Representation theory, discrete lattice subgroups, effective ergodic theorems, and applications

June 2, 2016

Geometric Analysis on Discrete Groups

RIMS workshop, Kyoto

Amos Nevo, Technion

Based on joint work with Alex Gorodnik, and on joint work with Anish Ghosh and Alex Gorodnik

Plan

- Talk I : Averaging operators in dynamical systems, operator norm estimates, and effective ergodic theorems for lattice subgroups

Plan

- Talk I : Averaging operators in dynamical systems, operator norm estimates, and effective ergodic theorems for lattice subgroups
- Talk II : Unitary representations, the automorphic representation of a lattice subgroup, counting lattice points and applications

Plan

- Talk I : Averaging operators in dynamical systems, operator norm estimates, and effective ergodic theorems for lattice subgroups
- Talk II : Unitary representations, the automorphic representation of a lattice subgroup, counting lattice points and applications
- Talk III : An effective form for the duality principle for homogeneous spaces and some of its applications : equidistribution of lattice orbits, and Diophantine approximation on algebraic varieties

The lattice point counting problem

- Let G be an lcsc group and $\Gamma \subset G$ a discrete lattice subgroup

The lattice point counting problem

- Let G be an lcsc group and $\Gamma \subset G$ a discrete lattice subgroup
- The lattice point counting problem in domains $B_t \subset G$ calls for obtaining an asymptotic for $|\Gamma \cap B_t|$, ideally so that

The lattice point counting problem

- Let G be an lcsc group and $\Gamma \subset G$ a discrete lattice subgroup
- The lattice point counting problem in domains $B_t \subset G$ calls for obtaining an asymptotic for $|\Gamma \cap B_t|$, ideally so that
- 1) Haar measure $m(B_t)$ is the main term in the asymptotic,

The lattice point counting problem

- Let G be an lcsc group and $\Gamma \subset G$ a discrete lattice subgroup
- The lattice point counting problem in domains $B_t \subset G$ calls for obtaining an asymptotic for $|\Gamma \cap B_t|$, ideally so that
- 1) Haar measure $m(B_t)$ is the main term in the asymptotic,
- 2) There is an error estimate of the form

$$\frac{|\Gamma \cap B_t|}{m(B_t)} = 1 + O\left(m(B_t)^{-\delta}\right)$$

where $\delta > 0$ and is as large as possible, and is given in an explicit form,

- 3) the solution should apply to general families of sets B_t in a general family of groups,

- 3) the solution should apply to general families of sets B_t in a general family of groups,
- 4) the solution should establish whether the error estimate can be taken as uniform over all (or some) finite index subgroups $\Lambda \subset \Gamma$, and over all their cosets, namely:

$$\frac{|\gamma\Lambda \cap B_T|}{m(B_T)} = \frac{1}{[\Gamma : \Lambda]} + O\left(m(B_T)^{-\delta}\right)$$

with δ and the implied constant independent of the finite index subgroup Λ, and the coset representative γ.

- 3) the solution should apply to general families of sets B_t in a general family of groups,
- 4) the solution should establish whether the error estimate can be taken as uniform over all (or some) finite index subgroups $\Lambda \subset \Gamma$, and over all their cosets, namely:

$$\frac{|\gamma\Lambda \cap B_T|}{m(B_T)} = \frac{1}{[\Gamma : \Lambda]} + O\left(m(B_T)^{-\delta}\right)$$

  with δ and the implied constant independent of the finite index subgroup Λ, and the coset representative γ.
- First main point: A general solution obeying the 4 requirements above can be given for lattices in simple algebraic groups and general domains B_t, using a method based on the effective mean ergodic theorem for G.

- 3) the solution should apply to general families of sets B_t in a general family of groups,
- 4) the solution should establish whether the error estimate can be taken as uniform over all (or some) finite index subgroups $\Lambda \subset \Gamma$, and over all their cosets, namely:

$$\frac{|\gamma\Lambda \cap B_T|}{m(B_T)} = \frac{1}{[\Gamma : \Lambda]} + O\left(m(B_T)^{-\delta}\right)$$

with δ and the implied constant independent of the finite index subgroup Λ, and the coset representative γ.

