MONOPOLES AND NAHM'S EQUATIONS

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Abstract. We give a different proof of Hitchin's result : a correspondence between SU(2)-monopoles and solutions of Nahm's equations. We also prove that this correspondence gives a hyper-Kähler isometry between the monopole moduli space and the space of equivalence classes of solutions of Nahm's equations, equipped with their natural metrics. Such a result was conjectured by Atiyah and Hitchin.

1. Introduction

In 1983 Hitchin [Hi3] gave an equivalence between

A) an SU(2) monopole satisfying certain asymptotic conditions,

B) a solution of Nahm's equation satisfying certain boundary conditions.

The correspondence B $\Rightarrow$ A is an adaptation of the Atiyah-Drinfeld-Hitchin-Manin construction [ADHM] of instantons on S^4, and was produced by Nahm [Na1]. Hitchin constructed the correspondence A $\Rightarrow$ B by relating A and B to the third object:

C) a compact algebraic curve in $T\mathbb{P}^1$ satisfying certain conditions.

The third object C is interesting to explore in itself, but for the purpose in giving the correspondence A $\Rightarrow$ B, this approach is indirect and it is not so easy to prove that the composition A $\Rightarrow$ B $\Rightarrow$ A gives back the same monopole.

Later Nahm [Na2] and Corrigan-Goddard [CG] pointed out a new approach which is more direct. From their point of view, the transform which produces B $\Rightarrow$ A and A $\Rightarrow$ B can be considered as analogous to a Fourier transform, so it seems very natural, at least philosophically, that two correspondences are mutually inverse. But they do not check the boundary behaviour of the solutions of Nahm's equations. This is the remaining part in their approach.

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Our aim is to fill the hole in their approach. But for the sake of the reader, we shall give the proofs (sometimes only in outline) in the whole steps.

Let us give the precise statement of Hitchin's result. Our objects are the following:

A) An SU(2) connection A on a rank 2 hermitian vector bundle E over $\mathbb{R}^3$ and a skew hermitian endomorphism Φ (the Higgs field) satisfying

A1) (the Bogomolny equation)

$$*R_A = d_A \Phi,$$

A2) the asymptotic expansion as $r = |x| \to \infty$, up to gauge transformation,

$$\Phi = \begin{pmatrix} i(1 - \frac{k}{2r}) & 0 \\ 0 & -i(1 - \frac{k}{2r}) \end{pmatrix} + O(r^{-2}),$$
$$|\nabla_A \Phi| = O(r^{-2}), \qquad \frac{\partial |\Phi|}{\partial \Omega} = O(r^{-2}),$$

where k is a positive integer.

B) A hermitian connection ∇ on a hermitian vector bundle V of rank k over the open interval $I = (-1, 1)$ and three skew-hermitian endomorphisms $T_\alpha \in \Gamma(I; \mathrm{Endskew}(V))$ satisfying

B1) (the Nahm's equation)

$$\nabla_t T_\alpha + \frac{1}{2} \sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma} [T_\beta, T_\gamma] = 0,$$

B2) T_α has at most simple poles at $t = \pm 1$ but is otherwise analytic,

B3) at each pole the residues of (T_1, T_2, T_3) define an irreducible representation of $\mathfrak{su}(2)$. Namely near the endpoint $t = 1$, in a covariant constant basis, we can write

$$T_\alpha(t) = \frac{a_\alpha}{t - 1} + b_\alpha(t),$$

where b_α is analytic in a neighbourhood of $t = 1$. Then

$$x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto -2(x_1 a_1 + x_2 a_2 + x_3 a_3)$$

defines a k-dimensional representation of $\mathfrak{su}(2)$. (This is a consequence of the Nahm's equation.) Here (e_1, e_2, e_3) is a basis for $\mathfrak{su}(2)$ defined by

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The last condition says this representation is *irreducible*, and similarly at the other pole $t = -1$.

Now our main result is

Theorem (Hitchin [Hi3])**.** *There is a natural equivalence between monopoles satisfying conditions* A *and Nahm data satisfying conditions* B*.*

The formal aspects of the proof is the same as the instanton case (see [Na2, CG]). In this case the similar proof but incorporating the complex geometry was given by Donaldson in [DK]. These two methods were used and presented side-by-side in [KN]. We shall adapt Nahm-Corrigan-Goddard's method in principle, but use the complex notation hoping that it makes the calculation familiar.

The paper is organized as follows. In Sect. 2 we give the correspondence A $\Rightarrow$ B. In Sect. 3 we review the construction of monopoles from Nahm data, i.e., B $\Rightarrow$ A. In Sects. 4 and 5 we prove that two correspondences are mutually inverse. Section 6 takes up an interesting side-issue: we show that our correspondence gives a hyper-Kähler isometry between the space of equivalence classes of solutions of Nahm's equations and the moduli space of monopoles, equipped with their natural hyper-Kähler structures. This was conjectured by Atiyah and Hitchin [AH]. Section 7 provides some remarks. In the appendix we shall give a proof of Lemma which we need in Sect. 2.

In a future work, the author hopes to extend Main Theorem to $\mathrm{SU}(m)$-monopoles. The only thing left is to study the boundary behaviour of the solutions of Nahm's equations. There are results of [HM] in this direction.

2. From Monopoles to Nahm's equations

The purpose of this section is to obtain from an $\mathrm{SU}(2)$-monopole (A, Φ) a solution T_α to Nahm's equations which satisfies the conditions B of the introduction.

For each $t \in I$ consider the following operators:

$$\begin{aligned} D_{A,t} &= D_A + (\Phi - it) \colon \Gamma(S \otimes E) \to \Gamma(S \otimes E), \\ D_{A,t}^* &= D_A - (\Phi - it) \colon \Gamma(S \otimes E) \to \Gamma(S \otimes E), \end{aligned}$$

where S is the spin bundle over $\mathbb{R}^3$ and D_A is the Dirac operator coupled with the connection A. Note that $D_{A,t}^*$ is the formal adjoint of $D_{A,t}$. Then the Weitzenböck formula shows

$$D_{A,t}^* D_{A,t} = 1_S \otimes (\nabla_A^* \nabla_A - (\Phi - it)^2), \tag{2.1}$$

which is a positive operator. In particular, $D_{A,t}$ has no L^2 kernel. Define

$$V_t = L^2 \text{ kernel of } D_{A,t}^*.$$

By an index theorem [Ca] the index of the operator $D_{A,t}$ is equal to $-k$, and we have $\operatorname{Ker} D_{A,t} = 0$. So V_t defines a vector bundle V of rank k on I which is a sub-bundle of the trivial bundle $\underline{L^2(\mathbb{R}^3; S \otimes E)}$ over I. (We denote by $\underline{W}$ the trivial bundle whose fiber is a vector space W.) Let π be the orthogonal projection onto V. Define a connection and three endomorphisms on V by

$$\nabla_t \psi = \pi(\frac{\partial \psi}{\partial t}), \qquad T_\alpha(\psi) = \pi(i x_\alpha \psi), \qquad \alpha = 1, 2, 3.$$

Note that $i x_\alpha \psi$ is in L^2 since ψ decays exponentially as $r = |x| \to \infty$.

The skew-hermiticity of T_α is automatic from the definition of T_α. So we first study the boundary condition B2. Since the behaviour when $t \to +1$ is similar, we only study the case $t \to -1$.

In the following calculation, we use the constant C in the generic sense. So the symbol C may mean different constants in different equations. The important point is that C must be independent of t, since we want to study the behaviour as $t \to -1$.

