Morse theory on moduli spaces of instantons on ALE scalar-flat Kähler surfaces

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Abstract. LeBrun constructed a scalar-flat Kähler metric on the total space of Chern class $-n$ line bundle $\mathcal{O}(-n) \to \mathbb{C}\mathrm{P}^1$. We study moduli spaces of ASD connections on it. It is known that the natural L^2-metrics on them are kählerian. We study them when the metric is complete. We give an algorithm to compute their Betti numbers. On the way of the proof, we also show that their homology groups have no torsion and vanish in odd degrees. Our method is the Morse theory, can be applied to a wider class of noncompact 4-manifolds.

1. Introduction

In the conference I talked about homology groups of moduli spaces of instantons on ALE hyper-Kähler 4-manifold constructed by Kronheimer [Kr1]. (The paper [Na2] will appear elsewhere.) After the talk, Y.S. Poon suggested me to apply the same technique to moduli spaces of instantons on other ALE Kähler surfaces (e.g., ALE scalar-flat Kähler metrics constructed by LeBrun [Le1, 2]). This paper gives an affirmative answer.

Let (X, g) be an ALE scalar-flat Kähler surface (see §2 for the definition). Take a hermitian vector bundle E over X and consider the moduli space $\mathfrak{M}$ of ASD connections on E (see §3 for the precise definition). It is known that $\mathfrak{M}$ has a natural Kähler structure induced from that on X [Na1]. We feel an interst in further geometric properties of $\mathfrak{M}$, e.g., the topology, when the metric is complete. Though the metric is incomplete in general, we shall give a criterion for the completeness of the metric (Proposition 3.4). Now suppose that the base space X has a $\mathrm{U}(1)$-action preserving the Kähler structure and satisfying the asymptotic condition (4.1). Then it induces a $\mathrm{U}(1)$-action on the moduli space. Following the approach due to Frankel [Fr] (see also [CS, At, Ki]), we use the corresponding moment map as a Morse function. The critical points set corresponds to the fixed points set of the action, and its components are submanifolds of $\mathfrak{M}$. Under the further assumption (4.3), we can perturb the Morse function to have only isolated critical points of even indices. Critical points and their indices have geometric meaning, thus we have an algorithm to compute homology groups of $\mathfrak{M}$ (Theorem 4.7).

Our previous result [Na2] gives an improvement of the algorithm in the case of ALE hyper-Kähler 4-manifolds, but relies on the ADHM description of the moduli space [KN]. On the other hand, our result is purely geometric.

Supported in part by Grant-in-Aid for Scientific Reserch (No. 03740014), Ministry of Education, Science and Culture, Japan

2. ALE scalar-flat Kähler surfaces

Let X be a complex surface. A Kähler metric g on X is called *scalar-flat* if its scalar curvature is identically zero. Such metrics attract our interest since they have anti-self-dual Weyl curvature [Ga]: hence twistor spaces with *integrable* complex structures [AHS].

Though lots of *compact* scalar-flat Kähler surfaces are known and studied extensively, we here will study noncompact spaces with the ALE condition. ALE stands for *asymptotically locally Euclidean* and means that our 4-manifold (X, g) is assumed to be complete, and that there exists a compact set K such that $X \setminus K$ is diffeomorphic to $(\mathbb{R}^4 \setminus \overline{B_R})/\Gamma$ for some finite subgroup Γ of $\mathrm{O}(4)$ acting freely on $\mathbb{R}^4 \setminus \overline{B_R}$, and the metric g approximates the Euclidean metric. Such a coordinate system is called *a coordinate system at infinity*. Since we are considering Kähler metrics, the group Γ is a subgroup of $\mathrm{U}(2)$, and the complex structure I on X approximates that on the Euclidean space $\mathbb{C}^2 = \mathbb{R}^4$.

Many examples of such spaces are known. A trivial example is, of course, the Euclidean space $\mathbb{C}^2$. Kronheimer [Kr1] constructed such metrics on the minimal resolution of $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $\mathrm{SU}(2)$. His spaces have the further property: hyper-Kähler structures. Other examples were given by LeBrun [Le1, 2]. He constructed a scalar-flat Kähler metric on the total space of any complex line bundle over $\mathbb{CP}^1$ with the first Chern class $c_1 < -2$. The fundamental group Γ of the end is the cyclic group of order $\#\Gamma = -c_1$. He also constructed such a metric on the blow-up of $\mathbb{C}^2$ at n-points situated along a straight complex line. The projection onto $\mathbb{C}^2$ gives a coordinate system at infinity. In particular, $\Gamma = \{e\}$.