- First main point: A general solution obeying the 4 requirements above can be given for lattices in simple algebraic groups and general domains B_t, using a method based on the effective mean ergodic theorem for G.
- Furthermore, this is the first essential step in proving mean ergodic theorems for lattice subgroup.

Admissible sets for the counting problem

- We will assume the following stability and regularity assumptions on the sets B_t (more general conditions are in fact possible).

Admissible sets for the counting problem

- We will assume the following stability and regularity assumptions on the sets B_t (more general conditions are in fact possible).
- Assume G is a simple Lie group, fix any left-invariant Riemannian metric on G, and let

$$\mathcal{O}_\varepsilon = \{g \in G : d(g, e) < \varepsilon\}.$$

Admissible sets for the counting problem

- We will assume the following stability and regularity assumptions on the sets B_t (more general conditions are in fact possible).

- Assume G is a simple Lie group, fix any left-invariant Riemannian metric on G, and let

$$\mathcal{O}_\varepsilon = \{g \in G : \ d(g,e) < \varepsilon\}.$$

- An increasing family of bounded Borel subset B_t, $t > 0$, of G will be called admissible if there exists $c > 0$, t_0 and ε_0 such that for all $t \geq t_0$ and $\varepsilon < \varepsilon_0$ we have :

$$\mathcal{O}_\varepsilon \cdot B_t \cdot \mathcal{O}_\varepsilon \subset B_{t+c\varepsilon}, \tag{1}$$

$$m_G(B_{t+\varepsilon}) \leq (1 + c\varepsilon) \cdot m_G(B_t). \tag{2}$$

Admissible sets for the counting problem

- We will assume the following stability and regularity assumptions on the sets B_t (more general conditions are in fact possible).

- Assume G is a simple Lie group, fix any left-invariant Riemannian metric on G, and let

$$\mathcal{O}_\varepsilon = \{g \in G : \ d(g, e) < \varepsilon\}.$$

- An increasing family of bounded Borel subset B_t, $t > 0$, of G will be called admissible if there exists $c > 0$, t_0 and ε_0 such that for all $t \geq t_0$ and $\varepsilon < \varepsilon_0$ we have :

$$\mathcal{O}_\varepsilon \cdot B_t \cdot \mathcal{O}_\varepsilon \subset B_{t+c\varepsilon}, \tag{1}$$

$$m_G(B_{t+\varepsilon}) \leq (1 + c\varepsilon) \cdot m_G(B_t). \tag{2}$$

- In the previous talk, we have proved the following.

Counting in connected Lie groups

- Lattice point counting. Gorodnik+N, 2006.

Counting in connected Lie groups

- Lattice point counting. Gorodnik+N, 2006.
- G a connected Lie group, $\Gamma \subset G$ a lattice, $B_t \subset G$ admissible.

Counting in connected Lie groups

- Lattice point counting. Gorodnik+N, 2006.
- G a connected Lie group, $\Gamma \subset G$ a lattice, $B_t \subset G$ admissible.
- 1) Assume the mean ergodic theorem holds for β_t in $L^2(m_{G/\Gamma})$:

$$\left\|\pi(\beta_t)f - \int_X f\, dm_{G/\Gamma}\right\|_{L^2} \to 0\,, \quad (m_{G/\Gamma}(G/\Gamma) = 1\,).$$

  Then

$$|\Gamma \cap B_t| = |\Gamma_t| \sim m_G(B_t) \quad \text{as} \quad t \to \infty.$$

Counting in connected Lie groups

- Lattice point counting. Gorodnik+N, 2006.
- G a connected Lie group, $\Gamma \subset G$ a lattice, $B_t \subset G$ admissible.
- 1) Assume the mean ergodic theorem holds for β_t in $L^2(m_{G/\Gamma})$:

$$\left\| \pi(\beta_t)f - \int_X f \, dm_{G/\Gamma} \right\|_{L^2} \to 0 \,, \quad (m_{G/\Gamma}(G/\Gamma) = 1 \,).$$

