left(X^{[n]}\right)^\Gamma = \{I \in X^{[n]} \mid \gamma \cdot I = I \text{ for any } \gamma \in \Gamma\}.$$

If $I \in \left(X^{[n]}\right)^\Gamma$, then $\mathbb{C}[x_1, x_2]/I$ is a Γ-module.

A typical example is the ideal I of functions vanishing at points in a Γ-orbit. The action of Γ on X is free outside the origin 0, therefore the orbit consists of $\#\Gamma$-elements. The Γ-module $\mathbb{C}[x_1, x_2]/I$ is the regular representation of Γ.

Since $X^{[n]}$ is smooth and Γ is finite, the fixed point set $\left(X^{[n]}\right)^\Gamma$ is a union of nonsingular submanifolds (of various dimensions). The Γ-module structure of $\mathbb{C}[x_1, x_2]/I$ is constant along each connected component of $\left(X^{[n]}\right)^\Gamma$. For a given Γ-module V we set

$$X(V) \overset{\text{def.}}{=} \left\{ I \in \left(X^{[n]}\right)^\Gamma \,\middle|\, \mathbb{C}[x_1, x_2]/I \cong V \right\}, \quad \text{where } n = \dim V.$$

A priori, this is a union of connected components of $\left(X^{[n]}\right)^\Gamma$. However, a stronger result is known: $X(V)$ is connected. This follows from a general result for quiver varieties by Crawley-Boevey [2] (See below). But in this special case, it is also possible to prove by computing Betti numbers along the discussion in §4.4. The detail is left for the reader.

6.2. **Example.** Let us give an example. This example is due to Kronheimer [5] (in a slightly different language), Ito-Nakamura (reference [8] in §4), and Ginzburg-Kapranov (unpublished).

Suppose that V is the regular representation of Γ. Then $X(V)$ of the fixed point set in the Hilbert scheme $X^{[n]}$ $(n = \#\Gamma)$, which contains ideals consisting of functions vanishing on a Γ-orbit. The corresponding fixed point set $(S^n X)^\Gamma$ in the symmetric product is isomorphic to X/Γ in this case. Thus we have a proper morphism $\pi\colon X(V) \to X/\Gamma$, which is a resolution of singularities. This is easy to check. In fact, it is an isomorphism on $\pi^{-1}((S^n_{(1^n)}X)^\Gamma) = \pi^{-1}(X \setminus \{0\})/\Gamma)$. And $X(V)$ is connected, so the complement is lower dimensional. (There is a very simple proof of the connectedness for this $X(V)$. See [Lecture, Chatper 4].)

Since $X(V)$ is a symplectic manifold, its canonical bundle is trivial. So $X(V)$ is the so-called *minimal resolution* of X/Γ. Such a resolution is unique, and has been studied from various points of view (much before the theory of quiver varieties is developed). In particular, it is known that the inverse image $\pi^{-1}(0)$ of the origin 0 under π is a union of projective lines, whose intersection graph is a Dynkin graph of type ADE. (Delete the black vertex from the affine Dynkin graph in the table.) This is the reason why the classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ is related to the classification of simple Lie algebras.

In fact, it is possible to study the exceptional fiber $\pi^{-1}(0)$ in the language of Hilbert schemes, or quiver varieties. See Exercise 8.1.

6.3. **McKay correspondence.** McKay obtained more direct connection between finite subgroups Γ of $\mathrm{SL}_2(\mathbb{C})$ and the Dynkin diagrams [7].

Let $\rho_0, \dots, \rho_n$ be (the isomorphism classes of) irreducible representations of Γ, where ρ_0 is the trivial representation. Let Q be the 2-dimensional representation defined by the inclusion $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ as above. Then we define nonnegative integers a_{ij} by

$$Q \otimes \rho_i = \bigoplus_j \rho_j^{\oplus a_{ij}},$$

i.e., $a_{ij} = \dim \mathrm{Hom}(\rho_j, Q \otimes \rho_i)^\Gamma$. Since Q is isomorphic to its dual Q^*, we see that $a_{ij} = a_{ji}$. Then we define a graph as follows. The vertices are irreducible representations ρ_i. We draw $a_{ij} = a_{ji}$ edges between the vertices ρ_i and ρ_j. A remarkable observation due to McKay is that this graph is of type affine ADE. The black vertex corresponds to the trivial representation ρ_0. Moreover, the graph obtained by removing the black vertex is same as one given by the intersection of irreducible components of the exceptional set. The original McKay's argument was based on the explicit calculation of characters of irreducible representations of Γ. There is a geometric approach to prove this assertion, due to Gonzalez-Sprinberg and Verdier [3], and also its generalization to the case $\Gamma \subset \mathrm{SL}_3(\mathbb{C})$ [4, 1]. However, we do not go into further detail in these lectures.

The McKay observation gives us another description of $X(V)$. Let

$$V = \bigoplus V_k \otimes \rho_k$$

be decomposition of the Γ-module V, i.e., V_k is the multiplicity of ρ_k in V. We consider the matrix description (B_1, B_2, i) for a point in $X(V)$. Since $i \in V$ is given by 1 mod I, it is fixed by the Γ-action, i.e., $i \in V_0 \otimes \rho_0$. We take a base for ρ_0, and identify ρ_0 with $\mathbb{C}$. So i is an element in V_0.

We consider the pair (B_1, B_2) as an element of $\mathrm{Hom}(V, Q \otimes V)$. Then it is clear that (B_1, B_2) is contained in $(Q \otimes \mathrm{Hom}(V, V))^\Gamma$. We have

$$(Q \otimes \mathrm{Hom}(V, V))^\Gamma = \bigoplus_{k,l} \mathrm{Hom}(V_l, V_k) \otimes \mathrm{Hom}(\rho_l, Q \otimes \rho_k)^\Gamma.$$

Choose and fix a base for $\mathrm{Hom}(\rho_l, Q \otimes \rho_k)^\Gamma$ for each pair (k, l). (In fact, if the graph is not $\widetilde{A}_1$, then the space is at most one dimensional.) Collecting the bases for all k, l, we denote the union by H. To each $h \in H$, we associate an *oriented* edge in the affine Dynkin diagram from the vertex l to k, if h is an element of the base of $\mathrm{Hom}(\rho_l, Q \otimes \rho_k)^\Gamma$. In this case, we denote k by $\mathrm{in}(h)$, l by $\mathrm{out}(h)$. For every edge in the affine Dynkin diagram, we can attach *two* orientations. In particular, the number of oriented edges is twice the number of unoriented edges. We decompose (B_1, B_2) as

$$(B_1, B_2) = \bigoplus B_h \otimes h, \qquad \text{where } B_h \in \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}).$$

The figure 3 represents the data, when Γ is of type A_n, where an oriented edge h is denoted by $\mathrm{in}(h), \mathrm{out}(h)$.

An oriented graph is called a *quiver*. This description for the example in §6.2, i.e., V is the regular representation, is Kronheimer's construction [5]. The description for a general V is a special case of the quiver variety in [N1].

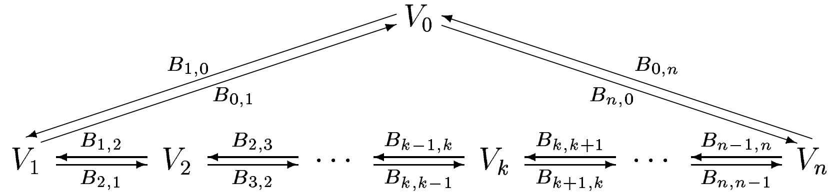


FIGURE 3

6.4. **Lagrangian subvarieties.** From now on, we assume Γ is *not* trivial subgroup $\{1\}$.

The inverse image $\pi^{-1}(0)$ of the origin plays an important role later. So we introduce the notation.

$$\mathfrak{L}(V) \overset{\text{def.}}{=} \pi^{-1}(0) \subset X(V).$$

We have

$$\mathfrak{L}(V) = X(V) \cap X_0^{[\dim V]},$$

where $X^{[\dim V]}$ is the punctual Hilbert scheme.

We have

Theorem 6.1. *$\mathfrak{L}(V)$ is a lagrangian subvariety in $X(V)$. In particular, it is middle dimensional.*

A proof, due to Lusztig, which is different from the original one, will be explained later.

When $X(V)$ is the minimal resolution of X/Γ as in an example above, $\mathfrak{L}(V)$ is the union of projective lines. So the result can be checked directly.

6.5. **Convolution.** Let

$$Z(V^1,V^2) \overset{\text{def.}}{=} \{(I^1,I^2) \in X(V^1)\times X(V^2) \mid \pi(I^1) = \pi(I^2)\}.$$

We need an explanation for the equality $\pi(I^1) = \pi(I^2)$. The left hand side is an element of $S^{\dim V^1}X$, while the right hand side is of $S^{\dim V^2}X$. For $m \le n$, we have an inclusion $S^mX \to S^nX$ defined by $C \mapsto C + (n-m)0$. We denote by $S^\infty X$ the direct limit $\lim_{n\to\infty} S^nX$. The notation π in the above equality is the composition of the previous π and the inclusion $S^{\dim V^1}X \to S^\infty X$, $S^{\dim V^2}X \to S^\infty X$. So both hand sides are elements of $S^\infty X$, and the equality makes sense.