As shown in [Hi2,p.591], under the condition A1, there is an asymptotic gauge in which the Higgs field is the form of B2 and the connection matrix has the following asymptotic behaviour:

$$\begin{pmatrix} A_0^* & 0 \\ 0 & A_0 \end{pmatrix} + O(r^{-2}),$$

where A_0 is the connection form for a homogeneous connection on a line bundle of degree k over $S^2 = \mathbb{P}^1$ extended radially to $\mathbb{R}^3 \setminus \{0\}$ (A_0^* is its dual). Take a radial coordinate system (r, θ). The spinor bundle of $S^2 \times (0, \infty)$ is isomorphic to that of S^2, hence decomposes as $S^+ \oplus S^-$. Then there exists a norm preserving bundle isomorphism between S and $S^+ \oplus S^-$ under which Dirac operators are related by (cf. [Hi1])

$$D\psi = r^{-2} D_{S^+ \oplus S^-}(r\psi) = \begin{pmatrix} i(\frac{\partial}{\partial r} + \frac{1}{r}) & \frac{1}{r} D^- \\ \frac{1}{r} D^+ & -i(\frac{\partial}{\partial r} + \frac{1}{r}) \end{pmatrix},$$

where $D^\pm$ is the Dirac operator on S^2. Let denote the Dirac operators on S^2 twisted by A_0, A_0^* by $D^\pm_{A_0}$, $D^\pm_{A_0^*}$. The operator $D^*_{A,t}$ can be represented as follows:

$$D^*_{A,t} = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} + O(r^{-2}),$$

where

$$B_1 = \begin{pmatrix} i(\frac{\partial}{\partial r} + t - 1 + \frac{k+2}{2r}) & \frac{1}{r} D^-_{A_0^*} \\ \frac{1}{r} D^+_{A_0^*} & -i(\frac{\partial}{\partial r} - t + 1 - \frac{k-2}{2r}) \end{pmatrix},$$
$$B_2 = \begin{pmatrix} i(\frac{\partial}{\partial r} + t + 1 - \frac{k-2}{2r}) & \frac{1}{r} D^-_{A_0} \\ \frac{1}{r} D^+_{A_0} & -i(\frac{\partial}{\partial r} - t - 1 + \frac{k+2}{2r}) \end{pmatrix}.$$

Let R be a fixed positive number and χ a cut-off function which is 0 on $[0, R]$ and 1 on $[R+1, \infty)$. Using the isomorphisms $S^+ \cong \Lambda^{0,0} \otimes H^*$, $S^- \cong \Lambda^{0,1} \otimes H^*$ (where H is a hyperplane bundle), we can define $\psi \in L^2(\mathbb{R}^3; S \otimes E)$ from $f \in H^0(\mathbb{P}^1; \mathcal{O}(k-1))$ by

$$\psi(r, \theta) = \begin{pmatrix} 0 & 0 & \chi(r) e^{-(t+1)r} r^{\frac{k-2}{2}} f(\theta) & 0 \end{pmatrix}^t.$$

Then it satisfies

$$|D^*_{A,t}\psi|_S \leq C e^{-(t+1)r} (r+1)^{\frac{k-2}{2} - 2},$$

for some constant C depending only on (A, Φ), χ and $\sup |f|$. This means that ψ is an approximate solution of $D^*_{A,t}\psi = 0$. A real solution is given by $\psi - D_{A,t}\varphi$ where φ is the unique solution

$$D^*_{A,t} D_{A,t}\varphi = 1_S \otimes (\nabla^*_A \nabla_A - (\Phi - it)^2)\varphi = D^*_{A,t}\psi. \tag{2.2}$$

We shall show that $D_{A,t}\varphi$ is small relative to ψ, so the boundary behaviour of T_α is determined from ψ.

The equation (2.2) can be uniquely solved by the same method as in [JT, Proposition IV.4.1]. Please see [JT] for details. The solution φ is the minimum of the functional

$$S(\varphi) = \|\nabla_A \varphi\|^2_{L^2} + \|(\Phi - it)\varphi\|^2_{L^2} - 2\langle \varphi, D^*_{A,t}\psi\rangle_{L^2}.$$

This is strictly convex, differential and coercive, so has a unique minimum. In particular, $S(\varphi) \leq S(0) = 0$. Hence,

$$\|D_{A,t}\varphi\|^2_{L^2} = \|\nabla_A \varphi\|^2_{L^2} + \|(\Phi - it)\varphi\|^2_{L^2} \leq 2\langle \varphi, D^*_{A,t}\psi\rangle_{L^2}. \tag{2.3}$$

If R is sufficiently large and t is near -1, we have an estimate

$$(1 + t)|\varphi| \leq 2|(\Phi - it)\varphi| \qquad \text{in } \mathbb{R}^3 \setminus B_{\frac{R}{1+t}}.$$

So we get

$$(1 + t)^2 \int_{\mathbb{R}^3 \setminus B_{\frac{R}{1+t}}} |\varphi|^2 \, dx \leq 4\|(\Phi - it)\varphi\|^2_{L^2}. \tag{2.4}$$

On the other hand, the integral over $B_{\frac{R}{1+t}}$ can be estimated by using the Hölder's and Sobolev inequalities as

$$(1 + t)^2 \int_{B_{\frac{R}{1+t}}} |\varphi|^2 \, dx \leq C\|\varphi\|^2_{L^6} \leq C\| \, d|\varphi| \, \|^2_{L^2} \leq C\|\nabla_A \varphi\|^2_{L^2}, \tag{2.5}$$

where we have used the Kato's inequality in the last step. Substituting (2.4) and (2.5) into (2.3), we get

$$\|D_{A,t}\varphi\|_{L^2} \leq C(1 + t)^{-1}\|D^*_{A,t}\psi\|_{L^2}$$

Direct calculation shows that

$$\|D^*_{A,t}\psi\|_{L^2} \leq C(1 + t)^2\|\psi\|_{L^2}.$$

Hence $D_{A,t}\varphi$ is small if t is sufficiently near to -1, as required.

Thus we have obtained a trivialization of the bundle V near $t = -1$ (after the Gram-Schmidt orthogonalization). This trivialization is not covariant constant, but the trace-free part of the connection form is bounded. Hence it is enough to study the asymptotic behaviour in this trivialization. So the condition B3 follows from

Lemma 2.6. *Let a_α be an endomorphism of $H^0(\mathbb{P}^1; \mathcal{O}(k-1))$ defined by*

$$\langle a_\alpha f_1, f_2 \rangle = \int_{\mathbb{P}^1} \langle i x_\alpha f_1, f_2 \rangle \, dV.$$

Then a non-zero constant multiple of a linear map $x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto x_1 a_1 + x_2 a_2 + x_3 a_3$ defines an irreducible k-dimensional representation of $\mathfrak{su}(2)$.

The proof will be given in the Appendix. Remark that we will prove that T_α's satisfy the Nahm's equations below, so the constant must be equal to -2.

Proposition 2.7. *The endomorphisms T_α and the connection ∇ satisfy the Nahm's equations*

$$\nabla_t T_\alpha + \frac{1}{2} \sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma} [T_\beta, T_\gamma] = 0, \qquad \alpha = 1, 2, 3.$$

Before entering the proof of this proposition, we prepare the complex notation as in [Do] by breaking the natural symmetry and choosing a particular isomorphism $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$.

Fixing a trivialization of the bundle V, we write the connection ∇ as $\frac{d}{dt} + T_0$. Put

$$\alpha = \frac{1}{2}(T_0 + iT_1), \qquad \beta = \frac{1}{2}(T_2 + iT_3).$$

Then the Nahm's equations become the following pair of equations:

$$\begin{aligned} &\frac{d\beta}{dt} + 2[\alpha, \beta] = 0 \qquad \text{(the complex equation)}, \\ &\frac{d}{dt}(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0 \qquad \text{(the real equation)}. \end{aligned}$$

To prove that T_α's satisfy the Nahm equations, it is not necessarily to check both the complex and real equations: If one can check the complex equation, then he/she also gets the real equation by changing the isomorphism $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$.