Then

$$|\Gamma \cap B_t| = |\Gamma_t| \sim m_G(B_t) \quad \text{as} \quad t \to \infty.$$

- Assume that the error term in the mean ergodic theorem for β_t in $L^2(m_{G/\Gamma})$ satisfies

$$\left\| \pi(\beta_t)f - \int_{G/\Gamma} f \, dm_{G/\Gamma} \right\|_{L^2} \leq Cm(B_t)^{-\theta} \left\| f \right\|_{L^2}$$

Then

$$\frac{|\Gamma_t|}{m_G(B_t)} = 1 + O\left(m(B_t)^{-\theta/(\dim G + 1)} \right).$$

Uniformity in the lattice point counting problem

- The ergodic theoretic proof immediately implies the important feature of uniformity of counting lattice points in finite index subgroups Λ

.

Uniformity in the lattice point counting problem

- The ergodic theoretic proof immediately implies the important feature of uniformity of counting lattice points in finite index subgroups Λ
- provided only that the averaging operators $\pi(\beta_t)$ satisfy the same norm decay estimate in the space $L^2(G/\Lambda)$.

.

Uniformity in the lattice point counting problem

- The ergodic theoretic proof immediately implies the important feature of uniformity of counting lattice points in finite index subgroups Λ
- provided only that the averaging operators $\pi(\beta_t)$ satisfy the same norm decay estimate in the space $L^2(G/\Lambda)$.
- This condition holds when the set of finite index subgroups satisfy property τ, namely when the spectral gap appearing in the representations $L_0^2(G/\Lambda)$ has a positive lower bound.

.

Uniformity in the lattice point counting problem

- The ergodic theoretic proof immediately implies the important feature of uniformity of counting lattice points in finite index subgroups Λ
- provided only that the averaging operators $\pi(\beta_t)$ satisfy the same norm decay estimate in the space $L^2(G/\Lambda)$.
- This condition holds when the set of finite index subgroups satisfy property τ, namely when the spectral gap appearing in the representations $L_0^2(G/\Lambda)$ has a positive lower bound.
- Property τ has been shown to hold for the set of congruence subgroups of any arithmetic lattice in a semisimple Lie group (Burger-Sarnak, Lubotzky, Clozel.....) generalizing the Selberg property.

Some previous results

- The classical non-Euclidean counting problem asks for the number of lattice points in a Riemannian ball in hyperbolic space $\mathbb{H}^d$. For this case, the best known bound is still the one due to Selberg (1940's) or Lax-Phillips (1970's). Bruggeman, Gruenwald and Miatello have established similar results for lattice points in Riemannian balls in products of $SL_2(\mathbb{R})$'s (2008).

Some previous results

- The classical non-Euclidean counting problem asks for the number of lattice points in a Riemannian ball in hyperbolic space $\mathbb{H}^d$. For this case, the best known bound is still the one due to Selberg (1940's) or Lax-Phillips (1970's). Bruggeman, Gruenwald and Miatello have established similar results for lattice points in Riemannian balls in products of $SL_2(\mathbb{R})$'s (2008).

- Riemannian balls in certain higher rank simple Lie groups were considered by Duke-Rudnik-Sarnak (1991), who obtained the best error estimate to date. This estimate is matched by the theorem stated above, (when adjusted for radial averages).

Some previous results

- The classical non-Euclidean counting problem asks for the number of lattice points in a Riemannian ball in hyperbolic space $\mathbb{H}^d$. For this case, the best known bound is still the one due to Selberg (1940's) or Lax-Phillips (1970's). Bruggeman, Gruenwald and Miatello have established similar results for lattice points in Riemannian balls in products of $SL_2(\mathbb{R})$'s (2008).

- Riemannian balls in certain higher rank simple Lie groups were considered by Duke-Rudnik-Sarnak (1991), who obtained the best error estimate to date. This estimate is matched by the theorem stated above, (when adjusted for radial averages).