The above constructions seems closely related to the geometry of the moduli spaces of ASD connections on them. However we do not review them here as we only give a *general* algorithm for the calculation of Betti numbers. If one wants to know about further properties of geometry, he/she certainly needs to know the constructions. For example, we describe our algorithm in terms of the Young tableaux when the space is Kronheimer's one [Na2]. We needed the detailed information about his construction.

3. Moduli spaces of ASD connections

Constructions of the moduli spaces of ASD connections on ALE spaces are discussed in [Na1] in detail. We must introduce weighted Sobolev norms in order to argue rigorously, but here we omit the analytic details for the sake of brevity. The interested reader should consult the above mentioned paper.

As in the previous section, (X, g) is assumed to be an ALE scalar-flat Kähler surface with a coordinate system at infinity $X \setminus K \to (\mathbb{R}^4 \setminus \overline{B_R})/\Gamma$. Let us take a representation $\rho\colon \Gamma \to \mathrm{U}(r)$. Take a hermitian vector bundle E over X and suppose that there exists a connection A_0 on E whose restriction to the end $X \setminus K$ is flat and corresponds to the representation ρ. Let $\mathcal{A}$ be the set of connections A on E such that

$$|\overbrace{\nabla_{A_0} \cdots \nabla_{A_0}}^{l \text{ times}}(A - A_0)| = O(r^{-3-l}),$$

where r is the distance function from a point in X. Let $\mathcal{G}_0$ be the group of gauge transformations s with

$$|\overbrace{\nabla_{A_0}\cdots\nabla_{A_0}}^{l\text{ times}}(s-\mathrm{id})| = O(r^{-2-l}).$$

This group acts on $\mathcal{A}$ by pull-back. Then the moduli space of ASD connections is defined by

$$\mathfrak{M} \overset{\text{def.}}{=} \{A \in \mathcal{A} \mid *R_A = -R_A\}/\mathcal{G}_0.$$

It is shown that the moduli space is a C^∞-manifold near the gauge equivalence class $[A]$, if

$$0 = L^2\text{–}\operatorname{Ker} d_A^*\colon \Omega^+(\operatorname{Endskew} E) \to \Omega^1(\operatorname{Endskew} E),$$

where $\operatorname{Endskew} E$ is the bundle of skew-adjoint endomorphisms of E. Thanks to the anti-self-duality of the Weyl tensor and the vanishing of the scalar curvature, we can prove that this condition is satisfied for any ASD connection A on X using Bochner-Weitzenböck formula (see [Na1, 5.1]). Hence the moduli space $\mathfrak{M}$ is a smooth manifold. Its dimension is given by the index formula (see [Na1, 2.7]). The tangent space at $[A]$ can be identified with

$$L^2\text{–}\operatorname{Ker}(d_A^+ \oplus d_A^*)\colon \Omega^1(\operatorname{Endskew} E) \to \Omega^+(\operatorname{Endskew} E) \oplus \Omega^0(\operatorname{Endskew} E).$$

The L^2-inner product gives a Riemannian metric on $\mathfrak{M}$. The almost complex structure I on X induces an endomorphism on the cotange bundle T^*X, and the $L^2\text{–}\operatorname{Ker}(d_A^+ \oplus d_A^*)$ is invariant under it. Hence we have an almost complex structure $I_{\mathfrak{M}}$ on $\mathfrak{M}$. And it is known that $I_{\mathfrak{M}}$ is covariant constant with respect to the Levi-Civita connection of the L^2-metric [Na1, 2.6]. Summarizing the above results, we have

Theorem 3.1. *The moduli space $\mathfrak{M}$ of ASD connections on the ALE scalar-flat Kähler surface X is a Kähler manifold.*

Before discussing the further properties of moduli spaces, we relate our moduli space to the moduli space of ASD connections on the one-point compactification $\widehat{X} = X \cup \{\infty\}$. The ALE condition allows us to give $\widehat{X}$ the structure of the orbifold. There exists an orbifold metric $\widehat{g}$ on $\widehat{X}$ which is conformal to g on X. (See [Kr2, p.686].) ASD connections with the above asymptotic condition extend to $\widehat{X}$. They all live a fixed orbifold vector bundle $\widehat{E}$. The fiber $\widehat{E}$ over ∞ has a Γ-action which is isomorphic to ρ. Then it is not hard to see