As is well-known, a monopole (A, Φ) on $\mathbb{R}^3 = \{(x_1, x_2, x_3)\}$ can be identified with an $\mathbb{R}$-invariant instanton B on $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3)\}$. The operators $D_{A,t}$, $D^*_{A,t}$ correspond to the Dirac operators $D^+_{B,t}$, $D^-_{B,t}$ respectively, where the subscript t means that the operators are twisted by a flat connection $itdx_0$. Using the isomorphism $\mathbb{R}^4 \cong \mathbb{C}^2$, we have isomorphisms

$$S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}, \qquad S^- = \Lambda^{0,1},$$

and the Dirac operators are written as

$$\begin{aligned} D^+_{B,t} &= \sqrt{2}(\overline{\partial}_{B,t}, \overline{\partial}^*_{B,t}) \colon \Omega^{0,0}(E) \oplus \Omega^{0,2}(E) \to \Omega^{0,1}(E), \\ D^-_{B,t} &= \sqrt{2} \begin{pmatrix} \overline{\partial}^*_{B,t} \\ \overline{\partial}_{B,t} \end{pmatrix} \colon \Omega^{0,1}(E) \to \Omega^{0,0}(E) \oplus \Omega^{0,2}(E). \end{aligned}$$

Correspondingly, we denote components of $D_{A,t}$, $D^*_{A,t}$ by Dolbeault operators:

$$D_{A,t} = \sqrt{2}(\overline{\partial}_{A,t}, \overline{\partial}^*_{A,t}), \qquad D^*_{A,t} = \sqrt{2} \begin{pmatrix} \overline{\partial}^*_{A,t} \\ \overline{\partial}_{A,t} \end{pmatrix}.$$

Then the followings are "key identities" in our calculation:

$$[\overline{\partial}_{A,t}, \frac{\partial}{\partial t} - x_1] = 0, \qquad [\overline{\partial}_{A,t}, x_2 + ix_3] = 0, \tag{2.8}$$

where x_α is the multiplication of a coordinate function. These identities means that "$z_1 = -i\frac{\partial}{\partial t} + ix_1$ and $z_2 = x_2 + ix_3$ are holomorphic" which is true on $\mathbb{C}^2 = \{(z_1, z_2)\}$. We shall use this funny notation, hoping this causes no confusion.

In this setting, the formula (2.1) is

$$(\overline{\partial}_{A,t}, \overline{\partial}^*_{A,t}) \begin{pmatrix} \overline{\partial}^*_{A,t} \\ \overline{\partial}_{A,t} \end{pmatrix} = \begin{pmatrix} \overline{\partial}_{A,t}\overline{\partial}^*_{A,t} & \overline{\partial}^*_{A,t}\overline{\partial}^*_{A,t} \\ \overline{\partial}_{A,t}\overline{\partial}_{A,t} & \overline{\partial}^*_{A,t}\overline{\partial}_{A,t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Delta_{A,t} & 0 \\ 0 & \Delta_{A,t} \end{pmatrix}, \tag{2.9}$$

where $\Delta_{A,t} = \nabla_A^* \nabla_A - (\Phi - it)^2$.

Proof of Proposition 2.7. Let $G_{A,t}$ denote the Green's operator $\Delta_{A,t}^{-1}$. Then the orthogonal projection π is given by

$$\pi = 1 - D_{A,t}(1_S \otimes G_{A,t}) D^*_{A,t}.$$

Let $\psi \in V_t$, i.e. an L^2-solution of $D^*_{A,t}\psi = 0$. By the definitions of ∇ and T_α

$$\begin{aligned} iz_1\psi - (\frac{d}{dt} + 2\alpha)\psi &= D_{A,t}(1_S \otimes G_{A,t}) D^*_{A,t}(iz_1\psi) \\ iz_2\psi - 2\beta\psi &= D_{A,t}(1_S \otimes G_{A,t}) D^*_{A,t}(iz_2\psi). \end{aligned} \tag{2.10}$$

Using the "Dolbeault" operators and the formula (2.8), (2.9), we find

$$iz_2(\frac{d}{dt} + 2\alpha)\psi - 2iz_1\beta\psi = 2\overline{\partial}_{A,t}\{iz_1 G_{A,t}\overline{\partial}^*_{A,t}(iz_2\psi) - iz_2 G_{A,t}\overline{\partial}^*_{A,t}(iz_1\psi)\}. \tag{2.11}$$

Projecting to V_t, we get the complex equation

$$\frac{d\beta}{dt} + 2[\alpha, \beta] = 0.$$

This completes the proof. $\square$

3. From Nahm's equations to Monopoles

In this section, we shall construct an SU(2)-monopole from a solution of the Nahm's equations.

Suppose that we are given Nahm data satisfying the conditions B in Sect. 1. Let consider the Sobolev space $\mathbb{C}^2 \otimes W_0^{1,2}(I; V)$ of sections of $\mathbb{C}^2 \otimes V$ whose derivatives are in L^2 and the boundary values are 0. Similarly let $\mathbb{C}^2 \otimes L^2(I; V)$ be the space of L^2 sections. For each $x \in \mathbb{R}^3$, define an operator $\mathfrak{D}_x \colon \mathbb{C}^2 \otimes W_0^{1,2}(I; V) \to \mathbb{C}^2 \otimes L^2(I; V)$ by

$$\mathfrak{D}_x = 1_{\mathbb{C}^2} \otimes \nabla_t + \sum_{\alpha=1}^{3} (e_\alpha \otimes T_\alpha - ix_\alpha e_\alpha \otimes 1_V),$$

where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathfrak{su}(2)$ (see Sect. 1). In the matrix notation, this is equal to

$$\mathfrak{D}_x = \begin{pmatrix} \frac{d}{dt} + 2\alpha & 2\beta^* \\ 2\beta & \frac{d}{dt} - 2\alpha^* \end{pmatrix} - \begin{pmatrix} -x_1 & -i\overline{z}_2 \\ iz_2 & x_1 \end{pmatrix},$$

where $z_2 = x_2 + ix_3$ as before. Let $\mathfrak{D}_x^*$ be the formal adjoint operator of $\mathfrak{D}_x$, which is given by

$$\mathfrak{D}_x^* = \begin{pmatrix} -\frac{d}{dt} + 2\alpha^* & 2\beta^* \\ 2\beta & -\frac{d}{dt} - 2\alpha \end{pmatrix} - \begin{pmatrix} -x_1 & -i\overline{z}_2 \\ iz_2 & x_1 \end{pmatrix}.$$

The Nahm's equations imply

$$\mathfrak{D}_x^*\mathfrak{D}_x = 1_{\mathbb{C}^2} \otimes \left(\nabla_t^*\nabla_t + \sum_{\alpha=1}^{3} (T_\alpha - ix_\alpha)^*(T_\alpha - ix_\alpha) \right). \tag{3.1}$$

This identity is an analogue of (2.1). Then one can show that $\operatorname{Ker}\mathfrak{D}_x = 0$ for all $x \in \mathbb{R}^3$, so $\operatorname{Ker}\mathfrak{D}_x^*$ forms a vector bundle E over $\mathbb{R}^3$. The index is equal to -2 [Hi3], so $\operatorname{rank} E = 2$. Since E is a subbundle of the trivial bundle $\underline{\mathbb{C}^2 \otimes L^2(I;V)}$ over $\mathbb{R}^3$, it inherits a hermitian metric and a connection A. More precisely, if p is the projection onto E,

$$d_A = p \circ d.$$

We define the Higgs field Φ by

$$\Phi = p \circ it.$$

Then Hitchin shows that

Theorem 3.2. *The connection A and Higgs field Φ satisfy the Bogomolny equation $*R_A = d_A\Phi$ and the boundary condition A2.*

Proof. We shall give the proof for the Bogomolny equation. Our proof is "dual" to that of Proposition 2.7. For the proof of the boundary condition, see [Hi3].

