Representation theory, discrete lattice subgroups, effective ergodic theorems, and applications

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Geometric Analysis on Discrete Groups

RIMS workshop, Kyoto

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Based on joint work with Alex Gorodnik, and on joint work with Anish Ghosh and Alex Gorodnik

Plan

- Talk I : Averaging operators in dynamical systems, operator norm estimates, and effective ergodic theorems for lattice subgroups

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- Talk III : An effective form for the duality principle for homogeneous spaces and some of its applications : equidistribution of lattice orbits, and Diophantine approximation on algebraic varieties

Ergodic theorems for general lattice actions

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- For an arbitrary ergodic Γ-action on a probability space (X, μ), the mean ergodic theorem holds: for every $f \in L^p$, $1 \leq p < \infty$

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- Furthermore, the pointwise ergodic theorem holds, namely for every $f \in L^p$, $p > 1$, and for almost every $x \in X$,

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$$\lim_{t\to\infty} \lambda_t f(x) = \int_X f d\mu\,.$$

- We emphasize that this result holds for all Γ-actions. The only connection to the original embedding of Γ in the group G is in the definition of the sets Γ_t.

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$$\left\| \lambda_t f - \int_X f d\mu \right\|_p \leq C_p m(B_t)^{-\theta_p} \|f\|_p \ ,$$

where $\theta_p = \theta_p(X) > 0$.

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- Under this condition, the effective pointwise ergodic theorem holds: for every $f \in L^p$, $p > 1$, for almost every x,

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- Let $\sigma : \Gamma \to U_n(\mathbb{C})$ be a unitary representation with dense image. Then for every unit vector u the sets $\sigma(\Gamma_t)u$ become equidistributed in the unit sphere, w.r.t. the rotation invariant measure, and the rate of equidistribution at every point is exponentially fast if the representations admits a spectral gap and the function is Holder.

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- Specializing further, in every action of Γ on a finite homogeneous space X, we have the following norm bound for the averaging operators

$$\left\|\lambda_t f - \int_X f d\mu\right\|_2 \leq Cm(B_t)^{-\theta_2} \|f\|_2 \ ,$$

Discrete groups : some steps in the spectral approach

- The problem of ergodic theorems for general discrete groups was raised already by Arnol'd and Krylov (1962). They proved an equidistribution theorem for dense free subgroups of isometries of the unit sphere $\mathbb{S}^2$ via a spectral argument similar to Weyl's equidistribution theorem on the circle (1918).

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- The distinctly non-amenable possibility of ergodic theorems with quantitative estimates on the rate of convergence was realized first by the Lubotzky-Phillips-Sarnak construction of a dense free group os isometries of $\mathbb{S}^2$ which has an optimal (!) spectral gap (1980's).

Averaging operators in general unitary representations

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- A key role is played by a fundamental norm estimate (and resulting mean ergodic theorem) which is satisfied by general families of averages β_t on G.

- This estimate, which we now state, utilizes the unitary representation theory of simple algebraic groups.

Spectral transfer principle

- Spectral transfer principle. [N 98], [Gorodnik+N 2005]. For every unitary representation π of a simple algebraic group G with a spectral gap and no finite-dimensional invariant subspaces, and for every family of probability measures $\beta_t = \chi_{B_t}/m_G(B_t)$, the following norm decay estimate holds (for every $\epsilon > 0$)

$$\|\pi(\beta_t)\| \le \|r_G(\beta_t)\|^{\frac{1}{n_e(\pi)}} \le C_\varepsilon m(B_t)^{-\frac{1}{2n_e(\pi)}+\varepsilon},$$

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- The norm estimate of the operator $\pi(\beta)$ in a general rep' π, has been reduced to a norm estimate for the convolution operator $r_G(\beta)$ in the regular rep' r_G. This establishes for simple groups an analog of the transfer(ence) principle for amenable groups.

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- In particular, we can bound the norm of the averaging operators of $\pi_X(\beta_t)$ acting on $L^2_0(X)$, when the action is ergodic and weak mixing, namely has no finite-dimensional invariant subspaces.

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- Thm. D. Effective mean ergodic theorem. [N 98], [Gorodnik+N 2005]. For any weak mixing action of a simple algebraic group G which has a spectral gap, and for any family $B_t \subset G$ with $m_G(B_t) \to \infty$, the convergence of the time averages $\pi_X(\beta_t)$ to the space average takes place at a definite rate :

$$\left\| \pi(\beta_t)f - \int_X f d\mu \right\|_{L^2(X)} \leq C_\theta \left(m_G(B_t)\right)^{-\theta} \|f\|_2 \ ,$$

for every $0 < \theta < \frac{1}{2n_e(\pi_X^0)}$.