- Eskin-McMullen (1990) devised the mixing method, which applies to general (well-rounded) sets, but have not produced an error estimate. Maucourant (2005) has obtained an error estimate using an effective form of the mixing method, and so have Benoist-Oh in the S-algebraic case (2012). The resulting estimates are weaker than those stated above

Counting rational points

- Counting rational points of bounded height in a simple algebraic group G defined over $\mathbb{Q}$ has been considered by Shalika, Takloo-Bighash and Tschinkel (2000's). They used direct spectral expansion of the height zeta function in the automorphic representation (using, in particular, regularization estimates for Eisenstein series).

Counting rational points

- Counting rational points of bounded height in a simple algebraic group G defined over $\mathbb{Q}$ has been considered by Shalika, Takloo-Bighash and Tschinkel (2000's). They used direct spectral expansion of the height zeta function in the automorphic representation (using, in particular, regularization estimates for Eisenstein series).

- Gorodnik, Maucourant and Oh (2008) have used the mixing method in the problem of counting rational points in group varieties.

Counting rational points

- Counting rational points of bounded height in a simple algebraic group G defined over $\mathbb{Q}$ has been considered by Shalika, Takloo-Bighash and Tschinkel (2000's). They used direct spectral expansion of the height zeta function in the automorphic representation (using, in particular, regularization estimates for Eisenstein series).

- Gorodnik, Maucourant and Oh (2008) have used the mixing method in the problem of counting rational points in group varieties.

- It is possible to consider the corresponding group G over the ring of adéles, in which the group of rational points is embedded as a lattice. Generalizing the operators norm estimates from $G(F)$ for all field completions F to the group of adéles, it is possible to use the method based on the effective mean ergodic theorem here too. The error estimate stated above is better that the estimate that both other methods produce.

Spectral gap and the ultimate ergodic theorem

Ergodic theorems for lattice subgroups, II. Gorodnik+N, ’08.

Spectral gap and the ultimate ergodic theorem

Ergodic theorems for lattice subgroups, II. Gorodnik+N, '08.

- If the Γ-action has a spectral gap then, the effective mean ergodic theorem holds : for every $f \in L^p$, $1 < p < \infty$

$$\left\| \lambda_t f - \int_X f d\mu \right\|_p \leq C_p m(B_t)^{-\theta_p} \|f\|_p \ ,$$

where $\theta_p = \theta_p(X) > 0$.

Spectral gap and the ultimate ergodic theorem

Ergodic theorems for lattice subgroups, II. Gorodnik+N, '08.

- If the Γ-action has a spectral gap then, the effective mean ergodic theorem holds : for every $f \in L^p$, $1 < p < \infty$

$$\left\| \lambda_t f - \int_X f d\mu \right\|_p \leq C_p m(B_t)^{-\theta_p} \left\| f \right\|_p ,$$

where $\theta_p = \theta_p(X) > 0$.

- Under this condition, the effective pointwise ergodic theorem holds: for every $f \in L^p$, $p > 1$, for almost every x,

$$\left| \lambda_t f(x) - \int_X f d\mu \right| \leq C_p(x, f) m(B_t)^{-\theta_p} .$$

where $\theta_p = \theta_p(X) > 0$.

Ergodic theorems for lattice groups : Induced actions

- Second main point : To prove the mean (and pointwise) ergodic theorems for lattice subgroups (stated above) we generalize the solution of the lattice point counting problem.

Ergodic theorems for lattice groups : Induced actions

- Second main point : To prove the mean (and pointwise) ergodic theorems for lattice subgroups (stated above) we generalize the solution of the lattice point counting problem.
- Indeed, there we have considered the action of G on G/Γ, namely the action induced to G from the trivial Γ-action on a point. We now analyze the action induced to G from a general ergodic action of Γ on (X, μ).

Ergodic theorems for lattice groups : Induced actions

- Second main point : To prove the mean (and pointwise) ergodic theorems for lattice subgroups (stated above) we generalize the solution of the lattice point counting problem.

- Indeed, there we have considered the action of G on G/Γ, namely the action induced to G from the trivial Γ-action on a point. We now analyze the action induced to G from a general ergodic action of Γ on (X, μ).

- Denote by $Y = G/\Gamma \times X = \frac{G \times X}{\Gamma}$, with the measure $m_{G/\Gamma} \times \mu$, the action of G induced from the Γ-action on X.