Let $p_{ij}\colon X(V^1)\times X(V^2)\times X(V^3) \to X(V^i)\times X(V^j)$ be the projection. We define the convolution

$$*\colon H_*(Z(V^1,V^2)) \otimes H_*(Z(V^2,V^3)) \to H_*(Z(V^1,V^3))$$

by

$$c * c' \overset{\text{def.}}{=} p_{13*}\left(p_{12}^*c \cap p_{23}^*c'\right) \qquad c \in H_*(Z(V^1,V^2)),\ c' \in H_*(Z(V^2,V^3)).$$

Let us check that this is well-defined. We have

$$p_{12}^*c \in H_*(p_{12}^{-1}(Z(V^1,V^2))), \quad p_{12}^{-1}(Z(V^1,V^2)) = \{(I_1,I_2,I_3) \mid \pi(I_1) = \pi(I_2)\},$$
$$p_{23}^*c' \in H_*(p_{23}^{-1}(Z(V^2,V^3))), \quad p_{23}^{-1}(Z(V^2,V^3)) = \{(I_1,I_2,I_3) \mid \pi(I_2) = \pi(I_3)\}.$$

Therefore

$$p_{12}^*c \cap p_{23}^*c' \in H_*(p_{12}^{-1}(Z(V^1,V^2)) \cap p_{23}^{-1}(Z(V^2,V^3))),$$
$$p_{12}^{-1}(Z(V^1,V^2)) \cap p_{23}^{-1}(Z(V^2,V^3)) = \{(I_1,I_2,I_3) \mid \pi(I_1) = \pi(I_2) = \pi(I_3)\}.$$

Finally the restriction of p_{13} to $p_{12}^{-1}(Z(V^1,V^2)) \cap p_{23}^{-1}(Z(V^2,V^3))$ is proper, and the image is contained in $Z(V^1,V^3)$. Thus the convolution is well-defined.

We will be intereseted in the case when degree is middle:

$$H_{d^1+d^2}(Z(V^1,V^2)) \otimes H_{d^2+d^3}(Z(V^2,V^3)) \to H_{d^1+d^3}(Z(V^1,V^3)),$$

where $d^i = \dim X(V^i)$. For abuse of notation, we denote these degrees by 'top', although they are different for different components.

Let

$$H_{\mathrm{top}}(Z) \overset{\text{def.}}{=} \prod_{V^1,V^2}{}' H_{\mathrm{top}}(Z(V^1,V^2)),$$

(V^1, V^2 run all pairs of isomorphism classes of Γ-modules) be the subspace of the direct product $\prod_{V^1,V^2} H_{d^1+d^2}(Z(V^1,V^2))$ consisting elements (c_{V^1,V^2}) such that

- for fixed V^1, $c_{V^1,V^2} = 0$ for all but finitely many choices of V^2,
- for fixed V^2, $c_{V^1,V^2} = 0$ for all but finitely many choices of V^1.

Then the convolution is well-defined on $H_{\mathrm{top}}(Z)$, which becomes an associative algebra. The unit is the product of diagonals.

Note also that the convolution defines

$$H_{d^1+d^2}(Z(V^1,V^2)) \otimes H_{d^2}(\mathfrak{L}(V^2)) \to H_{d^1}(\mathfrak{L}(V^1)).$$

We also denote these degrees by 'top'. The direct sum

$$H_{\mathrm{top}}(\mathfrak{L}) \overset{\text{def.}}{=} \bigoplus_{V^1} H_{\mathrm{top}}(\mathfrak{L}(V^1))$$

is a representation of $H_{\mathrm{top}}(Z)$.

6.6. **A generalization of Hilbert schemes to higher rank case.** We give slight generalizations of Hilbert schemes which are worth while mentioning. All quiver varities associated with the Dynkin diagram of type ADE, and affine ADE can be described as Γ-fixed point sets in these generalizations.

Let V, W be complex vector spaces. We consider

$$\mathbf{M}(V,W) \overset{\text{def.}}{=} (Q \otimes \mathrm{Hom}(V,V)) \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W),$$

where Q is a 2-dimensional complex vector space. We give the standard symplectic form on Q, which is considered as an identification $\bigwedge^2 Q = \mathbb{C}$.

An element of $\mathbf{M}(V,W)$ is denoted by (B,i,j). We define the action of $\mathrm{GL}(V)$ on $\mathbf{M}(V,W)$ by

$$(B,i,j) \longmapsto \left((\mathrm{id}_Q \otimes g)\, B\, (\mathrm{id}_Q \otimes g^{-1}), ig^{-1}, gj\right), \qquad g \in \mathrm{GL}(V).$$

We define a map $\mu\colon \mathbf{M}(V,W) \to \mathrm{End}(V)$ by

$$\mu(B,i,j) \overset{\text{def.}}{=} [B \wedge B] + ij,$$

where $[B \wedge B]$, which is a standard notation in the differential geometry, means $[B_1,B_2]$ where $B = B_1 e_1 + B_2 e_2$ under a basis $\{e_1,e_2\}$ of Q such that $e_1 \wedge e_2$ is identified with 1 under $\bigwedge^2 Q = \mathbb{C}$. Note that the affine algebraic variety $\mu^{-1}(0)$ is invariant under the action of $\mathrm{GL}(V)$.

The symplectic form on Q and the natural pairing between $\mathrm{Hom}(V,V) \leftrightarrow \mathrm{Hom}(V,V)$, $\mathrm{Hom}(W,V) \leftrightarrow \mathrm{Hom}(V,W)$ defines a symplectic structure on $\mathbf{M}(V,W)$. The $\mathrm{GL}(V)$-action preserves the symplectic structure, and the map μ is a moment map of this action.

We say (B,i,j) is *stable* if the following condition is satisfied:

If a subspace S contains $\mathrm{Im}\, i$ and satisfies $B(S) \subset Q \otimes S$, then $S = V$.

It is clear that this condition is invariant under the action of $\mathrm{GL}(V)$.

NB. The stability condition used here is different from the one used in [N1, N2]. The stability there is equivalent to that $({}^tB, {}^tj, {}^ti)$ is stable.

We define

$$\mathfrak{M}(V, W) \overset{\text{def.}}{=} \{(B, i, j) \mid \mu(B, i, j) = 0,\ (B, i, j) \text{ is stable}\} / \mathrm{GL}(V)$$

Exactly as in Theorem 4.7, we can prove the following:

Theorem 6.2. *$\mathfrak{M}(V, W)$ is a nonsingular complex manifold of dimension $2 \dim W \times \dim V$.*

Moreover, by a general theory of symplectic quotient, $\mathfrak{M}(V, W)$ has a natural symplectic form induced from that of $\mathbf{M}(V, W)$.

A relation to $\mathfrak{M}(V, W)$ to $X^{[n]}$ is the following. Suppose $\dim W = 1$ and we fix an identification $W \cong \mathbb{C}$. Then it is not difficult to show $j = 0$ ([Lecture, 2.8]). Therefore, we recover the matrix description of $X^{[n]}$. In fact, the extra space $\mathrm{Hom}(V, W) = V^*$ was implicit in the proof of Theorem 4.7. It is the cokernel of $d\mu$.

It is known that $\mathfrak{M}(V, W)$ is isomorphic to the space parametrizing the pair (E, Φ) such that

- E is a torsion free sheaf over $\mathbb{P}^2$ of $\operatorname{rank} E = \dim W$, $c_2(E) = \dim V$ which is locally free in a neighbourhood of $\ell_\infty = \{[0 : z_1 : z_2]\}$,
- Φ is an isomorphism $E|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ (framing at infinity)

up to isomorphism. (See [Lecture, Chapter 2].)

When $\dim W = 1$, then any torsion free sheaf E of rank 1 with above condition is a subsheaf of $\mathcal{O}_{\mathbb{P}^2}$ such that $\mathcal{O}_{\mathbb{P}^2}/E$ is supported in $\mathbb{C}^2$. Therefore, we recover the Hilbert scheme of points.

We define an analogue of $S^n X$ as

$$\mathfrak{M}_0(V, W) \overset{\text{def.}}{=} \text{the set of closed } \mathrm{GL}(V)\text{-orbits in } \mu^{-1}(0),$$

This is naturally identified with the quotient space

$$\mu^{-1}(0)/\!\sim,$$

where the equivalence relation $\sim$ is defined by

$$(B, i, j) \sim (B', i', j') \iff \overline{\mathrm{GL}(V) \cdot (B, i, j)} \cap \overline{\mathrm{GL}(V) \cdot (B', i', j')} \neq \emptyset,$$

since any orbit contains the unique closed orbit in its closure. We endow the quotient topology with $\mathfrak{M}_0(V, W)$.

Unfortunately, in order to define the structure of an affine algebraic variety to $\mathfrak{M}_0(V, W)$, we need the description of $\mathfrak{M}_0(V, W)$ as affine algebro-geometric quotient:

$$\mathfrak{M}_0(V, W) = \mu^{-1}(0)/\!\!/ \mathrm{GL}(V), \qquad \text{i.e., } \operatorname{Spec} \mathfrak{M}_0(V, W) = (\operatorname{Spec} \mu^{-1}(0))^{\mathrm{GL}(V)}.$$

We have a natural map

$$\pi \colon \mathfrak{M}(V, W) \to \mathfrak{M}_0(V, W)$$

sending a orbit $\mathrm{GL}(V) \cdot (B, i, j)$ to the closed orbit in its closure. This is an analogue of the Hilbert-Chow morphism. It can be shown that this is a projective morphism. (In particular, it is proper.)

Summary. $\mathfrak{M}(V, W)$ enjoys properties which one can define the convolution product, exactly as the Hilbert scheme $X^{[n]}$.