Proposition 3.2. *The moduli space $\mathfrak{M}$ is homeomorphic to the framed moduli space of ASD connections on $\widehat{E}$, that is the set of isomorphism classes of pairs:*

$$(\textit{ASD connection } A \textit{ on } \widehat{E},\ \Gamma\textit{-equivariant isomorphism } \varphi\colon \widehat{E}_\infty \to \mathbb{C}^r).$$

We return to study geometric properties of the moduli space $\mathfrak{M}$. The first is the natural 'symmetry'. The change of the framing induces an action on $\mathfrak{M}$:

Proposition 3.3. *Let G_ρ be the stabilizer of the representation ρ:*

$$G_\rho \overset{\text{def.}}{=} \{s \in \mathrm{U}(r) \mid s\rho s^{-1} = \rho\}.$$

Then there exists an action of G_ρ on the moduli space $\mathfrak{M}$ which preserves the L^2-metric and the complex structure.

Next we want to discuss the completeness of the metric. It relates to Uhlenbeck's compactness theorem applied to the orbifold $\widehat{X}$. Let $[A_i]$ be a sequence in $\mathfrak{M}$. Then we have a subsequence $[A_j]$ such that

(1) there exists a finite set $\{x_1, \dots, x_n\} \subset \widehat{X}$ such that A_j converges to an ASD connection A_∞ outside it after gauge transformations,
(2) there exist constants a_k $(k = 1, \dots, n)$ such that the curvature densities $|R_{A_j}|^2 dV$ converge to

$$|R_{A_\infty}|^2 dV + \sum_k a_k \delta_{x_k}.$$

The above constant a_k relates to the curvature integral of ASD connection bubbling out around x_k. If x_k is a regular point of $\widehat{X}$ (i.e., $x_k \in X$), a_k is an integer multiple of $8\pi^2$. On the other hand, if $x_k = \infty$, a_k is an integer multiple of $8\pi^2/\#\Gamma$, where $\#\Gamma$ is the order of Γ.

Proposition 3.4. *The L^2-metric on the moduli space $\mathfrak{M}$ is complete if we have $S = \emptyset$ or $S = \{\infty\}$ for any sequence $[A_i]$ as above.*

Proof. Let $[A_t]$ $(t \in [0, t_0))$ be an open curve of finite length. We want to show that $[A_t]$ has a limit point. As is shown in [Na1, 5.4], it is enough to show that

$$|R_{A_t}| \le C r^{-2}$$

for some constant C independent of t. But this can be proved exactly as in [Ba, Proposition 3]. We omit the detail. □

There are many examples satisfying the above condition. For example, if

$$\int_X |R_A|^2 dV < 8\pi^2 \qquad \text{for } [A] \in \mathfrak{M},$$

then $a_k < 8\pi^2$ for any k in the above statement (2). Thus the singular set S cannot contain regular points. For Kronheimer's ALE spaces, many examples are given by [KN, 9.2 and Remark following 9.2]. However, when Γ is the trivial group, the above condition is rarely satisfied. When $X = \mathbb{R}^4$, the L^2-metric is *never* complete.

4. Morse theory on the moduli space

From now on we assume that

(4.1) the ALE space X has a U(1)-action which preserves both the Riemannian metric and the complex structure, and approximates the following U(1)-action on $\mathbb{C}^2/\Gamma$ under the coordinate system at infinity:

$$(z_1, z_2) \bmod \Gamma \mapsto (tz_1, tz_2) \bmod \Gamma \qquad \text{for } (z_1, z_2) \bmod \Gamma \in \mathbb{C}^2/\Gamma,\ t \in \mathrm{U}(1),$$

(4.2) $H^1(X; \mathbb{R}) = 0$,
(4.3) the group Γ is a cyclic group.

The blow-up of $\mathbb{C}^2$ at n-points situated along a straight complex line has an U(1)-action, but it is asymptotically given by

$$(z_1, z_2) \mapsto (tz_1, z_2).$$

It does not satisfy the condition (4.1). The total space of complex line bundle over $\mathbb{C}\mathrm{P}^1$ with the first Chern class $c_1 < 2$ with LeBrun's metric space satisfies all these conditions. Kronheimer's space satisfies them if the space are *biholomorphic* to the minimal resolution of $\mathbb{C}^2/\Gamma$. (In general, his space is only *diffeomorphic* to it.)