Ergodic theorems for lattice groups : Induced actions

- Second main point : To prove the mean (and pointwise) ergodic theorems for lattice subgroups (stated above) we generalize the solution of the lattice point counting problem.

- Indeed, there we have considered the action of G on G/Γ, namely the action induced to G from the trivial Γ-action on a point. We now analyze the action induced to G from a general ergodic action of Γ on (X, μ).

- Denote by $Y = G/\Gamma \times X = \frac{G\times X}{\Gamma}$, with the measure $m_{G/\Gamma} \times \mu$, the action of G induced from the Γ-action on X.

- It is defined as the space $Y = \frac{G\times X}{\Gamma}$ of Γ-orbits in $G \times X$, where Γ acts via $(h, x)\gamma = (h\gamma, \gamma^{-1}x)$. G acts on $G \times X$ via $g \cdot (h, x) = (gh, x)$, an action which commutes with the Γ-action and is therefore well defined on Y. The measure $m_{G/\Gamma} \times \mu$ is G-invariant.

Ergodic theorems for lattice groups : proof overview

- The essence of the matter is to estimate the ergodic averages $\pi_X(\lambda_t)\phi(x)$ given by

$$\frac{1}{|\Gamma \cap B_t|} \sum_{\gamma \in \Gamma \cap B_t} \phi(\gamma^{-1}x)\,, \ \ \phi \in L^p(X),$$

above and below by the ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon(y)$, namely by

$$\frac{1}{m_G(B_{t\pm C})} \int_{g \in B_{t\pm C}} F_\varepsilon(g^{-1}y) dm_G(g)\ .$$

Ergodic theorems for lattice groups : proof overview

- The essence of the matter is to estimate the ergodic averages $\pi_X(\lambda_t)\phi(x)$ given by

$$\frac{1}{|\Gamma \cap B_t|} \sum_{\gamma \in \Gamma \cap B_t} \phi(\gamma^{-1}x)\,, \quad \phi \in L^p(X),$$

above and below by the ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon(y)$, namely by

$$\frac{1}{m_G(B_{t\pm C})} \int_{g \in B_{t\pm C}} F_\varepsilon(g^{-1}y)dm_G(g)\ .$$

- The link between the two expressions is given by setting $y = (h,x)\Gamma \in (G \times X)/\Gamma = Y$ and

$$F_\varepsilon((h,x)\Gamma) = \sum_{\gamma \in \Gamma} \chi_\varepsilon(h\gamma)\phi(\gamma^{-1}x)\ , \quad F \in L^p(Y),$$

where χ_ε is the normalized characteristic function of an identity neighborhood $\mathcal{O}_\varepsilon$. Assume from now on $\phi \geq 0$.

- The ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon$ can be rewritten in full as

$$\sum_{\gamma\in\Gamma}\left(\frac{1}{m_G(B_{t\pm C})}\int_{g\in B_{t\pm C}}\chi_\varepsilon(g^{-1}h\gamma)\right)\phi(\gamma^{-1}x)\ ,$$

- The ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon$ can be rewritten in full as

$$\sum_{\gamma\in\Gamma}\left(\frac{1}{m_G(B_{t\pm C})}\int_{g\in B_{t\pm C}}\chi_\varepsilon(g^{-1}h\gamma)\right)\phi(\gamma^{-1}x)\ ,$$

- We would like the expression in parentheses to be equal to one when (say) $\gamma\in\Gamma\cap B_{t-C}$ and equal to zero when (say) $\gamma\notin\Gamma\cap B_{t+C}$, in order to be able to compare it to $\pi_X(\lambda_t)\phi$. .

- The ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon$ can be rewritten in full as

$$\sum_{\gamma\in\Gamma}\left(\frac{1}{m_G(B_{t\pm C})}\int_{g\in B_{t\pm C}}\chi_\varepsilon(g^{-1}h\gamma)\right)\phi(\gamma^{-1}x)\ ,$$

- We would like the expression in parentheses to be equal to one when (say) $\gamma\in\Gamma\cap B_{t-C}$ and equal to zero when (say) $\gamma\notin\Gamma\cap B_{t+C}$, in order to be able to compare it to $\pi_X(\lambda_t)\phi$. .
- A favorable lower bound arises if $\chi_\varepsilon(g^{-1}h\gamma)\neq 0$ and $g\in B_{t-C}$ imply that $\gamma\in B_t$, and a favorable upper bound arises if for $\gamma\in\Gamma\cap B_t$ the support of $\chi_\varepsilon(g^{-1}h\gamma)$ contained in B_{t+C}.