Now we switch to the fixed point set of $\mathfrak{M}(V, W)$. We consider the 2-dimensional vector space Q as a Γ-module. The symplectic form is preserved. For Γ-modules V, W, let

$$\mathbf{M}(V, W) \overset{\text{def.}}{=} (Q \otimes \mathrm{Hom}(V, V))^\Gamma \oplus \mathrm{Hom}(W, V)^\Gamma \oplus \mathrm{Hom}(V, W)^\Gamma,$$

where $(\ )^\Gamma$ means the Γ-invariant part. When $\Gamma = \{1\}$, we recover the previous definition of $\mathbf{M}(V,W)$. (We hope that there are no confusion to use the same notation.) We have an action of $\mathrm{GL}(V)^\Gamma$ on $\mathbf{M}(V,W)$. We define the map μ as above

$$\mu\colon \mathbf{M}(V,W) \to \mathrm{End}(V)^\Gamma.$$

We define the stability as above, where the subspace $S \subset V$ is replaced by *submodules*. Then we define $\mathfrak{M}(V,W)$ and $\mathfrak{M}_0(V,W)$ exactly as above. When W is the trivial Γ-module ρ_0, then $\mathfrak{M}(V,W)$ coincides with $X(V)$.

Theorem 6.3. (1) *$\mathfrak{M}(V,W)$ is a nonsingular complex manifold, which has a symplectic form.*
(2) *$\mathfrak{M}_0(V,W)$ is an affine algebraic variety.*
(3) *$\pi\colon \mathfrak{M}(V,W) \to \mathfrak{M}_0(V,W)$ is a projective morphism.*
(4) *$\mathfrak{M}(V,W)$ is connected.*

The statement (4) is due to Crawley-Boevey [2] as we mentioned.

Let us give another description of $\mathfrak{M}(V,W)$ using the McKay correspondence. Let

$$V = \bigoplus V_k \otimes \rho_k, \qquad W = \bigoplus W_k \otimes \rho_k$$

be decomposition of Γ-modules V, W, i.e., V_k, W_k are multiplicities of ρ_k in V, W. We decompose elements $(B,i,j) \in \mathbf{M}(V,W)$ accordingly as follows. The i and j components are easy. We have

$$\mathrm{Hom}(W,V)^\Gamma = \bigoplus_k \mathrm{Hom}(W_k, V_k) \otimes \mathrm{Hom}(\rho_k, \rho_k)^\Gamma,$$
$$\mathrm{Hom}(V,W)^\Gamma = \bigoplus_k \mathrm{Hom}(V_k, W_k) \otimes \mathrm{Hom}(\rho_k, \rho_k)^\Gamma.$$

Then i, j decompose as

$$i = \bigoplus i_k \otimes \mathrm{id}_{\rho_k}, \quad j = \bigoplus j_k \otimes \mathrm{id}_{\rho_k}, \qquad \text{where } i_k \in \mathrm{Hom}(W_k, V_k), \quad j_k \in \mathrm{Hom}(V_k, W_k),$$

For B-component, we have

$$(Q \otimes \mathrm{Hom}(V,V))^\Gamma = \bigoplus_{k,l} \mathrm{Hom}(V_l, V_k) \otimes \mathrm{Hom}(\rho_l, Q \otimes \rho_k)^\Gamma.$$

Choose and fix a base for $\mathrm{Hom}(\rho_l, Q \otimes \rho_k)^\Gamma$ for each pair (k,l). (In fact, if the graph is not $\widetilde{A}_1$, then the space is at most one dimensional.) Choose bases for all k, l, we denote the union of the bases by H. We consider an element $h \in H$ as an *oriented* edge in the affine Dynkin diagram from the vertex l to k, if it is an element of the base of $\mathrm{Hom}(\rho_l, Q \otimes \rho_k)^\Gamma$. In this case, we denote k by $\mathrm{in}(h)$, l by $\mathrm{out}(h)$. For every edge in the affine Dynkin diagram, we can attach *two* orientations. In particular, the number of oriented edges is twice the number of unoriented edges. We decompose B as

$$B = \bigoplus B_h \otimes h, \qquad \text{where } B_h \in \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}).$$

The figure 4 represents the data, when Γ is of type A_n, where an oriented edge h is denoted by $\mathrm{in}(h), \mathrm{out}(h)$.

An oriented graph is called a *quiver*. This is the original description of the quiver variety in [N1].

Let $\mathrm{End}(V) \xrightarrow{\iota} Q \otimes \mathrm{End}(V) \oplus V \xrightarrow{d\mu} \mathrm{End}(V)$ be the deformation complex studied in the proof of Theorem 4.7. Then the tangent space of $X(V)$ is its Γ-invariant part:

$$(\mathrm{Ker}\, d\mu / \mathrm{Im}\, \iota)^\Gamma.$$

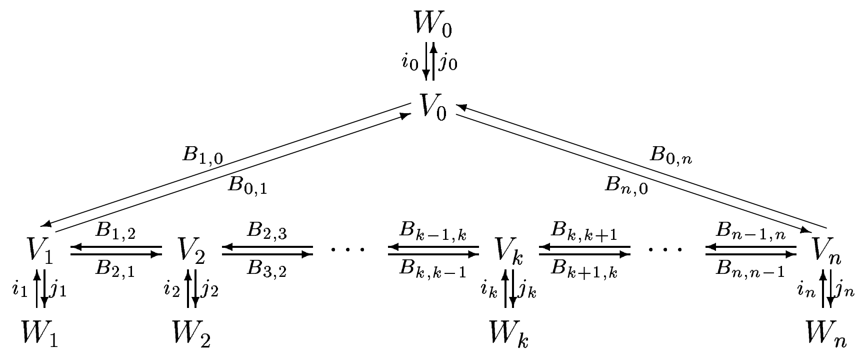


FIGURE 4

Since ι is injective and the cokernel of $d\mu$ is isomorphic to V, we have

$$\begin{aligned}\dim\,(\operatorname{Ker} d\mu/\operatorname{Im}\iota)^\Gamma &= \dim\,(Q\otimes\operatorname{End}(V)\oplus V)^\Gamma - 2\dim\operatorname{End}(V)^\Gamma + \dim V^\Gamma \\ &= 2\dim V_0 - \sum c_{ij}\dim V_i\,\dim V_j,\end{aligned} \tag{6.4}$$

where $V=\bigoplus V_i\otimes\rho_i$ as before.

6.7. **Lagrangian subvarieties.** We denote the inverse image $\pi^{-1}(0)$ of the origin by $\mathfrak{L}(V,W)$. It is not difficult to see the following (see [N1, 5.9]).

Proposition 6.5. *A point $[(B,i,j)]\in\mathfrak{M}(V,W)$ is contained in $\mathfrak{L}(V,W)$ if and only if $j=0$ and B is nilpotent.*

Furthermore, we have

Theorem 6.6. *Suppose $\Gamma\neq\{1\}$. Then $\mathfrak{L}(V,W)$ is a lagrangian subvariety in $\mathfrak{M}(V,W)$. In particular, it is middle dimensional.*

As we mentioned, we will give a proof later.

6.8. **Convolution.** Let

$$\begin{aligned}&Z(V^1,V^2;W)\\ &\overset{\text{def.}}{=}\{([B^1,i^1,j^1],[B^2,i^2,j^2])\in\mathfrak{M}(V^1,W)\times\mathfrak{M}(V^2,W)\mid \pi([B^1,i^1,j^1])=\pi([B^2,i^2,j^2])\}.\end{aligned}$$

We need an explanation for the equality $\pi([B^1,i^1,j^1])=\pi([B^2,i^2,j^2])$. The left hand side is an element of $\mathfrak{M}_0(V^1,W)$, while the right hand side is of $\mathfrak{M}_0(V^2,W)$. Extending the data by 0, we have inclusions $\mathbf{M}(V^1,W),\mathbf{M}(V^2,W)\subset\mathbf{M}(V^1\oplus V^2,W)$, which induce morphisms

$$\mathfrak{M}_0(V^1,W),\mathfrak{M}_0(V^2,W)\to\mathfrak{M}_0(V^1\oplus V^2,W).$$

The direct limit is denoted by $\mathfrak{M}_0(\infty,W)$. The notation π in the above equality is the composition of the previous π and the inclusion. So both hand sides are elements of $\mathfrak{M}_0(\infty,W)$, and the equality makes sense.

The convolution defines an operator

$$H_*(Z(V^1,V^2;W))\otimes H_*(Z(V^2,V^3;W))\to H_*(Z(V^1,V^3;W)).$$

We will be intereseted in the case when degree is middle:

$$H_{d^1+d^2}(Z(V^1,V^2;W))\otimes H_{d^2+d^3}(Z(V^2,V^3;W))\to H_{d^1+d^3}(Z(V^1,V^3;W)),$$

where $d^i=\dim\mathfrak{M}(V^i,W)$. We again write these degrees by 'top'. Let

$$H_{\text{top}}(Z(W))\overset{\text{def.}}{=}\prod_{V^1,V^2}{}' H_{\text{top}}(Z(V^1,V^2;W)),$$

(V^1, V^2 run all pairs of isomorphism classes of Γ-modules) be the subspace of the direct product $\prod_{V^1,V^2} H_{d^1+d^2}(Z(V^1,V^2;W))$ consisting elements (c_{V^1,V^2}) such that

- for fixed V^1, $c_{V^1,V^2} = 0$ for all but finitely many choices of V^2,
- for fixed V^2, $c_{V^1,V^2} = 0$ for all but finitely many choices of V^1.

Then the convolution is well-defined on $H_{\text{top}}(Z(W))$, which becomes an associative algebra. The unit is the product of diagonals.

Note also that the convolution defines

$$H_{d^1+d^2}(Z(V^1,V^2;W)) \otimes H_{d^2}(\mathfrak{L}(V^2,W)) \to H_{d^1}(\mathfrak{L}(V^1,W)).$$

Therefore, the direct sum

$$H_{\text{top}}(\mathfrak{L}(W)) \overset{\text{def.}}{=} \bigoplus_{V^1} H_{\text{top}}(\mathfrak{L}(V^1,W))$$

is a representation of $H_{\text{top}}(Z(W))$.