Let define

$$\sigma_x = \begin{pmatrix} \frac{d}{dt} + 2\alpha + x_1 \\ 2\beta - iz_2 \end{pmatrix}, \qquad \tau_x = (2\beta - iz_2, -\frac{d}{dt} - 2\alpha - x_1).$$

Let F_x be the inverse of $\nabla_t^*\nabla_t + \sum_{\alpha=1}^{3}(T_\alpha - ix_\alpha)^*(T_\alpha - ix_\alpha)$. Then the orthogonal projection p is given by

$$p = 1 - \mathfrak{D}_x(1_{\mathbb{C}^2} \otimes F_x)\mathfrak{D}_x^* = 1 - \sigma_x F_x \sigma_x^* - \tau_x^* F_x \tau_x.$$

Consider the following operators (cf. Sect. 2)

$$\begin{aligned} \mathbb{D} &= D + it \colon \Gamma(S \otimes \underline{\mathbb{C}^2 \otimes L^2(I;V)}) \to \Gamma(S \otimes \underline{\mathbb{C}^2 \otimes L^2(I;V)}), \\ \mathbb{D}^* &= D - it \colon \Gamma(S \otimes \underline{\mathbb{C}^2 \otimes L^2(I;V)}) \to \Gamma(S \otimes \underline{\mathbb{C}^2 \otimes L^2(I;V)}), \end{aligned}$$

where D is the Dirac operator associated with the trivial monopole on $\underline{\mathbb{C}^2 \otimes L^2(I;V)}$. Denote by $\overline{\partial}$, $\overline{\partial}^*$ the associated "Dolbeault" operators. Namely

$$\mathbb{D} = \sqrt{2}(\overline{\partial}, \overline{\partial}^*), \qquad \mathbb{D}^* = \sqrt{2}\begin{pmatrix} \overline{\partial}^* \\ \overline{\partial} \end{pmatrix}.$$

If we define $\overline{\partial}_{A,0}$ (we set the parameter $t = 0$.) from (A, Φ) as in Sect. 2, we find

$$\overline{\partial}_{A,0} = p\overline{\partial} = (1 - \sigma_x F_x \sigma_x^* - \tau_x^* F_x \tau_x)\overline{\partial}.$$

Then

$$\overline{\partial}_{A,0}\overline{\partial}_{A,0} = 0 \tag{3.3}$$

follows from $\overline{\partial}\,\overline{\partial} = 0$ and the identities

$$[\overline{\partial}, \sigma] = 0, \qquad [\overline{\partial}, \tau] = 0 \tag{3.4}$$

which mean that σ and τ are "holomorphic". These are analogue of (2.8). Now changing the complex structure, we get the full Bogomolny equation from (3.3). $\square$

4. Completeness

We now study the composition of the transformations given in previous sections.

$$\begin{matrix} \text{a monopole} & \overset{\S 2}{\Longrightarrow} & \text{Nahm data} & \overset{\S 3}{\Longrightarrow} & \text{a new monopole} \\ (A, \Phi) & & (\nabla, T_\alpha) & & (A', \Phi') \end{matrix}$$

Starting from a monopole (A, Φ) with the monopole charge k, we construct Nahm data T_α satisfying the conditions B in Sect. 2. Then we can construct another monopole (A', Φ') from this data as in Sect. 3. The aim of this section is to show that these data (A, Φ) and (A', Φ') are gauge equivalent. This will show that all monopoles arise by the construction given in Sect. 3.

First we shall construct a bundle map from the original bundle E to a new bundle E' on which (A', Φ') lives. Fix $t \in I$. Let $\psi \in \mathbb{C}^2 \otimes V_t$. Since V_t is a subspace of $L^2(\mathbb{R}^3; S \otimes E)$, we can define a section of $S \otimes \mathbb{C}^2 \otimes E$ by

$$K_t\psi = G_{A,t}[D_{A,t}^*\,, (-\frac{d}{dt} + i\underline{x}\,)]\,\psi, \tag{4.1}$$

where $\underline{x} = \sum_{\alpha=1}^3 x_\alpha e_\alpha$. The commutator $[D_{A,t}^*\,, (-\frac{d}{dt} + i\underline{x}\,)]$ is given by a Clifford multiplication of a constant vector, so can be applied to ψ which is defined only at t. Moreover, since $\psi \in \operatorname{Ker} D_{A,t}^*$, we have

$$K_t\psi = G_{A,t}D_{A,t}^*\,(-\frac{d}{dt} + i\underline{x}\,)\,\psi.$$

Using the identification of S with $\mathbb{C}^2$, we have a contraction map

$$\omega\colon S \otimes \mathbb{C}^2 \ni (s_1, s_2) \otimes (t_1, t_2) \mapsto s_1t_2 - s_2t_1 \in \mathbb{C}.$$

Then the map $V_t \ni \psi \mapsto (\omega K_t\psi)(x) \in E_x$ gives a bundle map from the trivial bundle $\mathbb{C}^2 \otimes \underline{V_t}$ over $\mathbb{R}^3$ to E. Taking the hermitian adjoint, and moving t, we finally obtain a bundle map

$$\kappa\colon E \to \mathbb{C}^2 \otimes \underline{\Gamma(I; V)}.$$

First we show that the image of κ is in $\mathbb{C}^2 \otimes W^{-1,2}(I;V)$. As is obtained in Sect. 2, we have an estimate

$$\|G_{A,t}\varphi\|_{L^6} \le C(1+t)^{-1}\|\varphi\|_{L^2} \qquad \text{for } \varphi \in L^2(S\otimes E),$$

if t is near -1. Since $\psi \in \mathbb{C}^2 \otimes V_t = \mathbb{C}^2 \otimes \operatorname{Ker} D^*_{A,t}$ satisfies an elliptic partial differential equation, the above estimate and the L^p-estimates (cf. [GT, Chapter 9]) give us

$$|(\omega K_t\psi)(x)| \le C(1+t)^{-1}\|\psi\|_{L^2}$$

for some constant C independent of t (which may depend on x). So if $f \in \mathbb{C}^2 \otimes W_0^{1,2}(I;V)$, we have

$$\left|\int_{-1}^{1}(\omega K_t f(t))(x)\,dt\right| \le C\|f\|_{C_0^{1/2}} \le C\|f\|_{W_0^{1,2}}.$$

This means that the image of κ is in the dual space of $W_0^{1,2}$, i.e. $W^{-1,2}$.

We will prove that the image of κ is, in fact, contained in L^2, later. So the following proposition merely means that the image of κ satisfies a certain differential equation at this moment, but later it will mean that κ is a bundle map from E to E'.

Proposition 4.2. *For each $x \in \mathbb{R}^3$ the image $\kappa(E_x)$ is contained in* $\operatorname{Ker} \mathfrak{D}^*_x$.

Proof. The calculation is the straightforward adaptation of that in [KN, Proposition 6.1].

We rewrite (4.1) by using the complex notation as in Sect. 2. For $\psi = (\psi_1,\psi_2) \in \mathbb{C}^2 \otimes C_0^\infty(I;V_t)$ the section $\omega K_t\psi$ can be rewritten as

$$\sqrt{2}\,G_{A,t}\left\{\overline{\partial}^*_{A,t}(iz_2\psi_1 - iz_1\psi_2) + \overline{\partial}_{A,t}(i\overline{z}_1\psi_1 + i\overline{z}_2\psi_2)\right\},$$

where z_1, $\overline{z}_1$ are as in Sect. 2. We have the following identities (cf. [KN, Lemma 6.2]):

$$(4.3) \qquad [\overline{\partial}_{A,t}\,, i\overline{z}_1] = [\overline{\partial}^*_{A,t}\,, iz_2], \qquad [\overline{\partial}^*_{A,t}\,, iz_1] = -[\overline{\partial}_{A,t}\,, i\overline{z}_2],$$

which can be checked easily. Hence we get

$$(4.4) \qquad \omega K_t\psi = 2\sqrt{2}\,G_{A,t}\overline{\partial}^*_{A,t}(iz_2\psi_1 - iz_1\psi_2) = 2\sqrt{2}\,G_{A,t}\overline{\partial}_{A,t}(i\overline{z}_1\psi_1 + i\overline{z}_2\psi_2).$$

From (2.11), we find

$$G_{A,t}\overline{\partial}^*_{A,t}\{iz_2(\frac{d}{dt}+2\alpha)\psi_1 - 2iz_1\beta\psi_1\} = iz_1G_{A,t}\overline{\partial}^*_{A,t}(iz_2\psi_1) - iz_2G_{A,t}\overline{\partial}^*_{A,t}(iz_1\psi_1).$$