- The ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon$ can be rewritten in full as

$$\sum_{\gamma\in\Gamma}\left(\frac{1}{m_G(B_{t\pm C})}\int_{g\in B_{t\pm C}}\chi_\varepsilon(g^{-1}h\gamma)\right)\phi(\gamma^{-1}x)\ ,$$

- We would like the expression in parentheses to be equal to one when (say) $\gamma\in\Gamma\cap B_{t-C}$ and equal to zero when (say) $\gamma\notin\Gamma\cap B_{t+C}$, in order to be able to compare it to $\pi_X(\lambda_t)\phi$. .
- A favorable lower bound arises if $\chi_\varepsilon(g^{-1}h\gamma)\neq 0$ and $g\in B_{t-C}$ imply that $\gamma\in B_t$, and a favorable upper bound arises if for $\gamma\in\Gamma\cap B_t$ the support of $\chi_\varepsilon(g^{-1}h\gamma)$ contained in B_{t+C}.
- Thus favorable lower and upper estimates depend only on the regularity properties of the sets B_t, specifically on the stability property under perturbations by elements h in a fixed neighborhood, and volume regularity of $m_G(B_t)$.

- Therefore taking some fixed $\mathcal{O}_\varepsilon$ and C, the usual strong maximal inequality for averaging over λ_t follows from the ordinary strong maximal inequality for averaging over β_t. It follows that for lattice actions, as for actions of the group G, the maximal inequality holds in great generality and requires only a coarse form of admissibility.

- Therefore taking some fixed $\mathcal{O}_\varepsilon$ and C, the usual strong maximal inequality for averaging over λ_t follows from the ordinary strong maximal inequality for averaging over β_t. It follows that for lattice actions, as for actions of the group G, the maximal inequality holds in great generality and requires only a coarse form of admissibility.

- The mean ergodic theorem for λ_t requires considerably sharper argument, and in particular requires passing to $\varepsilon \to 0$, namely $m_G(\mathcal{O}_\varepsilon) \to 0$.

- Therefore taking some fixed $\mathcal{O}_\varepsilon$ and C, the usual strong maximal inequality for averaging over λ_t follows from the ordinary strong maximal inequality for averaging over β_t. It follows that for lattice actions, as for actions of the group G, the maximal inequality holds in great generality and requires only a coarse form of admissibility.

- The mean ergodic theorem for λ_t requires considerably sharper argument, and in particular requires passing to $\varepsilon \to 0$, namely $m_G(\mathcal{O}_\varepsilon) \to 0$.

- The effective uniform volume estimate appearing in the definition of admissibility is utilized, and is matched against the unavoidable quantity $m_G(\mathcal{O}_\varepsilon)^{-1}$ which the approximation procedure introduces.

- The effective mean ergodic theorem requires in addition an effective estimate on the decay of the operator norms $\left\|\pi_Y^0(\beta_t)\right\|_{L_0^p(Y)}$.

- The effective mean ergodic theorem requires in addition an effective estimate on the decay of the operator norms $\left\|\pi_Y^0(\beta_t)\right\|_{L_0^p(Y)}$.

- This decay estimate plays an indispensable role, and allows a quantitative approximation argument to proceed, again using crucially that the averages are admissible.

- The effective mean ergodic theorem requires in addition an effective estimate on the decay of the operator norms $\left\|\pi_Y^0(\beta_t)\right\|_{L_0^p(Y)}$.
- This decay estimate plays an indispensable role, and allows a quantitative approximation argument to proceed, again using crucially that the averages are admissible.
- As a byproduct of the proof, we obtain an effective decay estimate on the norms $\left\|\pi_X^0(\lambda_t)\right\|_{L_0^p(X)}$, for $1 < p < \infty$.