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- Note that the only requirement needed to obtain the effective mean ergodic Thm' is simply that $m(B_t) \to \infty$, and the geometry of the sets is not relevant at all. This fact allows a great deal of flexibility in its application.

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- 1) Haar measure $m(B_t)$ is the main term in the asymptotic,

- 2) There is an error estimate of the form

$$\frac{|\Gamma \cap B_t|}{m(B_t)} = 1 + O\left(m(B_t)^{-\delta}\right)$$

where $\delta > 0$ and is as large as possible, and is given in an explicit form,

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- 4) the solution should establish whether the error estimate can be taken as uniform over all (or some) finite index subgroups $\Lambda \subset \Gamma$, and over all their cosets, namely:

$$\frac{|\gamma\Lambda \cap B_T|}{m(B_T)} = \frac{1}{[\Gamma : \Lambda]} + O\left(m(B_T)^{-\delta}\right)$$

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- First main point: A general solution obeying the 4 requirements above can be given for lattices in simple algebraic groups and general domains B_t, using a method based on the effective mean ergodic theorem for G.

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- First main point: A general solution obeying the 4 requirements above can be given for lattices in simple algebraic groups and general domains B_t, using a method based on the effective mean ergodic theorem for G.
- As we shall see, this is the first essential step in proving mean ergodic theorems for lattice.

Admissible sets for the counting problem

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- An increasing family of bounded Borel subset B_t, $t > 0$, of G will be called admissible if there exists $c > 0$, t_0 and ε_0 such that for all $t \geq t_0$ and $\varepsilon < \varepsilon_0$ we have :

$$\mathcal{O}_\varepsilon \cdot B_t \cdot \mathcal{O}_\varepsilon \subset B_{t+c\varepsilon}, \tag{1}$$

$$m_G(B_{t+\varepsilon}) \leq (1 + c\varepsilon) \cdot m_G(B_t). \tag{2}$$

Counting in connected Lie groups

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- 1) Assume the mean ergodic theorem holds for β_t in $L^2(m_{G/\Gamma})$:

$$\left\|\pi(\beta_t)f - \int_X f\, dm_{G/\Gamma}\right\|_{L^2} \to 0\,, \quad (m_{G/\Gamma}(G/\Gamma) = 1\,).$$

Then

$$|\Gamma \cap B_t| = |\Gamma_t| \sim m_G(B_t) \quad \text{as} \quad t \to \infty.$$

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$$|\Gamma \cap B_t| = |\Gamma_t| \sim m_G(B_t) \quad \text{as} \quad t \to \infty.$$

- Assume that the error term in the mean ergodic theorem for β_t in $L^2(m_{G/\Gamma})$ satisfies

$$\left\| \pi(\beta_t)f - \int_{G/\Gamma} f \, dm_{G/\Gamma} \right\|_{L^2} \leq C m(B_t)^{-\theta} \, \|f\|_{L^2}$$

Then

$$\frac{|\Gamma_t|}{m_G(B_t)} = 1 + O\left(m(B_t)^{-\theta/(\dim G + 1)} \right) .$$

Proof of the counting Theorem

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- Clearly ϕ is a bounded function on G/Γ with compact support,

$$\int_G \chi_\varepsilon \, dm_G = 1, \quad \text{and} \quad \int_{G/\Gamma} \phi_\varepsilon \, d\mu_{G/\Gamma} = 1.$$

- Let us apply the mean ergodic theorem to the function ϕ_ε. It follows from Chebycheff's inequality that for every $\delta > 0$,

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- In particular, for sufficiently large t, the measure of the deviation set is smaller than $m_{G/\Gamma}(\mathcal{O}_\varepsilon)$, and so there exists $g_t \in \mathcal{O}_\varepsilon$ such that

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Claim I. Given $\varepsilon, \delta > 0$, for t sufficiently large, there exists $g_t \in \mathcal{O}_\varepsilon$ satisfying

$$1 - \delta \leq \frac{1}{m_G(B_t)} \int_{B_t} \phi_\varepsilon(g^{-1} g_t \Gamma) dm_G \leq 1 + \delta\,.$$

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&= \frac{1}{m_G(B_t)} \int_{B_t} \sum_{\gamma\in\Gamma} \chi_\varepsilon(g^{-1}h\gamma)\,dm_G \\
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- Claim II. For sufficiently large t and any $h \in \mathcal{O}_\varepsilon$,

$$\int_{B_{t-c\varepsilon}} \phi_\varepsilon(g^{-1}h\Gamma)\, dm_G(g) \le |\Gamma_t| \le \int_{B_{t+c\varepsilon}} \phi_\varepsilon(g^{-1}h\Gamma)\, dm_G(g).$$