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We briefly recall the theory of untwisted affine Lie algebras in this section. See [1] for more detail.

7.1. **Definition.** The untwisted *affine Lie algebra* $\widehat{\mathfrak{g}}$ associated with a complex simple Lie algebra $\mathfrak{g}$ is

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with the Lie algebra structure given by

$$\begin{gathered}[\widehat{\mathfrak{g}}, K] = 0, \\ [X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n} + m\delta_{m+n,0}(X, Y)K, \\ [d, X \otimes z^m] = mX \otimes z^m,\end{gathered}$$

where (X, Y) is the Killing form of $\mathfrak{g}$. Note that $\widehat{\mathfrak{g}}$ contains $\mathfrak{g}$ as a Lie subalgebra by $\mathfrak{g} \ni X \mapsto X \otimes 1 \in \widehat{\mathfrak{g}}$.

Remark 7.1. The above definition makes sense for any Lie algbra $\mathfrak{g}$ with an invariant inner product $(\ ,\ )$. In particular, if $\mathfrak{g}$ is $\mathbb{C}$ (the 1-dimensional abelian Lie algebra) with the standard inner product, we get the infinite dimensional Heisenberg algebra, extended by d. On the Fock space $\Lambda = \bigoplus_n R(\mathfrak{S}_n)$, it acts by $\oplus - n\,\mathrm{id}_{R(\mathfrak{S}_n)}$.

We use an alternative description of $\widehat{\mathfrak{g}}$, as an example of a Kac-Moody Lie algebra. $\widehat{\mathfrak{g}}$ has generators e_i, f_i, h_i $(i = 0, 1, \dots, \operatorname{rank}\mathfrak{g})$, d and defining relations

$$[h_i, h_j] = 0, \qquad [h_i, d] = 0 \tag{7.2}$$

$$[h_i, e_j] = c_{ji}e_j, \qquad [h_i, f_j] = -c_{ji}f_j, \tag{7.3}$$

$$[d, e_i] = \delta_{0i}e_i, \qquad [d, f_i] = -\delta_{0i}f_i \tag{7.4}$$

$$[e_i, f_j] = \delta_{ij}h_i, \tag{7.5}$$

$$(\operatorname{ad} e_i)^{1-c_{ij}}\, e_j = 0, \qquad (\operatorname{ad} f_i)^{1-c_{ij}}\, f_j = 0 \qquad \text{for } i \neq j. \tag{7.6}$$

Here c_{ij} is the affine Cartan matrix. A subalgebra generated by e_i, f_i, h_i $(i \neq 0)$ is isomorphic to $\mathfrak{g}$. The isomorphism between this description and the above description is given by

$$\begin{aligned} e_i &\longleftrightarrow E_i \otimes 1, & f_i &\longleftrightarrow F_i \otimes 1, & h_i &\longleftrightarrow H_i \otimes 1, & &\text{for } i \neq 0, \\ e_0 &\longleftrightarrow E_\theta \otimes z, & f_0 &\longleftrightarrow F_\theta \otimes z^{-1}, & h_0 &\longleftrightarrow [E_\theta, F_\theta] \otimes 1 + (E_\theta, F_\theta)\, K, \end{aligned}$$

where θ is the highest root of $\mathfrak{g}$, and E_θ, F_θ are suitably normalized elements in the root spaces $\mathfrak{g}_{-\theta}$, $\mathfrak{g}_\theta$ respectively. Moreover, we denote the elements e_i, f_i, h_i $(i \neq 0)$ by E_i, F_i, H_i respectively when they are considered as elements of $\mathfrak{g}$.

Remark 7.7. The element d is called *the degree operator*. The subalgebra generated by e_i, f_i, h_i is also called an *affine Lie algebra* in some literature. It is denoted by $\mathfrak{g}'(A)$ in [1, §1.5].

Let $\mathfrak{h} = \bigoplus \mathbb{C}h_i \oplus \mathbb{C}d \subset \widehat{\mathfrak{g}}$. It is an abelian subalgebra, called the *Cartan subalgebra* of $\widehat{\mathfrak{g}}$. We define $\alpha_i \in \mathfrak{h}^*$ by

$$\langle \alpha_i, h_j \rangle = c_{ji}, \qquad \langle \alpha_i, d \rangle = \delta_{0i}.$$

The α_i are called *simple roots*.

7.2. **Integrable representations.** A $\mathfrak{g}$-module V is called a *weight module* if it admits a *weight space decomposition*: $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where

$$V_\lambda = \{v \in V \mid hv = \langle \lambda, h \rangle v \text{ for all } h \in \mathfrak{h}\}.$$

A weight module V is called a *highest weight module* of highest weight $\Lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v_\Lambda \in V$, called a *highest weight vector* such that

$$\begin{aligned} &e_i v_\Lambda = 0 \quad \text{for all } i, \\ &h v_\Lambda = \langle \Lambda, h \rangle v_\Lambda \quad \text{for all } h \in \mathfrak{h}, \\ &V = \mathbf{U}(\widehat{\mathfrak{g}}) v_\Lambda. \end{aligned}$$

For each $\Lambda \in \mathfrak{h}^*$, there exists a unique (up to isomorphism) irreducible highest weight module, denoteed by $L(\Lambda)$.

A weight module V is called *integrable* if all e_i and f_i are locally nilpotent on V. Integrable modules are counterparts of finite dimensional modules of $\mathfrak{g}$. We have the following result.

Theorem 7.8. (1) *The irreducible highest weight module $L(\Lambda)$ is integrable if and only if Λ satisfies $\langle \Lambda, h_i \rangle \in \mathbb{Z}_{\geq 0}$. (A weight Λ satisfying this condition is called* dominant.)

(2) *An integrable highest weight module V is automatically irreducbile.*

For the proof, see [1, §10].

Remark 7.9. A integrable highest weight module $L(\Lambda)$ is not *finite dimensional* unless $\Lambda = 0$.

In the next section, we give a construction of an integrable highest weight module using quiver varieties.

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8. Quiver varieties and affine Lie algebras

8.1. Hecke correspondences and Nested Hilbert schemes. A subvariety

$$X^{[n,n+1]} \overset{\text{def.}}{=} \{(I_1, I_2) \in X^{[n]} \times X^{[n+1]} \mid I_1 \supset I_2\}$$

of $X^{[n]} \times X^{[n+1]}$ is called a *nested Hilbert scheme*. A remarkable feature of the nested Hilbert scheme is that it is *nonsingular* of dimension $2n+2$. If we define a similar subvariety in $X^{[n]} \times X^{[n+k]}$ is *singular* for $k > 1$ [1, 2].

The nested Hilbert scheme can be considered as a subvariety of $X^{[n]} \times X \times X^{[n+1]}$ by

$$(I_1, I_2) \mapsto (I_1, \operatorname{Supp}(I_1/I_2), I_2).$$

Then it is clear that it is an irreducible component of Z studied in §5. Therefore the operator $P[\pm 1]$ can be represented by the convolution with respect to $X^{[n,n+1]}$:

$$P[-1]c = p_{3*}\left(p_1^* c \cap p_2^*[C] \cap [X^{[n,n+1]}]\right), \quad c \in H_{2n}(L^n C),$$
$$P[1]c = \pm p_{1*}\left([X^{[n,n+1]}] \cap p_2^*[\tau(C)] \cap p_3^* c\right), \quad c \in H_{2n+2}(L^{n+1} C),$$

where p_1, p_2, p_3 are the projections from $X^{[n]} \times X \times X^{[n+1]}$ to the first, second and third components.

We want to consider similar subvarieties for fixed point sets with respect to the action of a finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ as in the previous section. For nested Hilbert schemes, I_1, I_2 are Γ-invariant. For Hecke correspondences, V and S are Γ-modules. The condition $\dim I_1/I_2 = 1$ is *not* natural in this setting, we suppose I_1/I_2 is *irreducible* Γ*-modules.* Therefore we can define subvarieties for each ρ_i:

$$\mathfrak{P}_i(V) \overset{\text{def.}}{=} \{(I_1, I_2) \in X(V \ominus \rho_i) \times X(V) \mid I_1 \supset I_2\},$$

The *non-standard* notation $V \ominus \rho_i$ means that it is (an isomorphism class of) a representation V' of Γ such that $V' \oplus \rho_i$ is isomorphic to V. If such a representation does not exist, then we define $X(V \ominus \rho_i)$ to be the empty set. Note that $\operatorname{Supp}(I_1/I_2) = 0$. So we do not have the second component X in this case.

Exercise 8.1. Consider $\mathfrak{P}_i(V)$ when V is a regular representation as in §6.2. Show that $X(V \ominus \rho_i)$ is a single point and $\mathfrak{P}_i(V)$ is isomorphic to a projective line. Also study the intersection $\mathfrak{P}_i(V) \cap \mathfrak{P}_j(V)$ and identify the intersection pairing with the tensor product decomposition in the McKay correspondence. (See also the remark in the end of §6.4).