The integration of the right hand side is equal to

$$(4.5) \qquad \int_{-1}^{1} -x_1G_{A,t}\overline{\partial}^*_{A,t}(iz_2\psi_1) - iz_2G_{A,t}\overline{\partial}^*_{A,t}(iz_1\psi_1)\,dt,$$

where the term

$$\int_{-1}^{1}\frac{\partial}{\partial t}G_{A,t}\overline{\partial}^*_{A,t}(iz_2\psi_1)\,dt = \lim_{t\to 1}G_{A,t}\overline{\partial}^*_{A,t}(iz_2\psi_1) - \lim_{t\to -1}G_{A,t}\overline{\partial}^*_{A,t}(iz_2\psi_1)$$

drops because ψ_1 vanishes near the boundary. (Remember $iz_1 = \frac{\partial}{\partial t} - x_1$.) Similarly we have formula

$$\begin{aligned}
&\int_{-1}^{1} G_{A,t}\overline{\partial}_{A,t}\{2i\overline{z}_1\beta^*\psi_2 + i\overline{z}_2(-\frac{d}{dt} + 2\alpha^*)\psi_2\}\, dt \\
&= \int_{-1}^{1} -i\overline{z}_2 G_{A,t}\overline{\partial}_{A,t}(i\overline{z}_1\psi_2) + x_1 G_{A,t}\overline{\partial}_{A,t}(i\overline{z}_2\psi_2)\, dt.
\end{aligned} \tag{4.6}$$

Then (4.5), (4.6) and (4.4) imply $\kappa^*\mathfrak{D}\psi = 0$. Since ψ is arbitrary, we have $\mathfrak{D}^*\kappa = 0$. $\square$

Proposition 4.7. $\operatorname{Im}\kappa \subset \mathbb{C}^2 \otimes L^2(I;V)$.

Proof. Elements represented in the form

$$\mathfrak{D}_x f + g \qquad f \in \mathbb{C}^2 \otimes C_0^\infty(I;V),\ g \in \operatorname{Ker}\mathfrak{D}_x^* \cap L^2$$

are dense in $\mathbb{C}^2 \otimes L^2(I;V)$. The $\operatorname{Im}\kappa$ is orthogonal to $\mathfrak{D}_x f$ by Proposition 4.2. As used in [Hi3], an L^2-solution of $\mathfrak{D}_x^* g = 0$ is $O((1-t)^{(k-1)/2})$ near $t = 1$. Hence when $k > 1$, the L^2-inner product

$$\int_{-1}^{1} \langle g, \kappa(e)\rangle\, dt$$

is finite. When $k = 1$, T_α is bounded. So the equation

$$0 = \mathfrak{D}_x^*\kappa(e) = \left(-1_{\mathbb{C}^2} \otimes \nabla_t + \sum_{\alpha=1}^{3}(e_\alpha \otimes T_\alpha - ix_\alpha e_\alpha \otimes 1_V)\right)\kappa(e) \quad (e \in E_x)$$

implies that $\kappa(e)$ is bounded. So in either case, we have $\kappa(e) \in L^2$. $\square$

Proposition 4.8. *The bundle map $\kappa\colon E \to E'$ respects the metric (up to constant), the connection and the Higgs field.*

Proof. Let define

$$L_x(g) = \int_{-1}^{1} (\omega K_t g)(x)\, dt \in E_x$$

for $g \in \mathbb{C}^2 \otimes L^2(I;V)$. Take a local section ψ of $E' = \operatorname{Ker}\mathfrak{D}^*$. We regard $L_x(\psi(y))$ as a local section of $p_1^*(E)$ over $\mathbb{R}^3 \times \mathbb{R}^3$. The section $\kappa^*(\psi)$ is obtained by restricting $L_x(\psi(y))$ to the diagonal $x = y$. Then we have

$$\overline{\partial}^x_{A,0}\kappa^*(\psi) = \overline{\partial}^x_{A,0} L_x(\psi(y))\Big|_{y=x} + \overline{\partial}^y_{A,0} L_x(\psi(y))\Big|_{y=x}, \tag{4.9}$$

where the superscript x or y for $\overline{\partial}_{A,0}$ indicates the variable with respect to the differentiation is done. Using (2.10), we have

$$\begin{aligned}
&2\overline{\partial}^x_{A,t} G^x_{A,t} \overline{\partial}^{x*}_{A,t}(iz_2\psi_1(y) - iz_1\psi_2(y))\Big|_{y=x} \\
&= (iz_2 - 2\beta)\psi_1(x) - (iz_1 - (\frac{d}{dt} + 2\alpha))\psi_2(x) = -\frac{d}{dt}\psi_2(x),
\end{aligned}$$

where we have used $\mathfrak{D}_x^*\psi(x) = 0$ in the latter equality. The integration of the right hand side over I vanishes because

$$\lim_{t\to\pm 1} \frac{\psi_2(x)}{\|\psi_2\|_{L^2}} = 0.$$

(This follows from the study of the asymptotic behaviour as $t \to \pm 1$ in Sect. 2.) Noticing $\overline{\partial}_{A,t} = \overline{\partial}_{A,0} - it$, $\Phi' = p \circ it$ and $\kappa^*\mathfrak{D} = 0$, we get

$$\overline{\partial}_{A,0}^x L_x(\psi(y))\Big|_{y=x} = 2\sqrt{2}\int_{-1}^{1} it\, G_{A,t}^x \overline{\partial}_{A,t}^{x*}(iz_2\psi_1(y) - iz_1\psi_2(y))\Big|_{y=x} dt$$

$$= 2\sqrt{2}\int_{-1}^{1} G_{A,t}^x \overline{\partial}_{A,t}^{x*}(iz_2\Phi'(y)\psi_1(y) - iz_1\Phi'(y)\psi_2(y))\Big|_{y=x} dt.$$

Substituting into (4.9) and using $\kappa^*\mathfrak{D} = 0$ again, we obtain

$$\overline{\partial}_{A,0}\kappa^*(\psi) = \kappa^*(\overline{\partial}_{A',0}\psi).$$

Changing the isomorphism $\mathbb{R}^3 \cong \mathbb{R}\times\mathbb{C}$, we can conclude that κ respects the connection and the Higgs field. If κ is a zero map, $\omega K_t\psi = 0$ for all $\psi \in V_t$. But the equation implies $\psi = 0$, and it is a contradition. Since monopole connections are irreducible, κ preserves the fiber metrics up to a constant factor. □

5. Uniqueness

We now finish the proof of our main theorem.

$$\begin{array}{ccccc} \text{Nahm data} & \overset{\S 3}{\Longrightarrow} & \text{a monopole} & \overset{\S 2}{\Longrightarrow} & \text{new Nahm data} \\ (\nabla, T_\alpha) & & (A,\Phi) & & (\nabla', T'_\alpha) \end{array}$$

Starting from Nahm data V, ∇, T_α $(\alpha = 1,2,3)$, we construct a rank 2 vector bundle E over $\mathbb{R}^3$ with a connection A and a Higgs field Φ which satisfy the Bogomolny equation in Sect. 3. We then get new Nahm data V', ∇', T'_α by the transform in Sect. 2. The aim of this section is to show that there exists a isomorphism $V \cong V'$ under which ∇ and T_α correspond to ∇' and T'_α. This shows the uniqueness of the Nahm data corresponding to a monopole.

The proof is exactly "dual" to that of the completeness. Fix $x \in \mathbb{R}^3$. Let $f \in S\otimes E_x$. Since E_x is a subspace of $\mathbb{C}^2\otimes L^2(I;V)$, we can define a section of $S\otimes\mathbb{C}^2\otimes V$ by

$$F_x\,[\mathfrak{D}_x^*, \mathbb{D}^*]\,f,$$

where $\mathbb{D}^*$ is the operator acting on sections of the bundle $S\otimes\underline{\mathbb{C}^2\otimes L^2(I;V)}$ over $\mathbb{R}^3$ defined in Sect. 3. Although f is defined only at x, the commutator $[\mathfrak{D}_x^*, \mathbb{D}^*]$ is equal to multiplication by a constant vector, so can be applied to f. Contracting the $S\otimes\mathbb{C}^2$-factor by ω, taking the hermitian adjoint and moving $x \in \mathbb{R}^3$, we obtain a bundle map

$$\lambda\colon V \to \underline{\Gamma(\mathbb{R}^3; S\otimes E)}$$

over I.