By the condition (4.2), there exists a function $\mu: X \to \mathbb{R}$ such that $I\mathrm{grad}\ \mu$ is the vector generating the U(1)-action in (4.1). This is essentially the moment map of the U(1)-action.

As in §3, we take a hermitian vector bundle E over X admitting a flat connection A_0 on the end.

It is not clear that the U(1)-action in (4.1) can lift to an action on E at first sight, but easy to see the infinitesimal action is always liftable: (In fact, we shall see that the action is liftable. See Remark after Lemma 4.5 and Remark 4.9(1).)

Lemma 4.4 (see [GP, 4.3], [Ma, §4])**.** *The* U(1)*-action in* (4.1) *induces an infinitesimal* U(1)*-action V on $\mathfrak{M}$ given by the formula*

$$V([A]) \overset{\text{def.}}{=} (I\mathrm{grad}\ \mu) \lrcorner R_A \in L^2\text{–}\mathrm{Ker}(d_A^+ \oplus d_A^*) \cong T_{[A]}\mathfrak{M},$$

where $\lrcorner$ denotes the interior product.

The vector field V is holomorphic and Killing. The corresponding moment map is given by [Ma]

$$F_0([A]) \overset{\text{def.}}{=} \int_X \mu |R_A|^2 dV.$$

So we have

$$\mathrm{grad}\ F_0 = IV.$$

Lemma 4.5. *The function F_0 is proper if the L^2-metric is complete.*

Proof. Since the U(1)-action approximates the standard action on $\mathbb{C}^2$, the moment map μ has the following asymptotic behaviour:

$$\mu \approx \frac{\sqrt{-1}}{2} r^2.$$

Hence if F is bounded,

$$\int_X r^2 |R_A|^2 dV$$

is also bounded. Hence the curvature density cannot converges to $|R_{A_\infty}|^2 dV + a\delta_\infty$ with $a > 0$. Proposition 3.4 ensures that $[A]$ stays in a compact set. □

Remark. The above implies that the vector field V is complete.

The function F is a non-degenereate Morse function on $\mathfrak{M}$ in the sense of Bott. The critical points are the fixed points of the U(1)-action. This is, in general, a union of submanifolds of $\mathfrak{M}$, and the index along a critical submanifold is an even integer.

It seems that it is not so easy to determine the fixed point set explicitly. So we use the G_ρ action (see (3.3)) to perturb F_0 as follows. Take a maximal torus T^r of G_ρ. Under the assumption (4.3), we have $r = \operatorname{rank} E$. Consider the corresponding moment map coupled with an element $\varepsilon \in \mathfrak{t}^r$:

$$\lim_{r\to\infty} \int_{S_r} i(\frac{\partial}{\partial r}) \operatorname{tr}(\varepsilon R_A) \wedge \omega,$$

where S_r is the distance sphere of the radius r, i denotes the interior product, ω is the Kähler form, and ε is considered as a section of $\operatorname{End} E$ near infinity. Let

$$F([A]) \overset{\text{def.}}{=} F_0([A]) + \lim_{r\to\infty} \int_{S_r} i(\frac{\partial}{\partial r}) \operatorname{tr}(\varepsilon R_A) \wedge \omega. \tag{4.6}$$

Theorem 4.7. *Assume* (4.1)–(4.3). *Suppose that the* L^2*-metric is complete and* ε *is sufficiently small and generic. Then the function* F *satisfies the following properties.*

(1) F *is proper.*

(2) *The gauge equivalence class* $[A]$ *is a critical point of* F *if and only if there exists a* T^r*-invariant decomposition* $E = L_1 \oplus \cdots \oplus L_r$ *into sum of line bundles such that the connection* A *decomposes accordingly.*

(3) F *is a Morse function* (*in the usual sense*) *and the index at each critical point is an even number. In particular, the homology of* $\mathfrak{M}$ *has no torsion and vanishes in odd degrees, and every component of* $\mathfrak{M}$ *is simply-connected.*

Proof. (1) If ε are sufficiently small, $F \le c$ implies F_0 is bounded. Hence $F \le c$ is compact.