We give a generalization following [N2, §5]. Let V, W be Γ-modules as above. Let $\mathfrak{P}(V, W)$ be the set of all pairs (B, i, j) and S modulo $\mathrm{GL}(V)^\Gamma$-action satisfying the following conditions

- $(B, i, j) \in \mu^{-1}(0)$ is stable, and
- S is a Γ-submodule of V, isomorphic to ρ_i, such that $B(S) \subset Q \otimes S$, $S \subset \operatorname{Ker} j$

If such a subspace is given, then (B, i, j) induces an element in $\mathbf{M}(V/S, W)$. The element, denoted by (B', i', j'), satisfies $\mu(B', i', j') = 0$ and the stability condition. Moreover, the construction is compatible with the actions of $\mathrm{GL}(V)^\Gamma$ and $\mathrm{GL}(V/S)^\Gamma$. Therefore we have a map

$$\mathfrak{P}(V, W) \to \mathfrak{M}(V/S, W) \times \mathfrak{M}(V, W).$$

If $\Gamma = \{1\}$, the trace of the restriction of B to S defines an element of $Q = \mathbb{C}^2 = X$. Therefore, we have

$$\mathfrak{P}(V, W) \to \mathfrak{M}(V/S, W) \times X \times \mathfrak{M}(V, W).$$

It is not difficult to show that these are embeddings, using the stability condition.

Definition 8.2. We call $\mathfrak{P}(V, W)$ the *Hecke correspondence.*

The relation to the Hecke correspondence and the nested Hilbert scheme is the following. If $I_1 \supset I_2$, then we have a surjective homomorphism of $\mathbb{C}[x_1, x_2]$-algebras

$$\frac{\mathbb{C}[x_1,x_2]}{I_2} \to \frac{\mathbb{C}[x_1,x_2]}{I_1}.$$

The kernel is a 1-dimension subspace of $\mathbb{C}[x_1, x_2]/I_2$ invariant under the multiplications of x_1, x_2. Thus $\mathfrak{P}(V, W)$ is the nested Hilbert scheme in the special case $\dim W = 1$.

Under the identification of $\mathfrak{M}(V, W)$ with the moduli space of pairs (E, Φ), the Hecke correspondence is identified with the space of $((E_1, \Phi_1), (E_2, \Phi_2))$ such that

- $E_1 \supset E_2$, and the inclusion is an isomorphism outside a compact subset of $\mathbb{C}^2$,
- the framings Φ_1, Φ_2 at ℓ_∞ are the same under the above isomorphism $E_1 \cong E_2$,

up to isomorphisms.

The following result was proved in [N2, §5].

Theorem 8.3. *$\mathfrak{P}(V, W)$ is a nonsingular complex manifold. Moreover, it is lagrangian in $\mathfrak{M}(V, W)$ if $\Gamma \neq \{1\}$.*

8.2. We assume $\Gamma \neq \{1\}$ from now.

We define an algebra homomorphism $\mathbf{U}(\widehat{\mathfrak{g}}) \to H_{\mathrm{top}}(Z(W))$. We define the image of generators and check the defining relations. We set

$$\begin{gathered} d \mapsto \prod_V (-\dim V_0)\ \left[\Delta_{\mathfrak{M}(V,W)}\right], \\ h_i \mapsto \prod_V (\dim W_i - \sum_j c_{ij} \dim V_j)\ \left[\Delta_{\mathfrak{M}(V,W)}\right], \\ e_i \mapsto \prod_V [\mathfrak{P}_i(V,W)], \qquad f_i \mapsto (-1)^{r(V,W)} \prod_V [\omega \mathfrak{P}_i(V,W)] \end{gathered}$$

where $\Delta_{\mathfrak{M}(V,W)}$ is the diagonal in $\mathfrak{M}(V, W) \times \mathfrak{M}(V, W)$, $\omega\colon \mathfrak{M}(V \ominus \rho_i, W) \times \mathfrak{M}(V, W) \to \mathfrak{M}(V, W) \times \mathfrak{M}(V \ominus \rho_i, W)$ is the interchange of the factors, and $r(V, W) = \frac{1}{2}(\dim \mathfrak{M}(V \ominus \rho_i, W) - \dim \mathfrak{M}(V, W))$.

Theorem 8.4. *The above assignment extends (uniquely) to an algebra homomorphism $\mathbf{U}(\widehat{\mathfrak{g}}) \to H_{\mathrm{top}}(Z(W))$.*

It is clear that d and h_i's make a commuting family. Thus we have the relation (7.2).

Since

$$\left[\Delta_{\mathfrak{M}(V\ominus\rho_i,W)}\right] * [\mathfrak{P}_i(V,W)] = [\mathfrak{P}_i(V,W)] * \left[\Delta_{\mathfrak{M}(V,W)}\right],$$

the relations (7.3, 7.4) follow.

Thus the relations (7.5, 7.6) are remained to be checked.

Before checking these relations, we introduce a complex which will play an important role later. For each point in $[(B, i, j)] \in \mathfrak{M}(V, W)$, consider

$$V \xrightarrow[\alpha]{\begin{bmatrix} B_1 \\ B_2 \\ j \end{bmatrix}} Q \otimes V \oplus W \xrightarrow[\beta]{[-B_1 B_2 i]} V. \tag{8.5}$$

If W is the trivial representation, this can be written in terms of the ideal $I \in X(V)$:

$$\begin{array}{ccccc} \mathbb{C}[x_1,x_2]/I & \xrightarrow{\alpha} & Q\otimes \mathbb{C}[x_1,x_2]/I \oplus \mathbb{C} & \xrightarrow{\beta} & \mathbb{C}[x_1,x_2]/I \\ f & \mapsto & \begin{bmatrix} x_1 f \bmod I \\ x_2 f \bmod I \\ 0 \end{bmatrix} & & \\ & & \begin{bmatrix} f_1 \bmod I \\ f_2 \bmod I \\ a \end{bmatrix} & \mapsto & (x_1 f_2 - x_2 f_1 + a) \bmod I. \end{array}$$

This is an analogue of the Koszul complex. The cyclicity or the stability implies β is surjective.

As an application of β, we have the following:

Lemma 8.6. *For a fixed V, $e_i^N \left[\Delta_{\mathfrak{M}(V,W)}\right]$ and $f_i^N \left[\Delta_{\mathfrak{M}(V,W)}\right]$ are 0 for sufficiently large N. In particular, the operators e_i, f_i are locally nilpotent on $H_{\mathrm{top}}(\mathfrak{L}(W))$.*

Proof. The first case is obvious since

$$e_i^N \left[\Delta_{\mathfrak{M}(V,W)}\right] \in H_{\mathrm{top}}(Z(V\ominus N\rho_i, V, W))$$

and $\mathfrak{M}(V\ominus N\rho_i, W) = \emptyset$ if N is greater than the multiplicity of ρ_i in V.

The second case follows from the assertion that if

$$\dim W_i - \sum_j c_{ij}\dim V_j + \dim V_i = \dim W_i - \sum_{j:j\neq i} c_{ij}\dim V_j - \dim V_i < 0$$

then $\mathfrak{M}(V,W) = \emptyset$. This assertion follows from the surjectivity of β, as its ρ_i-component

$$W_i \oplus \bigoplus_{j:j\neq i} V_j^{\oplus(-c_{ij})} \to V_i$$

must be surjective. □

It is known that the relation (7.6) follows the rest of relations and the property in Lemma 8.6 (see e.g., §3.3 of the reference [1] in §7). Thus the only remaining relation is (7.5). I explain only the key point in the proof. See the original paper [N2] for the complete proof. We consider $\left[\Delta_{\mathfrak{M}(V^1,W)}\right] e_i f_j$ and $\left[\Delta_{\mathfrak{M}(V^1,W)}\right] f_j e_i$. Let us consider two triple products

$$\mathfrak{M}(V_1,W)\times\mathfrak{M}(V_2,W)\times\mathfrak{M}(V_3,W), \qquad \mathfrak{M}(V_1,W)\times\mathfrak{M}(V_2',W)\times\mathfrak{M}(V_3,W),$$

where

$$V_2 = V_1\oplus\rho_i = V_3\oplus\rho_j, \quad V_2' = V_1\ominus\rho_j = V_3\ominus\rho_i.$$

Note that these equations are compatible since

$$V_2 = V_2'\oplus\rho_i\oplus\rho_j.$$

Let p_{ij} be the projection as usual. (We use the same notation for two triple products for brevity.) Then we have

$$\begin{aligned} \left[\Delta_{\mathfrak{M}(V^1,W)}\right] e_i f_j &= \pm p_{13*}\left(p_{12}^*\left[\mathfrak{P}_i(V_2,W)\right]\cap p_{23}^*\left[\omega\mathfrak{P}_j(V_2,W)\right]\right), \\ \left[\Delta_{\mathfrak{M}(V^1,W)}\right] f_j e_i &= \pm p_{13*}\left(p_{12}^*\left[\omega\mathfrak{P}_j(V_1,W)\right]\cap p_{23}^*\left[\mathfrak{P}_i(V_3,W)\right]\right). \end{aligned}$$

Let us consider the set theoretical interesections in the special case $W = \rho_0$:

$$\begin{aligned} p_{12}^{-1}(\mathfrak{P}_i(V_2))\cap p_{23}^{-1}(\omega\mathfrak{P}_j(V_2)) &= \{(I_1,I_2,I_3)\mid I_1\supset I_2\subset I_3\}, \\ p_{12}^{-1}(\omega\mathfrak{P}_j(V_1))\cap p_{23}^{-1}(\mathfrak{P}_i(V_3)) &= \{(I_1,I_2',I_3)\mid I_1\subset I_2'\supset I_3\}. \end{aligned} \tag{8.7}$$

The crusial observation is the following: if $I_1\neq I_3$, I_2 and I_2' are determined by I_1, I_3 as

$$I_2 = I_1\cap I_3, \qquad I_2' = I_1 + I_3.$$

Moreover, $I_1 \cap I_3 \in X(V_2)$ if and only if $I_1 + I_3 \in X(V_2')$. Let U be the open subset given by $X(V_1) \times X(V_3)$ given by $I_1 \neq I_3$. If $i \neq j$, then $U = \emptyset$. The above means that on the open set $p_{13}^{-1}(U)$, the intersections (8.7) and their images under the projection p_{13} are all isomorphic. Let us draw a picture when $\Gamma = \{1\}$ and all I_i are ideals of functions vanishing at distinct points, although the case $\Gamma = \{1\}$ is excluded from our discussion:

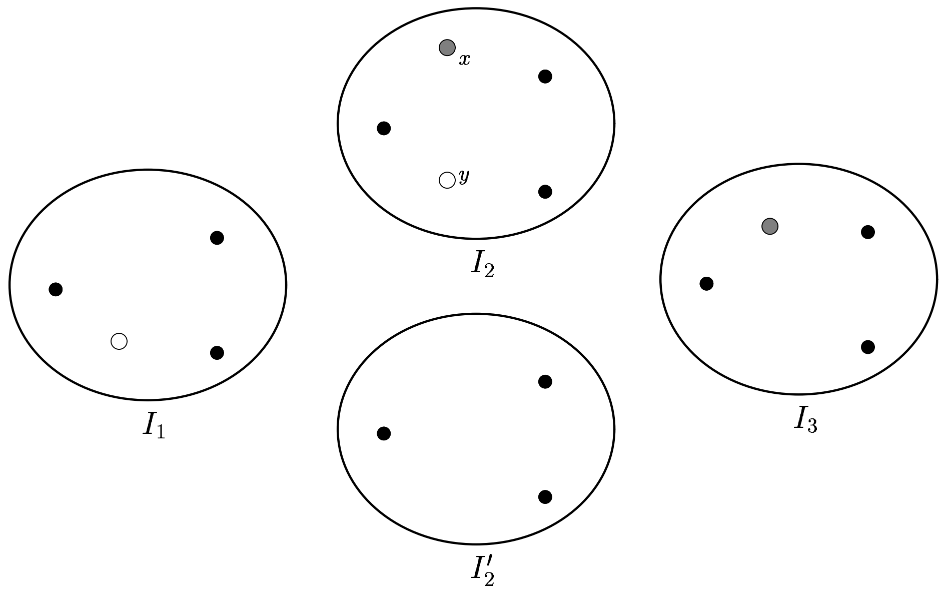

FIGURE 5. Correspondence between $I_1 \supset I_2 \subset I_3$ and $I_1 \subset I_2' \supset I_3$

The general case is the same. The set theoretical intersection is

$$\begin{aligned}
&p_{12}^{-1}(\mathfrak{P}_i(V_2, W)) \cap p_{23}^{-1}(\omega\mathfrak{P}_j(V_2, W)) \\
&= \left\{ ([(B^1, i^1, j^1)], [(B^2, i^2, j^2)], [(B^3, i^3, j^3)] \;\middle|\; \begin{array}{l}\text{there exist } \Gamma\text{-equivariant surjections} \\ V^1 \twoheadleftarrow V^2 \twoheadrightarrow V^3 \text{ compatible with data}\end{array} \right\}, \\
&p_{12}^{-1}(\omega\mathfrak{P}_j(V_1, W)) \cap p_{23}^{-1}(\mathfrak{P}_j(V_3, W)) \\
&= \left\{ ([(B^1, i^1, j^1)], [(B'^2, i'^2, j'^2)], [(B^3, i^3, j^3)] \;\middle|\; \begin{array}{l}\text{there exist } \Gamma\text{-equivariant surjections} \\ V^1 \twoheadrightarrow V'^2 \twoheadleftarrow V^3 \text{ compatible with data}\end{array} \right\}.
\end{aligned}$$

Here 'compatible with data' means that the kernel of the surjection $V^p \twoheadrightarrow V^q$ is invariant under B^p and is contained $\operatorname{Ker} j^p$ and the data induced on the quotient V^q is isomorphic to (B^q, i^q, j^q) $((p,q) = (2,1), (2,3), (1,\prime 2), (3,\prime 2))$. As above, in the complement of the diagonal, the two intersection are isomorphic, and also isomorphic to their image under the projection to $\mathfrak{M}(V^1, W) \times \mathfrak{M}(V^3, W)$. We have already encountered the analogues result in §1.3.

One can check the transversality of the intersections on $p_{13}^{-1}(U)$ (see [N2, Appendix]). If $j\colon U \to \mathfrak{M}(V_1, W) \times \mathfrak{M}(V_3, W)$ denotes the inclusion, we get

$$j^* \left[\Delta_{\mathfrak{M}(V^1,W)}\right] e_i f_j = j^* \left[\Delta_{\mathfrak{M}(V^1,W)}\right] f_j e_i.$$

Thus we have checked the relation (7.5) for $i \neq j$. Consider the case $i = j$. By the above and the long exact sequence in the homology groups, we know that $\left[\Delta_{\mathfrak{M}(V^1,W)}\right] e_i f_j - \left[\Delta_{\mathfrak{M}(V^1,W)}\right] f_j e_i$ is contained in the image of

$$H_{\text{top}}(\Delta_{\mathfrak{M}(V^1,W)}) \to H_{\text{top}}(Z(W)).$$

Since $\mathfrak{M}(V^1, W)$ is connected and has dimension equal to 'top', we have

$$e_i f_j - f_j e_i = c_V \left[\Delta_{\mathfrak{M}(V,W)}\right]$$

for some constant $c_V \in \mathbb{Z}$. The last step in the proof is the calculation of a self-intersection product to compute the constant c_V. For this, see [N2, §9].

References

[1] J. Cheah, *Cellular decompositions for nested Hilbert schemes of points*, Pacific J. Math. **183** (1998), 39–90.
[2] A. Tikhomirov, ???

9. Irreducibility of the representation and the crystal structure

Theorem 9.1. *As a representation of* $\mathbf{U}(\widehat{\mathfrak{g}})$, $\bigoplus H_{\mathrm{top}}(\mathfrak{L}(V,W))$ *is integrable and highest weight (hence irreducible). Its highest weight vector is the fundamental class of* $\mathfrak{M}(0,W) =$ *point. Its highest weight* Λ *is given by* $\langle \Lambda, d\rangle = 0$, $\langle \Lambda, h_i\rangle = \dim W_i$.

When W is the trivial representation, the corresponding integrable representation is called the basic representation.

We have already checked that $\bigoplus H_{\mathrm{top}}(\mathfrak{L}(V,W))$ is integrable in Lemma 8.6. It is also clear that the weight of $[\mathfrak{M}(0,W)]$ is given by the above Λ. In order to show that it is a highest weight module, we introduce a structure of a crystal to the set of irreducible components of $\bigsqcup_V \mathfrak{L}(V,W)$. Note that $\bigsqcup_V \mathfrak{L}(V,W)$ gives a base of $\bigoplus_V H_{\mathrm{top}}(\mathfrak{L}(V,W))$.

9.1. **Crystal.** Let us review the notion of crystals briefly. See [1, KS] for detail.

Let

$$I \overset{\text{def.}}{=} \{0,1,\dots,n\} \quad \text{(the index set of simple roots)},$$
$$P^\vee \overset{\text{def.}}{=} \mathbb{Z}h_0 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d, \qquad P \overset{\text{def.}}{=} \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, P^\vee\rangle \in \mathbb{Z}\}.$$

Definition 9.2. A *crystal* $\mathcal{B}$ associated with the affine Lie algebra $\widehat{\mathfrak{g}}$ is a set together with maps $\mathrm{wt}\colon \mathcal{B} \to P$, $\varepsilon_i, \varphi_i\colon \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\}$, $\widetilde{e}_i, \widetilde{f}_i\colon \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ $(i \in I)$ satisfying the following properties

(9.3a) $\quad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle,$

(9.3b) $\quad \mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i,\ \varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1,\ \varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1, \quad \text{if } \widetilde{e}_i b \in \mathcal{B},$

(9.3c) $\quad \mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i,\ \varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1,\ \varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1, \quad \text{if } \widetilde{f}_i b \in \mathcal{B},$

(9.3d) $\quad b' = \widetilde{f}_i b \Longleftrightarrow b = \widetilde{e}_i b' \quad \text{for } b,\ b' \in \mathcal{B},$

(9.3e) $\quad \text{if } \varphi_i(b) = -\infty \text{ for } b \in \mathcal{B}, \text{ then } \widetilde{e}_i b = \widetilde{f}_i b = 0$

We set $\mathrm{wt}_i(b) = \langle h_i, \mathrm{wt}(b)\rangle$.

We give simple examples.