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- *Proof*. If $\chi_\varepsilon(g^{-1}h\gamma) \neq 0$ for some $g \in B_{t-c\varepsilon}$ and $h \in \mathcal{O}_\varepsilon$, then clearly $g^{-1}h\gamma \in \text{supp } \chi_\varepsilon$, and so

$$\gamma \in h^{-1} \cdot B_{t-c\varepsilon} \cdot (\text{supp } \chi_\varepsilon) \subset B_t.$$

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- Hence,

$$\int_{B_{t-c\varepsilon}} \phi_\varepsilon(g^{-1}h\Gamma)\, dm_G(g) \leq$$

$$\leq \sum_{\gamma \in \Gamma_t} \int_{B_t} \chi_\varepsilon(g^{-1}h\gamma)\, dm_G(g) \leq |\Gamma_t|.$$

- On the other hand, for $\gamma \in \Gamma_t$ and $h \in \mathcal{O}_\varepsilon$,

$$\mathrm{supp}(g \mapsto \chi_\varepsilon(g^{-1}h\gamma)) = h\gamma(\mathrm{supp}\ \chi_\varepsilon)^{-1} \subset B_{t+c\varepsilon}$$

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- and since $\chi_\varepsilon \geq 0$ and $\int_G \chi_\varepsilon dm = 1$:

$$\int_{B_{t+c\varepsilon}} \phi_\varepsilon(g^{-1}h\Gamma)\, dm_G(g) \geq$$

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$$\geq \sum_{\gamma\in\Gamma_t} \int_{B_{t+c\varepsilon}} \chi_\varepsilon(g^{-1}h\gamma)\, dm_G(g) \geq |\Gamma_t|.$$

- Now taking t sufficiently large, $h = g_t$ and using Claims I and II

$$|\Gamma_t| \leq (1+\delta)m(B_{t+\varepsilon}) \leq$$

$$\leq (1+\delta)(1+c\varepsilon)m(B_t),$$

by admissibility. The lower estimate is proved similarly.

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- By the mean ergodic theorem and Chebycheff's inequality :

$$m_{G/\Gamma}(\{x \in G/\Gamma : |\pi_{G/\Gamma}(\beta_t)\phi_\varepsilon(x) - 1| > \delta\})$$

$$\leq C\delta^{-2}\varepsilon^{-n} m(B_t)^{-2\theta}.$$

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- The estimate of the measure of the deviation set holds for all t, ε and δ, since the mean ergodic theorem with error term is a statement about the rate of convergence in operator norm, and is thus uniform over all functions.
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- Taking $\delta \sim \varepsilon \sim m(B_t)^{-\theta/(n+1)}$ to balance the two significant parts of the error appearing in the estimate $(1+\delta)(1+c\varepsilon)$, the result follows.

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- Indeed all that is needed is that the averaging operators $\pi(\beta_t)$ satisfy the same norm decay estimate in the space $L^2(G/\Lambda)$.
- This holds when the set of finite index subgroups satisfy property τ, namely when the spectral gap appearing in the representations $L^2_0(G/\Lambda)$ has a positive lower bound.
- Property τ has been shown to hold for the set of congruence subgroups of any arithmetic lattice in a semisimple Lie group (Burger-Sarnak, Lubotzky, Clozel..... generalizing the Selberg property).

Some previous results

- The classical non-Euclidean counting problem asks for the number of lattice points in a Riemannian ball in hyperbolic space $\mathbb{H}^d$. For this case, the best known bound is still the one due to Selberg (1940's) or Lax-Phillips (1970's). Bruggeman, Gruenwald and Miatello have established similar results for lattice points in Riemannian balls in products of $SL_2(\mathbb{R})$'s (2008).

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- Eskin-McMullen (1990) devised the mixing method, which applies to general (well-rounded) sets, but have not produced an error estimate. Maucourant (2005) has obtained an error estimate using an effective form of the mixing method, and so have Benoist-Oh in the S-algebraic case (2012). The resulting estimates are weaker than those stated above

Counting rational points

- Counting rational points on algebraic varieties homogeneous under a simple algebraic group G defined over $\mathbb{Q}$ has been considered by Shalika, Takloo-Bighash and Tschinkel (2000's). They used direct spectral expansion of the height zeta function in the automorphic representation (using, in particular, regularization estimates for Eisenstein series).