(2) Since F is essentially the moment map of the torus $\mathrm{U}(1) \times T^r = T^{r+1}$-action coupling with ε, the critical points of F are precisely the fixed points if ε are generic. Take a gauge equivalence class of A. It is fixed by T^r if and only if for each $h \in T^r$ there exists a gauge transformation γ such that $\gamma^* A = A$ and

$$\lim_{x\to\infty} \gamma(x) = h.$$

Then A decomposes as the bundle decomposes into eigenspaces of γ. If the eigenvalues of h are all distinct, the bundle is a direct sum of line bundles, that is $L_1 \oplus \cdots \oplus L_r$.

Since $H^1(X; \mathbb{R}) = 0$, the gauge equivalence classes of connections on a line bundles are classified by their curvature form. If the curvature form is ASD, it is uniquely determined by its cohomology class, that is the first Chern class of the line bundle. In particular, the moduli space on L_a consists of one point, so the point must be fixed by the $\mathrm{U}(1)$-action. Therefore the direct sum is also a fixed point.

(3) This statement holds for a general function arising from a moment map (see [At, Ki]). But we give the proof for our situation.

Take a fixed point $[A]$ in $\mathfrak{M}$. The complex structure I on X induces a complex structure $I_{\mathfrak{M}}$ on the tangent space of $\mathfrak{M}$ at $[A]$. This complex tangent space of $\mathfrak{M}$ at $[A]$ is identified with the L^2-kernel of the operator

$$\overline{\partial}_A^* \oplus \overline{\partial}_A \colon \Omega^{0,1}(\operatorname{End} E) \to \Omega^{0,0}(\operatorname{End} E) \oplus \Omega^{0,2}(\operatorname{End} E).$$

Since $[A]$ corresponds to the sum of line bundles $L_1 \oplus \cdots \oplus L_r$, the L^2 kernel has a $\mathbb{C}$-vector space decomposition

$$\bigoplus_{a,b} (L^2\text{-kernel of } \overline{\partial}_A^* \oplus \overline{\partial}_A) \cap \Omega^{0,1}(L_a^* \otimes L_b).$$

Since $[A]$ is a fixed point, there exists a lift $\tilde{t}$ to E of $t: X \to X$ which respects the connection A, preserves the decomposition $E = L_1 \oplus \cdots \oplus L_r$ and acts as the identity on $E_\infty = \oplus (L_a)_\infty$. Hence $T_{[A]}\mathfrak{M}_\zeta$ becomes a U(1)-module and decomposes into the sum

$$\bigoplus_{a,b} \bigoplus_{m \in \mathbb{Z}} H_{a,b}^m$$

of complex subspaces where U(1) acts on $H_{a,b}^m$ with weight m. Then the hessian of F_0 acts on $H_{a,b}^m$ as multiplication by m. Suppose ε, regarded as an endomorphism on E_∞, acts on $(L_a)_\infty$ as the multiplication by $\sqrt{-1}\varepsilon_a$. Then the hessian of the second term in (4.6) acts on $H_{a,b}^m$ as multiplication by $\varepsilon_b - \varepsilon_a$. So the hessian of F is non-degenerate, if all ε_a's are distinct, as we have been assuming. The index is given by

$$\sum_{a,b} \left(\sum_{m<0} \dim_{\mathbb{R}} H_{a,b}^m + \sum_{m=0, \varepsilon_a > \varepsilon_b} \dim_{\mathbb{R}} H_{a,b}^m \right). \tag{4.8}$$

Since $H_{a,b}^m$ is a complex vector space, the index is even. $\square$

Remarks 4.9. (1) Since F takes a minimum at a point, the moduli space $\mathfrak{M}$ contains at least one point which comes from the direct sum of line bundles. Since the moduli space on a line bundle consists of one point, the U(1)-action lifts to the line bundle. Therefore the U(1)-action lifts to the direct sum E.

(2) The index (4.8) can be calculated as follows:

i) Determine the fixed points set of the U(1)-action on the base manifold X.
ii) Calculate weights for the normal bundles of components $\mathfrak{F}$ of the fixed points set.
iii) Calculate the weights for the fiber of $L_a^* \otimes L_b$ over $\mathfrak{F}$.
iv) Calculate the eta invariants for the Dirac operators on S^3/Γ twisted by the flat connection, to which the ASD connection on $L_a^* \otimes L_b$ is asymptotic.
v) Substituting the above data to the Lefschetz fixed points formula, we get the weights space decomposition of the L^2-kernel of the Dolbeault operator.