Example 9.4. (1) For all $i \in I$, we define the crystal $\mathcal{B}_i$ as follows:

$$\mathcal{B}_i = \{b_i(n) \mid n \in \mathbb{Z}\},$$
$$\mathrm{wt}(b_i(n)) = n\alpha_i, \quad \varphi_i(b_i(n)) = n, \quad \varepsilon_i(b_i(n)) = -n,$$
$$\varphi_j(b_i(n)) = \varepsilon_j(b_i(n)) = -\infty \quad (i \neq j),$$
$$\widetilde{e}_i(b_i(n)) = b_i(n+1), \quad \widetilde{f}_i(b_i(n)) = b_i(n-1),$$
$$\widetilde{e}_j(b_i(n)) = \widetilde{f}_j(b_i(n)) = 0 \quad (i \neq j).$$

(2) For $\lambda \in P^+$, we define the crystal T_λ by

$$T_\lambda = \{t_\lambda\},$$
$$\mathrm{wt}(t_\lambda) = \lambda, \quad \varphi_i(t_\lambda) = \varepsilon_i(t_\lambda) = -\infty,$$
$$\widetilde{e}_i(t_\lambda) = \widetilde{f}_i(t_\lambda) = 0.$$

A crystal $\mathcal{B}$ is called *normal* if

$$\varepsilon_i(b) = \max\{n \mid \widetilde{e}_i^{\,n} b \neq 0\}, \quad \varphi_i(b) = \max\{n \mid \widetilde{f}_i^{\,n} b \neq 0\}.$$

For given two crystals $\mathcal{B}_1$, $\mathcal{B}_2$, a *morphism* ψ of crystal from $\mathcal{B}_1$ to $\mathcal{B}_2$ is a map $\mathcal{B}_1 \sqcup \{0\} \to \mathcal{B}_2 \sqcup \{0\}$ satisfying $\psi(0) = 0$ and the following conditions for all $b \in \mathcal{B}_1$, $i \in I$:

(9.5a) $\quad \mathrm{wt}(\psi(b)) = \mathrm{wt}(b),\ \varepsilon_i(\psi(b)) = \varepsilon_i(b),\ \varphi_i(\psi(b)) = \varphi_i(b) \quad \text{if } \psi(b) \in \mathcal{B}_2,$

(9.5b) $\quad \widetilde{e}_i\psi(b) = \psi(\widetilde{e}_i b) \quad \text{if } \psi(b) \in \mathcal{B}_2,\ \widetilde{e}_i b \in \mathcal{B}_1,$

(9.5c) $\quad \widetilde{f}_i\psi(b) = \psi(\widetilde{f}_i b) \quad \text{if } \psi(b) \in \mathcal{B}_2,\ \widetilde{f}_i b \in \mathcal{B}_1.$

A morphism ψ is called *strict* if ψ commutes with $\widetilde{e}_i$, $\widetilde{f}_i$ for all $i \in I$ without any restriction. A morphism ψ is called an *embedding* if ψ is an injective map from $B_1 \sqcup \{0\}$ to $B_2 \sqcup \{0\}$.

Definition 9.6. The *tensor product* $\mathcal{B}_1 \otimes \mathcal{B}_2$ of crystals $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined to be the set $\mathcal{B}_1 \times \mathcal{B}_2$ with maps defined by

(9.7a) $$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),$$

(9.7b) $$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}_i(b_1)),$$

(9.7c) $$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \mathrm{wt}_i(b_2)),$$

(9.7d) $$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{e}_i b_2 & \text{otherwise,} \end{cases}$$

(9.7e) $$\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i b_2 & \text{otherwise.} \end{cases}$$

Here (b_1, b_2) is denoted by $b_1 \otimes b_2$ and $0 \otimes b_2$, $b_1 \otimes 0$ are identified with 0.

It is easy to check that these satisfy the axioms in Definition 9.2. It is also easy to check that the tensor product of two normal crystals is again normal.

It is easy to check $(\mathcal{B}_1 \otimes \mathcal{B}_2) \otimes \mathcal{B}_3 = \mathcal{B}_1 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_3)$. We denote it by $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_3$. Similarly we can define $\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$.

The crystal was introduced by abstracting the notion of crystal bases constructed by Kashiwara [1]. Thus we have the following examples of crystals.

Example 9.8. (1) The lower half $\mathbf{U}_q(\mathfrak{g})^-$ of the quantized universal enveloping algebra has a base which has a structure of the crystal. Let $\mathcal{B}(\infty)$ denote this crystal. Let b_0 be the vector corresponding to $1 \in \mathbf{U}_q(\mathfrak{g})^-$.

(2) Similarly the simple $\mathbf{U}_q(\mathfrak{g})$-module $L(\Lambda)$ with highest weight λ has a base which has a structure of the crystal. Let $\mathcal{B}(\Lambda)$ denote this crystal. Let b_λ denote the highest weight vector considered as an element of $B(\Lambda)$. It is known that $\mathcal{B}(\Lambda)$ is normal. It is also known that the map

$$\pi\colon \mathcal{B}(\infty) \otimes T_\Lambda \ni \widetilde{f}_{i_1} \dots \widetilde{f}_{i_n} b_0 \otimes t_\Lambda \longmapsto \widetilde{f}_{i_1} \dots \widetilde{f}_{i_n} b_\Lambda \in \mathcal{B}(\Lambda) \sqcup \{0\}$$

is well-defined and is a strict morphism. Furthermore, $L(\Lambda_1) \otimes L(\Lambda_2)$ has a base which has a structure of crystal isomorphic to $\mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_2)$.

Remark that the character of $L(\Lambda)$ is given by

$$\mathrm{ch}\, L(\Lambda) \overset{\text{def.}}{=} \sum_\lambda \dim L(\Lambda)_\lambda e^\lambda = \sum_{b \in \mathcal{B}(\Lambda)} e^{\mathrm{wt}(b)}.$$

We also have the tensor product decomposition (generalized Littlewood-Rechiardson rule):

$$L(\Lambda_1) \otimes L(\Lambda_2) = \bigoplus L(\mathrm{wt}(b_1) + \mathrm{wt}(b_2)),$$

where the summation runs over all $b_1 \otimes b_2 \in \mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_2)$ such that $e_i(b_1 \otimes b_2) = 0$ for all $i \in I$.

9.2. Let us consider the complex (8.5). Let $\bigoplus K_i \otimes \rho_i$ be the decomposition of $\operatorname{Ker}\alpha$ to irreducible representations. (Note that $\operatorname{Ker}\alpha$ is invariant under the Γ-action. Note that the kernel is nonzero if $[(B,i,j)] \in \mathfrak{L}(V,W)$, since B_1 and B_2 are commuting nilpotent elements.

Let $\mathfrak{M}_{i;r}(V,W)$ be the subset of $\mathfrak{M}(V,W)$ consisting of elements $[(B,i,j)]$ whose K_i has dimension r. It is a locally closed subvariety since $\bigcup_{s:s\le r}\mathfrak{M}_{i;s}(V,W)$ is an open subset of $X(V)$. Let us define a map

$$p\colon \mathfrak{M}_{i;r}(V,W) \to \mathfrak{M}_{i;0}(V \ominus r\rho_i, W)$$

as follows. We replace V_i by V_i/K_i and consider the induced map (B', i', j'). Other components V_j are unchanged.

If $W = \rho_0$ and the point $[(B,i,j)]$ corresponds to an ideal $I \in X_{i;r}(V)$, then $p(I)$ is an ideal generated by I and (representatives of) $K_i \otimes \rho_i$.

Lemma 9.9. (1) *Let α', β' be homomorphism defined as above for points $[(B', i', j')] \in \mathfrak{M}_{i;0}(V \ominus r\rho_i, W)$. Then $\operatorname{Hom}_\Gamma(\rho_i, \operatorname{Ker}\beta'/\operatorname{Im}\alpha')$ forms a vector bundle over $\mathfrak{M}_{i;0}(V\ominus r\rho_i, W)$. Its rank is equal to*

$$\dim W_i - \sum_j c_{ij}(\dim V_j - r\delta_{ij}),$$

which is $\langle$the weight of $H_{\mathrm{top}}(\mathfrak{L}(V\ominus r\rho_i, W)), h_i\rangle$. (Recall the definition of h_i.)

(2) *$p\colon \mathfrak{M}_{i;r}(V,W) \to \mathfrak{M}_{i;0}(V\ominus r\rho_i, W)$ is isomorphic to the Grassmann bundle of r-planes in $\operatorname{Hom}_\Gamma(\rho_i, \operatorname{Ker}\beta'/\operatorname{Im}\alpha')$.*

In fact, the isomorphism is given by mapping $[(B,i,j)]$ to $\operatorname{Hom}_\Gamma(\rho_i, \operatorname{Ker}\beta/\operatorname{Im}\alpha)$. The latter can be considered as a subspace of $\operatorname{Hom}_\Gamma(\rho_i, \operatorname{Ker}\beta'/\operatorname{Im}\alpha')$ since $\operatorname{Ker}\beta' \subset \operatorname{Ker}\beta$ and $\operatorname{Im}\alpha = \operatorname{Im}\alpha'$ by the definition of $[(B_1', i', j')]$.

It is clear that $\pi(([B,i,j])) = \pi([B', i', j'])$. Therefore, the restriction of the Grassmann bundle to $\mathfrak{L}(V\ominus r\rho_i, W)$ gives us

$$p\colon \mathfrak{M}_{i;r}(V,W)\cap\mathfrak{L}(V,W) \to \mathfrak{M}_{i;0}(V\ominus r\rho_i, W)\cap\mathfrak{L}(V\ominus r\rho_i, W),$$

which is still isomorphic to a Grassmann bundle.

Using (6.4), we find

$$\frac{1}{2}\dim\mathfrak{M}(V\ominus r\rho_i, W) + \dim(\text{Grassmann}) = \frac{1}{2}\dim\mathfrak{M}(V,W).$$

This means that the dimension of the fiber is just half of the difference of dimensions of total space. This remarkable observation is due to Lusztig.

Let us show that $\dim\mathfrak{M}_{i;r}(V,W)\cap\mathfrak{L}(V,W)$ is equal to the half of $\dim\mathfrak{M}(V,W)$ by induction, by using this observation. A little bit more effort shows that $\mathfrak{M}_{i;r}(V,W)\cap\mathfrak{L}(V,W)$ is a lagrangian subvariety.