Operator norm estimates in the automorphic representation

- Let π and σ be unitary representations of an lcsc group G, and assume σ is (isomorphic to) a subrepresentation of π.

Operator norm estimates in the automorphic representation

- Let π and σ be unitary representations of an lcsc group *G*, and assume σ is (isomorphic to) a subrepresentation of π.

- Then for for every probability measure β on *G*, clearly $\|\sigma(\beta)\| \leq \|\pi(\beta)\|$.

Operator norm estimates in the automorphic representation

- Let π and σ be unitary representations of an lcsc group G, and assume σ is (isomorphic to) a subrepresentation of π.
- Then for for every probability measure β on G, clearly $\|\sigma(\beta)\| \le \|\pi(\beta)\|$.
- This inequality holds whenever diagonal matrix coeff' $\langle\sigma(g)v, v\rangle$ of σ can be approximated uniformly on compact sets in G by convex combinations of diagonal matrix coeff' $\langle\pi(g)w, w\rangle$ of π.

Operator norm estimates in the automorphic representation

- Let π and σ be unitary representations of an lcsc group G, and assume σ is (isomorphic to) a subrepresentation of π.
- Then for for every probability measure β on G, clearly $\|\sigma(\beta)\| \leq \|\pi(\beta)\|$.
- This inequality holds whenever diagonal matrix coeff' $\langle\sigma(g)v, v\rangle$ of σ can be approximated uniformly on compact sets in G by convex combinations of diagonal matrix coeff' $\langle\pi(g)w, w\rangle$ of π.
- The latter condition is called weak containment, denoted $\sigma \leq_w \pi$

Operator norm estimates in the automorphic representation

- Let π and σ be unitary representations of an lcsc group G, and assume σ is (isomorphic to) a subrepresentation of π.
- Then for for every probability measure β on G, clearly $\|\sigma(\beta)\| \leq \|\pi(\beta)\|$.
- This inequality holds whenever diagonal matrix coeff' $\langle\sigma(g)v, v\rangle$ of σ can be approximated uniformly on compact sets in G by convex combinations of diagonal matrix coeff' $\langle\pi(g)w, w\rangle$ of π.
- The latter condition is called weak containment, denoted $\sigma \leq_w \pi$
- It was established by Diximier (1969) that weak containment is equivalent to the norm inequality $\|\sigma(f)\| \leq \|\pi(f)\|$ for every $f \in L^1(G)$.

Tempered representations

- We will call a unitary rep' π of G a tempered representation if $\pi \leq_w r_G$. It then follows that $\|\pi(\beta)\| \leq \|r_G(\beta)\|$ for all abs. cont. prob' measures β on G.

Tempered representations

- We will call a unitary rep' π of G a tempered representation if $\pi \leq_w r_G$. It then follows that $\|\pi(\beta)\| \leq \|r_G(\beta)\|$ for all abs. cont. prob' measures β on G.

- Define (G, H, Γ) to be a tempered triple if the restriction of the automorphic representation of G on $L_0^2(G/\Gamma)$ is a tempered representation of H.

Tempered representations

- We will call a unitary rep' π of G a tempered representation if $\pi \leq_w r_G$. It then follows that $\|\pi(\beta)\| \leq \|r_G(\beta)\|$ for all abs. cont. prob' measures β on G.

- Define (G, H, Γ) to be a tempered triple if the restriction of the automorphic representation of G on $L_0^2(G/\Gamma)$ is a tempered representation of H.

  • It is a surprising and useful fact that the foregoing condition of "subgroup temperedness" holds in considerable generality for a large class of triples (G, H, Γ).

Subgroup temperedness

- There are several general principles which can be used to establish subgroup temperedness.

Subgroup temperedness

• There are several general principles which can be used to establish subgroup temperedness.

• Kazhdan's original argument : When $H = SL_2(\mathbb{R}) \ltimes \mathbb{R}^2 \subset G$, any unitary representation of G without $\mathbb{R}^2$-invariant unit vectors is tempered when restricted to $SL_2(\mathbb{R})$.