First we show that the image of λ is in $L^{2+\mu}$ for any $\mu > 0$. We use the notation C to denote a general constant; C may be different in different equations. If $g \in W_0^{1,2}(I;V)$, we have an estimate

$$
\begin{aligned}
&(\nabla_t^* \nabla_t g + \sum_{\alpha=1}^{3} (T_\alpha - i x_\alpha)^* (T_\alpha - i x_\alpha) g, g)_{L^2(I;V)} \\
&= \|\nabla_t g\|^2_{L^2(I;V)} + \sum_{\alpha=1}^{3} \|(T_\alpha - i x_\alpha) g\|^2_{L^2(I;V)} \\
&\geq \|\nabla_t g\|^2_{L^2(I;V)} - \delta \sum_{\alpha=1}^{3} \|T_\alpha g\|^2_{L^2(I;V)} + \frac{r^2 \delta}{2} \|g\|^2_{L^2(I;V)},
\end{aligned} \tag{5.1}
$$

where $\delta < 1$ is a positive number, which will be fixed later. Using the Sobolev inequlity

$$\|g\|_{C^{1/2}(I;V)} \leq C \|\nabla_t g\|_{L^2(I;V)}$$

and the asymptotic behaivour of T_α as $t \to \pm 1$, we find

$$\|T_\alpha g\|^2_{L^2(I;V)} \leq C \|\nabla_t g\|^2_{L^2(I;V)}.$$

Substituting this inequality into (5.1) and choosing δ sufficiently small, we get

$$(\nabla_t^* \nabla_t g + \sum_{\alpha=1}^{3} (T_\alpha - i x_\alpha)^* (T_\alpha - i x_\alpha) g, g)_{L^2(I;V)} \geq \frac{1}{C} (\|\nabla_t g\|^2_{L^2(I;V)} + r^2 \|g\|^2_{L^2(I;V)}).$$

Hence for $g = F_x h$, we have

$$\|F_x h\|_{C^{1/2}(I;V)} \leq \frac{C}{r} \|h\|_{L^2(I;V)}.$$

Fix an $\varepsilon > 0$ and take $f \in L^{2-\varepsilon}(\mathbb{R}^3; S \otimes E)$. Then for $v \in V_t$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \langle \lambda(v), f(x) \rangle \, dx = \int_{\mathbb{R}^3} \langle v, (\omega F_x \, [\mathfrak{D}_x^*, \mathbb{D}^*] \, f(x))(t) \rangle \, dx \\
&\leq C \|v\|_{V_t} \int_{\mathbb{R}^3} (1 + r)^{-1} \|f(x)\|_{S \otimes E_x} \, dx \leq C_\varepsilon \|v\|_{V_t} \, \|f\|_{L^{2-\varepsilon}(\mathbb{R}^3; S \otimes E)},
\end{aligned}
$$

where C_ε is a constant depending on ε. This shows that $\lambda(v) \in L^{2+\mu}$ for any $\mu > 0$.

Proposition 5.2. *For each $t \in I$ the image $\lambda(V_t)$ is contained in* Ker $D^*_{A,t}$.

The proof is exactly "dual" to that of (4.2), and we skip it. Once we obtain the above, we deduce the following since any $L^{2+\mu}$-solution of $D^*_{A,t}\varphi = 0$ decays exponentially.

Corollary 5.3. *λ defines a bundle map from V to V'.*

Finally we get the following which can be proved by the argument similar to (4.8).

Proposition 5.4. *The bundle map $\lambda\colon V \to V'$ intertwines the Nahm data.*

6. Metrics on moduli spaces

Our transform identifies the (framed) moduli space of SU(2)-monopoles of charge k with the moduli space of the solutions of Nahm's equations of rank k. These moduli spaces are well-known to admit hyper-Kähler metrics. Atiyah and Hitchin conjectured that our transform is actually a hyper-Kähler isometry [AH, p.126]. We shall verify this conjecture. The corresponding results for the Fourier transforms of instantons on 4-tori and on ALE spaces are proved respectively in [BB] and [KN] (in the case of $\mathbb{R}^4$ independently in [Ma]).

We shall review the construction of a hyper-Kähler structure on the monopole moduli space very quickly. See [AH] and the reference therein for detail. (The Analytical footing was established by Taubes [Ta].)

We introduce an equivalence relation $\sim$ on the space of SU(2)-monopoles of charge k by defining $(A, \Phi) \sim (A', \Phi')$ if and only if (A, Φ) and (A', Φ') are gauge-equivalent under a gauge transformation converging to the identity as $x \to \infty$. Let denote $\mathfrak{M}_k$ the set of equivalence classes. Then the following is well-known:

Proposition 6.1. *The space $\mathfrak{M}_k$ has a structure of a smooth manifold and its tangent space at $[(A, \Phi)]$ is identified with the space of (a, ϕ) which are in L^2 and satisfy the equations*

$$* \, d_A a - d_A \phi + [\Phi, a] = 0$$

$$* \, d_A * a - [\Phi, \phi] = 0.$$

Here a and ϕ are an $\mathfrak{su}(2)$-valued 1-form and function respectively.

The second equation is a linearization of Bogomolny equations, while the first one means that (a, ϕ) is orthogonal to the orbit of gauge group action.

The space of all pairs (a, ϕ) has a structure of quaternion module. In fact, if $a = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$, then (a, ϕ) corresponds to the $\mathfrak{su}(2) \otimes \mathbb{H}$-valued function $\phi + a_1 I + a_2 J + a_3 K$, where I, J, K are the usual basis of imaginary quaternions. The equations in (6.1) are $\mathbb{H}$-invariant.

The L^2-inner product induces a Riemannian metric on $\mathfrak{M}_k$. Then one can show that

Proposition 6.2. *The almost complex structures I, J, K are parallel with respect to the Levi-Civita connection of the Riemannian metric. Hence $\mathfrak{M}_k$ has a structure of hyper-Kähler manifold.*

In fact, the Bogomolny equation can be viewed as a hyper-Kähler moment map (see [HKLR]) associated with the action of the gauge group, and the moduli space is a hyper-Kähler quotient of an infinite dimensional quaternion module.

The construction of a hyper-Kähler structure on the moduli space of the solutions to Nahm's equations of rank k is similar to the above. We fix a trivialization of a bundle V so that the connection is given by $\nabla = \frac{d}{dt} + T_0$ where T_0 is a skew-adjoint endomorphism. We say two Nahm data (T_α), (T'_α) $(\alpha = 0, 1, 2, 3)$ are equivalent if they are gauge equivalent under a gauge transformation converging to the identity at the end points of the interval. We denote by $\mathfrak{N}_k$ the set of equivalence classes. Then

Proposition 6.3. *The space $\mathfrak{N}_k$ has a structure of a smooth manifold and its tangent space at $[(T_\alpha)]$ is identified with the space of (t_0, t_1, t_2, t_3) which are in L^2*

and satisfy the equations

$$\frac{dt_0}{dt} + [T_0, t_0] + [T_1, t_1] + [T_2, t_2] + [T_3, t_3] = 0$$
$$\frac{dt_\alpha}{dt} + [T_0, t_\alpha] - [T_\alpha, t_0] + \sum_{\beta,\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma}[T_\beta, t_\gamma] = 0, \qquad \alpha = 1, 2, 3.$$

The metric on $\mathfrak{N}_k$ is defined by the L^2-inner product. We give an $\mathbb{H}$-module structure to the tangent space by identifying (t_0, t_1, t_2, t_3) with $t_0 + t_1 I + t_2 J + t_3 K$.

Proposition 6.4. *The manifold $\mathfrak{N}_k$ together with the above structures is hyper-Kähler.*

Now our main result in this section is

Theorem 6.5. *The transform $\Xi\colon \mathfrak{M}_k \to \mathfrak{N}_k$ given in Sect. 2 is a hyper-Kähler isometry up to a constant factor.*

The proof of Theorem 6.5 is very similar to that for instantons on ALE spaces [KN].

