Representation theory, discrete lattice subgroups, effective ergodic theorems, and applications

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Geometric Analysis on Discrete Groups

RIMS workshop, Kyoto

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Based on joint work with Alex Gorodnik, and on joint work with Anish Ghosh and Alex Gorodnik

Plan

- Talk I : Averaging operators in dynamical systems, operator norm estimates, and effective ergodic theorems for lattice subgroups

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- Talk II : Unitary representations, the automorphic representation of a lattice subgroup, counting lattice points and applications

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- Talk III : An effective form for the duality principle for homogeneous spaces and some of its applications : equidistribution of lattice orbits, and Diophantine approximation on algebraic varieties

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- Do the time averages converge ? If so, what is their limit ?

Ergodicity and the ergodic hypothesis

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- Bolzmann's Ergodic Hypothesis : for an ergodic flow, the time averages of an observable f converge to the space average of f on phase space, namely to $\int_M f\, d\mathrm{vol}$.

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- For the proof, von-Neumann utilized his recently established spectral theorem for unitary operators.

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- and finally that the span of $\{a_s h - h\,;\, s \in \mathbb{R}, h \in \mathcal{H}\}$ is dense in the orthogonal complement of the space of invariants.

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- von-Neumann established this class as a common generalization of compact groups and Abelian groups by proving the existence of Haar measure for compact groups, and the existence of invariant means (Banach limits) for general Abelian groups.

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- Følner showed that the existence of an asymptotically invariant family is equivalent to the existence of an invariant mean, namely it characterizes meanable groups, subsequently renamed amenable groups.

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- We will mention an important consequence of asymptotic invariance below, but first let us introduce the general set-up of ergodic theorems and the averaging operators which will be our main subject.

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- Basic problem : study the averaging operators

$$\pi_X(\beta_t)f(x) = \frac{1}{|B_t|}\int_{B_t} f(g^{-1}x)dm_G(g)$$

and their convergence properties

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- In particular, when G is amenable all averaging operators satisfy $\|\pi_X(\beta)\|_{L^2_0(X)} = 1$, in every properly ergodic action.
- Corollary : in properly ergodic actions of amenable groups no rate of convergence to the ergodic mean can be established, in the operator norm.

- We conclude that when G is non-amenable, at least in some ergodic actions, at least some of the averaging operators $\pi_X(\beta)$ are strict contractions on $L_0^2(X)$.

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- We now turn to a systematic study of averaging operators which are strict contractions.

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- For every generating probability measure β on G

$$\left\|\pi_X(\beta)f - \int_X f d\mu\right\| < (1-\eta)\,\|f\|$$

  for all $f \in L^2(X)$ and a fixed $\eta(\beta) > 0$.
  Here β is generating if the support of $\beta^* * \beta$ generates a dense subgroup of G.

Spectral estimates associated with a spectral gap

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- Let Γ be countable and finitely generated, d the left-invariant metric associated with a finite symmetric generating set, and B_n the balls of of radius n and center e w.r.t. d. Let β_n be the uniform measure on B_n.

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- Namely, the property of rapid decay of the convolution norms $\|\lambda_\Gamma(\beta_n)\|$ holds, which implies that exponential decay in n holds, with the best possible rate, namely $Cn^k\left|B_n\right|^{-1/2}$.
- But for $\Gamma = SL_3(\mathbb{Z})$ for example, even the weakest statement, namely Conjecture III (and certainly Conjecture I) seem to be completely open for any choice of word metric.

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- In fact, an even more remarkable property holds, namely the following uniform operator norm estimate.
- G has property T if and only if for every absolutely continuous generating measure β there exists $\alpha(\beta) < 1$, such that in every ergodic action of G on X, the following uniform operator norm estimate holds : $\|\pi_X(\beta)\|_{L^2_0(X)} \leq \alpha(\beta)$.

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- λ_t = uniform measures on $\Gamma \cap B_t = \Gamma_t$.

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- We emphasize that this result holds for all Γ-actions. The only connection to the original embedding of Γ in the group G is in the definition of the sets Γ_t.

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$$\left\| \lambda_t f - \int_X f d\mu \right\|_p \leq C_p m(B_t)^{-\theta_p} \|f\|_p \ ,$$

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- Under this condition, the effective pointwise ergodic theorem holds: for every $f \in L^p$, $p > 1$, for almost every x,

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Examples

- In particular, if Γ has property T, then the quantitative mean and pointwise ergodic theorems hold in every ergodic measure-preserving action with a fixed $\theta_p = \theta_p(G) > 0$ independent of X.

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- Specializing further, in every action of Γ on a finite homogeneous space X, we have the following norm bound for the averaging operators

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$$\left\| \lambda_t f - \int_X f d\mu \right\|_2 \leq Cm(B_t)^{-\theta_2} \|f\|_2 \ ,$$

- As already noted, this estimate goes well beyond the contraction property guaranteed by the special gap, and holds uniformly over families of finite-index subgroups provided they satisfy property T, or more generally Lubotzky-Zimmer's property τ ('85).

Some steps in the spectral approach

- The problem of ergodic theorems for general discrete groups was raised already by Arnol'd and Krylov (1962). They proved an equidistribution theorem for dense free subgroups of isometries of the unit sphere $\mathbb{S}^2$ via a spectral argument similar to Weyl's equidistribution theorem on the circle (1918).

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- Tempelman has proved mean ergodic theorems for averages on semisimple Lie group using spectral theory, namely the Howe-Moore vanishing of matrix coefficients theorem (1980's),
- The exciting, and distinctly non-amenable, possibility of ergodic theorems with quantitative estimates on the rate of convergence was realized by the Lubotzky-Phillips-Sarnak construction of a dense free group os isometries of $\mathbb{S}^2$ which has an optimal (!) spectral gap (1980's).