(For Kronheimer's ALE spaces, see [Na2]).

Examples 4.10. (1) Examples for Kronheimer's ALE spaces can be found in [Na2].

(2) Let X be the total space of the Chern class $-n$-bundle $\mathcal{O}(-n) \to \mathbb{C}\mathrm{P}^1$ $(n = 2, 3, \dots)$ with LeBrun's metric. The zero section of $\mathcal{O}(-n)$, considered as a divisor of X, produces the line bundle L. Then L has a unique ASD connection which is asymptotic to the trivial connection. Set $E = \underline{\mathbb{C}} \oplus L$ and consider the moduli space $\mathfrak{M}$ of ASD connections asymptotic to the trivial connection. Then the dimension formula [Na1, 2.7] shows $\dim_{\mathbb{R}} \mathfrak{M} = 2n$.

Claim. *The L^2-metric is complete.*

Proof. We use Proposition 3.4. Suppose we have a sequence $[A_i]$ in $\mathfrak{M}$ with $\{\infty\} \neq S$. We may assume that $\infty \notin S$. By Uhlenbeck's removable singularities theorem the limit $[A_\infty]$ extends to a connection on a possibly different bundle E' over X. Let $\mathfrak{M}'$ be the moduli space containing $[A_\infty]$. By Remark 4.9(1), $\mathfrak{M}'$ contains a reducible connection. Then

a) A_∞ is also asymptotic to the trivial connection since $\infty \notin S$,
b) the first Chern class is preserved under the weak convergence, so we have $c_1(E') = c_1(E)$.

Therefore the reducible connection must be in the form:

$$L^{\otimes m} \oplus L^{\otimes 1-m},$$

where $L^{\otimes -m} = (L^*)^{\otimes m}$ for $m > 0$. We have

$$\int_X c_2(E') = \int_X c_2(L^{\otimes m} \oplus L^{\otimes 1-m}) \geq \int_X c_2(E)$$

with the equality if and only if $m = 0, 1$. However, by the lower semi-continuity of the action under the weak convergence, we have the inequality of the opposite direction. Hence $c_2(E') = c_2(E)$ and $S = \emptyset$. $\square$

We have two fixed points of the T^3-action corresponding to $\underline{\mathbb{C}} \oplus L$ and $L \oplus \underline{\mathbb{C}}$. (Since we are discussing the *framed* moduli space, these two points are different !) As in [Na2] we can show that one has index 0 and the other has 2. Hence the Poincaré polynomial is $1 + t^2$.

(3) Let X be as in (2). Take the line bundle L such that the first Chern class $c_1(L)$ is a generator of $H^2(X; \mathbb{Z}) \cong \mathbb{Z}$. It has an ASD connection A_0 asymptotic to a flat connection with the associated representation

$$\rho\left(\exp(\frac{2\pi i k}{n})\right) = \exp(\frac{2\pi i k}{n}) \qquad \text{for } k = 0, 1, \ldots, n-1.$$

Set $E = L \oplus L^*$ and consider the moduli space $\mathfrak{M}$ of ASD connections asymptotic to $\rho \oplus \rho^*$. Since

$$\int_X c_2(E) = -\int_X c_1(L)^2 = \frac{1}{n} < 1,$$

the L^2-metric is complete. The dimension formula [Na1, 2.7] shows $\dim_{\mathbb{R}} \mathfrak{M} = 2$ if $n > 2$, and $\dim_{\mathbb{R}} \mathfrak{M} = 4$ if $n = 2$.

When $n = 2$, we have two fixed points of T^3-action corresponding to $L \oplus L^*$, $L^* \oplus L$. (Note that $\rho^* \cong \rho$ if $n = 2$.) One is of index 0 and the other of 2. Hence the Poincaré polynomial is $1 + t^2$. In fact, it can be shown that the moduli space is isomorphic to the cotangent bundle of $\mathbb{CP}^1$ (see [KN]).

When $n > 2$, we have only one fixed point $L \oplus L^*$. Hence $\mathfrak{M}$ is diffeomorphic to the 2-ball B^2.

Acknowledgement. I would like to thank Y.S. Poon for suggesting me the problem.

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