Suppose that a family (T_α^s) $(-\varepsilon < s < \varepsilon)$ of solutions of Nahm's equations is given. For brevity, we omit the superscript s. Let e_μ be a unitary frame field for $E = \operatorname{Ker}\mathfrak{D}^*$. Then the derivative δe_μ with respect s satisfies $\mathfrak{D}^*\delta e_\mu = -(\delta\mathfrak{D}^*)e_\mu$. If we normalize δe_μ by requiring $\delta e_\mu \perp E$, this equation implies

$$\delta e_\mu = -\mathfrak{D}(1_{\mathbb{C}^2} \otimes F)(\delta\mathfrak{D}^*)e_\mu.$$

The derivative of the connection $A_{\mu\nu} = \langle de_\mu, e_\nu\rangle$ and the Higgs field $\Phi_{\mu\nu} = \langle ite_\mu, e_\nu\rangle$ are given by

$$\begin{aligned}\delta A_{\mu\nu} &= \langle(\delta\mathfrak{D}^*)e_\mu, (1_{\mathbb{C}^2}\otimes F)\mathfrak{D}^* de_\nu\rangle - \langle(1_{\mathbb{C}^2}\otimes F)\mathfrak{D}^* de_\mu, (\delta\mathfrak{D}^*)e_\nu\rangle \\ \delta\Phi_{\mu\nu} &= \langle(\delta\mathfrak{D}^*)e_\mu, (1_{\mathbb{C}^2}\otimes F)\mathfrak{D}^* ite_\nu\rangle - \langle(1_{\mathbb{C}^2}\otimes F)\mathfrak{D}^* ite_\mu, (\delta\mathfrak{D}^*)e_\nu\rangle.\end{aligned} \tag{6.6}$$

Next suppose that a family (A^s, Φ^s) of monopoles is given. We omit the superscript s. Let v_i be a unitary frame field for $V = \bigcup_t \operatorname{Ker} D_{A,t}^*$. Then the derivative of $(T_\alpha)_{ij} = \langle ix_\alpha v_i, v_j\rangle$ and of $(T_0)_{ij} = \langle(\nabla_t - \frac{d}{dt})v_i, v_j\rangle$ are given by

$$\begin{aligned}\delta(T_\alpha)_{ij} &= \langle(\delta A)\cdot v_i - (\delta\Phi)v_i\,,\,(1_S\otimes G_{A,t})D_{A,t}^*(ix_\alpha v_j)\rangle \\ &\qquad - \langle(1_S\otimes G_{A,t})D_{A,t}^*(ix_\alpha v_i)\,,\,(\delta A)\cdot v_j - (\delta\Phi)v_j\rangle \\ \delta(T_0)_{ij} &= \langle(\delta A)\cdot v_i - (\delta\Phi)v_i\,,\,(1_S\otimes G_{A,t})D_{A,t}^*\frac{dv_j}{dt}\rangle \\ &\qquad - \langle(1_S\otimes G_{A,t})D_{A,t}^*\frac{dv_i}{dt}\,,\,(\delta A)\cdot v_j - (\delta\Phi)v_j\rangle,\end{aligned} \tag{6.7}$$

where $(\delta A)\cdot v_i = \sum_{\alpha=1}^{3}\delta A(\frac{\partial}{\partial x_\alpha})\frac{\partial}{\partial x_\alpha}\cdot v_i$.

We fix a particular complex structure I, and regard the moduli spaces $\mathfrak{M}_k$ and $\mathfrak{N}_k$ as Kähler manifolds. First we shall show that the differential $d\Xi$ of our transformation respects the almost complex structures, i.e., Ξ is a holomorphic map. Then changing the complex structure, one can show that Ξ is holomorphic with respect to each complex structure I, J, K.

We rewrite (6.6) in the complex notation. We identify the monopole (A, Φ) with an invariant instanton on $\mathbb{R}^4 \cong \mathbb{C}^2$, hence $(\delta A, \delta\Phi)$ can be considered as a 1-form on $\mathbb{C}^2$. Then $(0,1)$-part of (6.6) (up to a constant factor) is given by

$$\begin{aligned} &\langle (\delta\mathfrak{D}^*)e_\mu, (1_{\mathbb{C}^2}\otimes F)\,\mathfrak{D}^*\overline{\partial}^*(e_\nu\omega_{\mathbb{C}})\rangle + \langle (1_{\mathbb{C}^2}\otimes F)\mathfrak{D}^*\overline{\partial}e_\mu, (\delta\mathfrak{D}^*)e_\nu\rangle \\ &= \langle (\delta\tau)e_\mu, F\tau\overline{\partial}^*(e_\nu\omega_{\mathbb{C}})\rangle + \langle F\sigma^*\overline{\partial}e_\mu, (\delta\sigma^*)e_\nu\rangle, \end{aligned} \tag{6.8}$$

where the inner product is taken over the fiber component, have nothing to do with the form component, and $\omega_{\mathbb{C}}$ is the $(0,2)$-form of unit length. Recall that $\{v_i\}$ be a unitary frame for V. Substituting $\lambda = (\omega\, F[\mathfrak{D}^*, \mathbb{D}^*])^*$, we find that (6.8) is equal to (up to a constant factor)

$$\langle (\delta\tau)e_\mu, v_i\rangle\langle\lambda(v_i), e_\nu\rangle + \langle v_i, (\delta\sigma^*)e_\nu\rangle\langle e_\mu, \varepsilon\lambda(v_i)\rangle. \tag{6.9}$$

Here ε is an endomorphism defined by

$$\Lambda^{0,1} \ni a d\overline{z}_1 + b d\overline{z}_2 \mapsto b d\overline{z}_1 - a d\overline{z}_2.$$

Similarly the $(0,1)$-component of (6.7) (up to a constant factor) is given by

$$\begin{aligned} &\langle (\delta A_2 + i\delta A_3, -\delta\Phi - i\delta A_1)\, v_i, e_\mu\rangle\langle\kappa(e_\mu), v_j\rangle \\ &\quad + \langle v_i, \varepsilon\kappa(e_\mu)\rangle\langle e_\mu, (-\delta\Phi + i\delta A_1, -\delta A_2 + i\delta A_3)v_j\rangle, \end{aligned} \tag{6.10}$$

where we take a unitary basis $\{\frac{1}{\sqrt{2}}d\overline{z}_1, \frac{1}{\sqrt{2}}d\overline{z}_2\}$ for $\Lambda^{0,1}$ and $\kappa(e_\mu) \in \mathbb{C}^2 \otimes \underline{\Gamma(I;V)}$ is considered as a $(0,1)$-form. Here we have used the decomposition of the matrix $\delta A \cdot -\delta\Phi\colon S\otimes E \to S\otimes E$:

$$\begin{pmatrix} -\delta\Phi + i\delta A_1 & -\delta A_2 + i\delta A_3 \\ \delta A_2 + i\delta A_3 & -\delta\Phi - i\delta A_1 \end{pmatrix}.$$

If we apply I to (δT_α), $\delta\tau$ is multiplied by i and $\delta\sigma^*$ by $-i$. Hence the (6.8) is multiplied by i, and we verify the assertion. (REMARK that when we consider $(\delta A, \delta\Phi)$ as a tangent vector in the moduli space, its $(1,0)$-part is given by the $(0,1)$-part of $(\delta A, \delta\Phi)$, considered as 1-form.)

The only thing left to be proved is whether the map $d\Xi$ is isometry, i.e.

$$\langle d\Xi(\delta T_0, \delta T_1, \delta T_2, \delta T_3)\,,\,(\delta A, \delta\Phi)\rangle = c\langle(\delta T_0, \delta T_1, \delta T_2, \delta T_3)\,,\, d\Xi^{-1}(\delta A, \delta\Phi)\rangle. \tag{6.11}$$

holds for some constant c. This can be checked by using (6.9) and (6.10) and the fact : λ and κ are isometries up to constant factors. Rigorously speaking, we have not proved that the constants, up to which λ and κ are isometries, are independent of (A, Φ), T_α. So the constant c in (6.11) may change if we move (A, Φ). But we already observed that Ξ respects almost complex structures I, J, K. If a map between hyper-Kähler manifolds respects almost complex structures, it also respects the Levi-Civita connection. Hence c is a constant function.

7. Remark

A Fourier transform of invariant instantons. We explain, briefly and without proofs, how the transformation in Sects. 2 and 3 can be generalized for anti-self-dual connections on $\mathbb{R}^4$, invariant under a subgroup of translation $\Lambda \subset \mathbb{R}^4$. This is already noticed in [BB, p. 272], but it is worth while explaining again.