When $V=0$, then $\mathfrak{L}(0,W) = \mathfrak{M}(0,W)$ is a point. So the assertion is obvious. Assume that we have $\dim\mathfrak{M}_{i;r}(V',W)\cap\mathfrak{L}(V',W) = \frac{1}{2}\dim\mathfrak{M}(V',W)$ if $\dim V' < \dim V$. If $V\neq 0$, then $K_i \neq 0$ for some i for any point in $\mathfrak{L}(V,W)$. That is

$$\mathfrak{L}(V,W) = \bigcup_{i\in I, r\neq 0}\mathfrak{M}_{i;r}(V,W)\cap\mathfrak{L}(V,W)$$

By the induction hypothesis and the above observation, $\mathfrak{M}_{i;r}\cap\mathfrak{L}(V,W)$ is a half-dimensional subvariety. Since the above is a finite union, the total set $\mathfrak{L}(V,W)$ is also half-dimensional. Since $\mathfrak{M}_{i;0}(V,W)\cap\mathfrak{L}(V,W)$ is an open subset of $\mathfrak{L}(V,W)$, it is also half-dimensional. This completes the induction.

Let Y be an irreducible component of $\mathfrak{L}(V,W)$. We define $\operatorname{wt}(Y)$ as a weight of $H_{\mathrm{top}}(\mathfrak{L}(V,W))$, i.e.,

$$\dim W_i - \sum_j c_{ij}\dim V_j.$$

We define $\varepsilon_i(Y)$ so that

$$\begin{aligned}\varepsilon_i(Y) &= \dim K_i \quad \text{for a generic point } [(B,i,j)] \text{ in } Y,\\ &= \min_{[(B,i,j)]\in Y} \dim K_i.\end{aligned}$$

As we remarked above, $\varepsilon_i(Y) > 0$ for some i if $V \neq 0$. We set $\varphi_i(Y) = \varepsilon_i(Y) + \langle \mathrm{wt}(Y), h_i\rangle$.

Let $r \overset{\text{def.}}{=} \varepsilon_i(Y)$. We define an irreducible component Y' of $\mathfrak{L}(V\ominus r\rho_i, W)$ by

$$Y' \overset{\text{def.}}{=} \overline{p\left(Y \cap \mathfrak{M}_{i;r}(V,W)\right)},$$

where $p\colon \mathfrak{M}_{i;r}(V,W) \to \mathfrak{M}_{i;0}(V\ominus r\rho_i, W)$ is the Grassmann bundle above. We have

$$\varepsilon_i(Y') = 0$$

Conversely, we can recover Y from Y' as

$$Y = \overline{p^{-1}\left(Y' \cap \mathfrak{M}_{i;0}(V\ominus r\rho_i, W)\right)}.$$

Therefore we have a bijection

$$\{Y \in \operatorname{Irr}\mathfrak{L}(V,W) \mid \varepsilon_i(Y) = r\} \longleftrightarrow \{Y' \in \operatorname{Irr}\mathfrak{L}(V\ominus r\rho_i, W) \mid \varepsilon_i(Y') = 0\}.$$

Using above observation, we want to define maps

$$\widetilde{e}_i, \widetilde{f}_i\colon \bigsqcup \operatorname{Irr}\mathfrak{L}(V,W) \to \bigsqcup \operatorname{Irr}\mathfrak{L}(V,W) \sqcup \{0\}.$$

If $\varepsilon_i(Y) = 0$, then we define $\widetilde{e}_i Y = 0$. Otherwise, we define $\widetilde{e}_i Y$ as the image of Y under the composition of bijections

$$\begin{aligned}\{Y \in \operatorname{Irr}\mathfrak{L}(V,W) \mid \varepsilon_i(Y) = r\} &\longleftrightarrow \{Y' \in \operatorname{Irr}\mathfrak{L}(V\ominus r\rho_i, W) \mid \varepsilon_i(Y') = 0\}\\ &\longleftrightarrow \{Y'' \in \operatorname{Irr}\mathfrak{L}(V\ominus \rho_i, W) \mid \varepsilon_i(Y') = r-1\},\end{aligned}$$

where the latter bijection is given by the Grassmann bundle $\mathfrak{M}_{i;r-1}(V\ominus\rho_i, W) \to \mathfrak{M}_{i;0}(V\ominus r\rho_i, W)$.

Similarly we define $\widetilde{f}_i Y$ as the image of Y under the composition of bijections

$$\begin{aligned}\{Y \in \operatorname{Irr}\mathfrak{L}(V,W) \mid \varepsilon_i(Y) = r\} &\longleftrightarrow \{Y' \in \operatorname{Irr}\mathfrak{L}(V\ominus r\rho_i, W) \mid \varepsilon_i(Y') = 0\}\\ &\longleftrightarrow \{Y''' \in \operatorname{Irr}\mathfrak{L}(V\oplus \rho_i, W) \mid \varepsilon_i(Y') = r+1\}.\end{aligned}$$

However we must be careful. Since

$$\dim \operatorname{Hom}_\Gamma\left(\rho_i, \operatorname{Ker}\beta'/\operatorname{Im}\alpha'\right) = \langle h_i, \mathrm{wt}(Y')\rangle = \langle h_i, \mathrm{wt}(Y)\rangle + 2r,$$

the Grassmann bundle of $(r+1)$-planes in $\operatorname{Hom}_\Gamma\left(\rho_i, \operatorname{Ker}\beta'/\operatorname{Im}\alpha'\right)$ is empty if $\langle h_i, \mathrm{wt}(Y)\rangle + r \leq 0$. We set $\widetilde{f}_i Y = 0$ in this case. Otherwise, $\widetilde{f}_i Y$ is defined by the bijections.

Theorem 9.10. *The above ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ on $\bigsqcup_V \operatorname{Irr}\mathfrak{L}(V,W)$ is a crystal.*

Using the exact sequence in homology groups, it is not difficult to show that

$$f_i\,[Y] = c\left[\widetilde{f}_i Y\right] + \sum_{Y':\varepsilon_i(Y)>\varepsilon_i(Y)+1} c_{Y'}\,[Y']$$

for some constants c, $c_{Y'}$. (Use the open set $\bigcup_{s:s\leq\varepsilon(Y)} \mathfrak{M}_{i;s}(V,W)$.) In order to determine the constant c, we pullback both hand sides to $\bigcup_{s:s\leq\varepsilon(Y)+1} \mathfrak{M}_{i;s}(V,W)$. In the right hand side, only $c\left[\widetilde{f}_i Y\right]$ survives. Then it is not difficult to determine c by using the self-intersection formula. It is given by $\pm(\varepsilon(Y)+1)$.

Using the above formula, we prove that $\bigoplus H_{\mathrm{top}}(\mathfrak{L}(V,W))$ is a highest weight module by induction on $\dim V$ and ε_i. If $V = 0$, $\mathfrak{L}(0,W) = \mathfrak{M}(0,W)$ is a point. We have nothing to prove. Let Y be an irreducible component of $\mathfrak{L}(V,W)$. There exists i such that $\varepsilon_i(Y) > 0$. Suppose that we already know that

(1) if $\dim V' < \dim V$, then $H_{\text{top}}(\mathfrak{L}(V', W))$ is contained in $\mathbf{U}(\widehat{\mathfrak{g}}) \cdot [\mathfrak{M}(0, W)]$.

(2) if $Y' \in \operatorname{Irr} \mathfrak{L}(V, W)$ satisfies $\varepsilon_i(Y') > \varepsilon_i(Y)$, then $[Y']$ is contained in $\mathbf{U}(\widehat{\mathfrak{g}}) \cdot [\mathfrak{M}(0, W)]$.

Since the value of ε_i on $\operatorname{Irr} \mathfrak{L}(V, W)$ is bounded from above, we may assume the second condition by the descending induction. By the above formula, we have

$$f_i\,[\widetilde{e}_i Y] = \pm\varepsilon_i(Y)[Y] + \sum_{Y' : \varepsilon_i(Y) > \varepsilon_i(Y)} c_{Y'}\,[Y'] .$$

By (1), the left hand side is contained in $\mathbf{U}(\widehat{\mathfrak{g}}) \cdot [\mathfrak{M}(0, W)]$. By (2), terms in the right hand side, except $\pm\varepsilon_i(Y)[Y]$ are contained in $\mathbf{U}(\widehat{\mathfrak{g}}) \cdot [\mathfrak{M}(0, W)]$. Therefore $[Y]$ is also contained in $\mathbf{U}(\widehat{\mathfrak{g}}) \cdot [\mathfrak{M}(0, W)]$. This completes the proof.

Remark 9.11. It is known that the crystal defined above (the definition is due to Lusztig) is isomorphic to the crystal of the highest weight module of the quantum affine algebra. See [KS, 3, 6] for the proof.

Further study. The quiver varieties are relevant for studies of q-analogues of loop algebras. In our case when quiver varieties are realized as fixed point set of Hilbert schemes, it is the q-analogue of the loop algebra of the affine Lie algebra:

$$\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}}).$$

Note that the affine Lie algebra $\widehat{\mathfrak{g}}$ already contains the loop algebra $\mathbf{L}\mathfrak{g}$ of a simple finite dimensional Lie algebra $\mathfrak{g}$. Therefore, $\mathbf{L}\widehat{\mathfrak{g}}$ is sometimes called a *double loop algebra* or a *toroidal algebra*.

In fact, a restricted class of quiver varieties ($W_0 = V_0 = 0$ in our notation) corresponds to a q-analogue $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ of a *single* loop algebra $\mathbf{L}\mathfrak{g}$ of the simple Lie algebra $\mathfrak{g}$. It already contains an information on its representation theory, which cannot be accesible by purely algebraic techniques [4, 5]. So far, the representation theory of $\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$ is *not* so studied, but basically all results in [4, 5] can be generalized to the case of $\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$. One of missing pieces is the definition of a coproduct.

I have already explained basic results for quiver varieties. Materials which are necessarily to read [4, 5], e.g., K-theory of coherent sheaves, perverse sheaves, can be studied in lots of nice textbooks. So you are ready to wellcome a study of representation theory by a geometric method !

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