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- It is possible to consider the corresponding group G over the ring of adéles, in which the group of rational points is embedded as a lattice. Generalizing the operators norm estimates from $G(F)$ for all field completions F to the group of adéles, it is possible to use the method based on the effective mean ergodic theorem here too. The error estimate statedabove is better that the estimate that both other methods produce.

- Second main point : To prove the mean (and pointwise) ergodic theorems for lattice subgroups (stated above) we generalize the solution of the lattice point counting problem.

Ergodic theorems for lattice groups : Induced actions

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- Denote by $Y = G/\Gamma \times X = \frac{G \times X}{\Gamma}$, with the measure $m_{G/\Gamma} \times \mu$, the action of G induced from the Γ-action on X.

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- It is defined as the space $Y = \frac{G \times X}{\Gamma}$ of Γ-orbits in $G \times X$, where Γ acts via $(h, x)\gamma = (h\gamma, \gamma^{-1}x)$. G acts on $G \times X$ via $g \cdot (h, x) = (gh, x)$, an action which commutes with the Γ-action and is therefore well defined on Y. The measure $m_{G/\Gamma} \times \mu$ is G-invariant.

Ergodic theorems for lattice groups : proof overview

- The essence of the matter is to estimate the ergodic averages $\pi_X(\lambda_t)\phi(x)$ given by

$$\frac{1}{|\Gamma \cap B_t|} \sum_{\gamma \in \Gamma \cap B_t} \phi(\gamma^{-1}x)\,, \quad \phi \in L^p(X),$$

  above and below by the ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon(y)$, namely by

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- The link between the two expressions is given by setting $y = (h, x)\Gamma \in (G \times X)/\Gamma = Y$ and

$$F_\varepsilon((h, x)\Gamma) = \sum_{\gamma \in \Gamma} \chi_\varepsilon(h\gamma)\phi(\gamma^{-1}x)\ , \quad F \in L^p(Y),$$

  where χ_ε is the normalized characteristic function of an identity neighborhood $\mathcal{O}_\varepsilon$. Assume from now on $\phi \geq 0$.

- The ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon$ can be rewritten in full as

$$\sum_{\gamma\in\Gamma}\left(\frac{1}{m_G(B_{t\pm C})}\int_{g\in B_{t\pm C}}\chi_\varepsilon(g^{-1}h\gamma)\right)\phi(\gamma^{-1}x)\ ,$$

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- We would like the expression in parentheses to be equal to one when (say) $\gamma\in\Gamma\cap B_{t-C}$ and equal to zero when (say) $\gamma\notin\Gamma\cap B_{t+C}$, in order to be able to compare it to $\pi_X(\lambda_t)\phi$. .

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- A favorable lower bound arises if $\chi_\varepsilon(g^{-1}h\gamma)\neq 0$ and $g\in B_{t-C}$ imply that $\gamma\in B_t$, and a favorable upper bound arises if for $\gamma\in\Gamma\cap B_t$ the support of $\chi_\varepsilon(g^{-1}h\gamma)$ contained in B_{t+C}.

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- Thus favorable lower and upper estimates depend only on the regularity properties of the sets B_t, specifically on the stability property under perturbations by elements h in a fixed neighborhood, and volume regularity of $m_G(B_t)$.

- Therefore taking some fixed $\mathcal{O}_\varepsilon$ and C, the usual strong maximal inequality for averaging over λ_t follows from the ordinary strong maximal inequality for averaging over β_t. It follows that for lattice actions, as for actions of the group G, the maximal inequality holds in great generality and requires only a coarse form of admissibility.

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- The mean ergodic theorem for λ_t requires considerably sharper argument, and in particular requires passing to $\varepsilon \to 0$, namely $m_G(\mathcal{O}_\varepsilon) \to 0$.

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- The mean ergodic theorem for λ_t requires considerably sharper argument, and in particular requires passing to $\varepsilon \to 0$, namely $m_G(\mathcal{O}_\varepsilon) \to 0$.

- The effective uniform volume estimate appearing in the definition of admissibility is utilized, and is matched against the unavoidable quantity $m_G(\mathcal{O}_\varepsilon)^{-1}$ which the approximation procedure introduces.

- The effective mean ergodic theorem requires in addition an effective estimate on the decay of the operator norms $\left\|\pi_Y^0(\beta_t)\right\|_{L_0^p(Y)}$.

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- This decay estimate plays an indispensable role, and allows a quantitative approximation argument to proceed, again using crucially that the averages are admissible.
- As a byproduct of the proof, we obtain an effective decay estimate on the norms $\left\|\pi_X^0(\lambda_t)\right\|_{L_0^p(X)}$, for $1 < p < \infty$.