Let $(\mathbb{R}^4)^*$ denote the dual space of $\mathbb{R}^4$ and define

$$\Lambda^* = \{\lambda^* \in (\mathbb{R}^4)^* \mid \lambda^*(\lambda) \in \mathbb{Z}, \forall \lambda \in \Lambda\}.$$

For example, when $\Lambda = \mathbb{R}$ (this is our case), $\Lambda^* = \mathbb{R}^3$. Define a connection 1-form $\mathbb{A}$ on the trivial line bundle $\mathbb{L} \to \mathbb{R}^4 \times (\mathbb{R}^4)^*$ by

$$\mathbb{A} = -2\pi i \sum_{\alpha=0}^{3} q_\alpha dx_\alpha,$$

where x_α and q_α are dual linear coordinates on $\mathbb{R}^4$ and $(\mathbb{R}^4)^*$. The action of $\Lambda \times \Lambda^*$ on $\mathbb{R}^4 \times (\mathbb{R}^4)^*$ lifts to that on $\mathbb{L}$ by

$$\mathbb{L} = \mathbb{R}^4 \times (\mathbb{R}^4)^* \times \mathbb{C} \ni (x, q, \zeta) \mapsto (x + \lambda, q + \lambda^*, e^{2\pi i \lambda^*(x)}\zeta) \quad \text{for } (\lambda, \lambda^*) \in \Lambda \times \Lambda^*.$$

This action preserves $\mathbb{A}$.

Now suppose that we have a connection A on a bundle E over $\mathbb{R}^4$ invariant under Λ. For each $q \in (\mathbb{R}^4)^*$, consider the Dirac operator twisted by the connection $\mathbb{A}$ and A:

$$D^{\pm}_{A,q} \colon \Gamma(S^{\pm}_{\mathbb{R}^4} \otimes E \otimes \mathbb{L}|_{\mathbb{R}^4 \times \{q\}}) \to \Gamma(S^{\mp}_{\mathbb{R}^4} \otimes E \otimes \mathbb{L}|_{\mathbb{R}^4 \times \{q\}}),$$

where $S^{\pm}_{\mathbb{R}^4}$ is the spinor bundle over $\mathbb{R}^4$. Define

$$\hat{E}_q = \Lambda\text{-invariant part of the } L^2\text{-kernel of } D^{-}_{A,q},$$

where the L^2-metric is taken over $\mathbb{R}^4/\Lambda$. Assume that

a) Λ-invariant part of the L^2-kernel of $D^{+}_{A,q} = 0$,

b) $\hat{E} = \bigcup_q \hat{E}_q$ forms a vector bundle over $(\mathbb{R}^4)^*$.

Then considering $\hat{E}$ as a subbundle of a (may be infinite rank) vector bundle

$$\mathcal{H} = \bigcup_q \Lambda\text{-invariant part of } L^2(S^{-}_{\mathbb{R}^4} \otimes E \otimes \mathbb{L}|_{\mathbb{R}^4 \times \{q\}}),$$

we induce a metric and a connection $\hat{A}$ on $\hat{E}$. Here the connection on $\mathcal{H}$ is defined from A and $\mathbb{A}$. The action of $\Lambda \times \Lambda^*$ on $\mathbb{L}$ naturally induces an action of Λ^* on $\hat{E}$ and $\hat{A}$ is invariant under this action. Changing the role of x and q and using the dual connection $\mathbb{A}^*$ instead of $\mathbb{A}$, we define a similar transform (denoted by $\check{}$) from a Λ^*-invariant connection satisfying

a') Λ^*-invariant part of the L^2-kernel of $D^{+}_{A,x} = 0$,

b') $\check{E} = \bigcup_x \check{E}_x$ forms a vector bundle over $\mathbb{R}^4$.

to a Λ-invariant connection. Then one has

Theorem 7.1. *If A is anti-self-dual and satisfies a), b), then $\hat{A}$ is anti-self-dual and satisfies a'), b'). Moreover $\check{\hat{A}}$ is gauge equivalent to A.*

This theorem is not proven in full generality. In fact, we must put the condition on the asymptotic behaviour of the connection in order to ensure that the Fredholm theory is valid. In some cases, the connection is not defined over the whole space and may have singularities (as is observed in this paper). Such a modification and the precise proof are given only in the cases of $\Lambda = 0$ (ordinary instantons on $\mathbb{R}^4$ [ADHM, CG, DK]), $\Lambda = \mathbb{R}$ (monopoles on $\mathbb{R}^3$) and $\Lambda \cong \mathbb{Z}^4$ (instantons on torus $\mathbb{R}^4/\Lambda$ [BB, Sc, DK]).

Appendix

In this appendix, we shall give the proof of Lemma 2.6. The following proof is due to Toshiyuki Kobayashi.

Proof. Let denote by V_k the unique $(k+1)$-dimensional irreducible representation of SU(2). By the theorem of Peter-Weyl the space $L^2(\mathbb{P}^1; H^k)$ of L^2-sections of H^k (by which we denote the k-times tensor product of the hyperplane bundle) over $\mathbb{P}^1$ decomposes into

$$L^2(\mathbb{P}^1; H^k) = \bigoplus_{l \geq 0} V_{k+2l},$$

and the space $H^0(\mathbb{P}^1; \mathcal{O}(k))$ of holomorphic sections is the component V_k. The set of coordinate functions $\{x_1, x_2, x_3\}$ induces the 3-dimensional representation V_2. Then the multiplication of the coordinate function gives an SU(2)-equivariant map

$$m\colon L^2(\mathbb{P}^1; H^k) \otimes V_2 \to L^2(\mathbb{P}^1; H^k).$$

By the Clebsch-Gordan rule

$$V_k \otimes V_2 \cong V_{k+2} \oplus V_k \oplus V_{k-2}. \tag{A.1}$$

The map defined in Lemma 2.6 is the composition of

$$V_k \otimes V_2 \xrightarrow{\text{inclusion}\otimes\text{id}} L^2(\mathbb{P}^1; H^k) \otimes V_2 \xrightarrow{m} L^2(\mathbb{P}^1; H^k) \xrightarrow{\text{projection}} V_k. \tag{A.2}$$

It is SU(2)-equivariant, and must be a constant multiple of the projection onto the second component in (A.1).

On the other hand, the adjoint representation of SU(2) is also V_2. So we have a linear map

$$V_k \otimes V_2 \to V_k; \quad v \otimes X \mapsto Xv$$

where V_2 is regarded as (the complexfication of) the Lie algebra $\mathfrak{su}(2)$ and it acts on V_k by the differential of the action of SU(2) on V_k. This map is also SU(2)-equivariant, so must be a constant multiple of the projection onto the second component in (A.1).

Finally we must check that the map (A.2) is non-zero. It is sufficient to show that the map

$$V_k \otimes V_2 \xrightarrow{\text{inclusion}\otimes\text{id}} L^2(\mathbb{P}^1; H^k) \otimes V_2 \xrightarrow{m} L^2(\mathbb{P}^1; H^k) = \bigoplus_{l \geq 0} V_{k+2l}$$

has rank strictly greater that $\dim V_{k+2}$. (Since it is SU(2)-equivariant, the image is contained in $V_k \oplus V_{k+2}$.) Elements in V_k can be represented by homogeneous polynomial in z_0 and z_1 of degree k:

$$z_0^k, z_0^{k-1} z_1, \ldots, z_0 z_1^{k-1}, z_1^k.$$

It is easy to see that if we multiply the above functions by $x_1 = \frac{2\mathrm{Re} z_0}{1+|z_0|^2}$ and $x_2 = \frac{2\mathrm{Im} z_0}{1+|z_0|^2}$ (where we take an affine coordinate $[z_0 : 1] \in \mathbb{P}^1$ by setting $z_1 = 1$), we obtain $2(k+1)$ linearly independent functions. Hence if $2(k+1) > k+3$, we are done. And in the exceptional case $k = 0, 1$, we can check the assertion case by case. $\square$

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