Subgroup temperedness

• There are several general principles which can be used to establish subgroup temperedness.

• Kazhdan's original argument : When $H = SL_2(\mathbb{R}) \ltimes \mathbb{R}^2 \subset G$, any unitary representation of G without $\mathbb{R}^2$-invariant unit vectors is tempered when restricted to $SL_2(\mathbb{R})$.

• When G is semi simple and had property T, there are universal pointwise bounds on the K-finite matrix coefficients of G in general unitary representations established by Cowling (1980) and Howe (1980), Cowling-Haagerup-Howe (1988), Borel-Wallach (1980's), Howe-Tan (1992), and further developed by Oh (1998). These estimates can be restricted to a subgroup H and are often in $L^{2+\eta}(H)$, which implies that the restriction to H is tempered.

• For example, this holds for (the images of) all the irreducible representations $SL_2(\mathbb{R}) \to SL_n(\mathbb{R})$, $n \geq 3$. This general phenomenon was first observed by Margulis (1997) and subsequently developed further by Oh (1998).

• For example, this holds for (the images of) all the irreducible representations $SL_2(\mathbb{R}) \to SL_n(\mathbb{R})$, $n \geq 3$. This general phenomenon was first observed by Margulis (1997) and subsequently developed further by Oh (1998).

• Unitary representations of simple algebraic groups have matrix coefficients in $L^{2k}(G)$ for some k, Howe-Moore (1978), Howe (1980), Cowling (1980). Restricting a representation of G^k to the diagonally embedded copy of G yields matrix coefficients which are in $L^{2+\eta}(G)$, so the diagonally embedded subgroup is tempered.

• For example, this holds for (the images of) all the irreducible representations $SL_2(\mathbb{R}) \to SL_n(\mathbb{R})$, $n \geq 3$. This general phenomenon was first observed by Margulis (1997) and subsequently developed further by Oh (1998).

• Unitary representations of simple algebraic groups have matrix coefficients in $L^{2k}(G)$ for some k, Howe-Moore (1978), Howe (1980), Cowling (1980). Restricting a representation of G^k to the diagonally embedded copy of G yields matrix coefficients which are in $L^{2+\eta}(G)$, so the diagonally embedded subgroup is tempered.

• For some lattices and their low level congruence subgroups the Selberg eigenvalue conjecture is known to hold, so that $L_0^2(\Gamma \setminus G)$ is known to be a tempered representation of G. This holds for example for $SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z}[i]) \subset SL_2(\mathbb{C})$.

• For example, this holds for (the images of) all the irreducible representations $SL_2(\mathbb{R}) \to SL_n(\mathbb{R})$, $n \geq 3$. This general phenomenon was first observed by Margulis (1997) and subsequently developed further by Oh (1998).

• Unitary representations of simple algebraic groups have matrix coefficients in $L^{2k}(G)$ for some k, Howe-Moore (1978), Howe (1980), Cowling (1980). Restricting a representation of G^k to the diagonally embedded copy of G yields matrix coefficients which are in $L^{2+\eta}(G)$, so the diagonally embedded subgroup is tempered.

• For some lattices and their low level congruence subgroups the Selberg eigenvalue conjecture is known to hold, so that $L_0^2(\Gamma \setminus G)$ is known to be a tempered representation of G. This holds for example for $SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z}[i]) \subset SL_2(\mathbb{C})$.

• Spectral estimates for subgroups of G acting in the automorphic representation on $L^2(G/\Gamma)$ have a very wide range of applications, due to a recently established effective form of the duality principle.

The duality principle

• The general classical method of duality in homogenous dynamics aims to establish properties of the Γ-orbits in $H \setminus G$ by using properties of the H-orbits in G/Γ.

The duality principle

• The general classical method of duality in homogenous dynamics aims to establish properties of the Γ-orbits in $H \setminus G$ by using properties of the H-orbits in G/Γ.

• This approach has been applied to solve many problem in homogeneous dynamics from its beginning (Hedlund, Hopf, Furstenberg, Veech, Zimmer, Dani, Margulis, Margulis-Kleinbock.......)

The duality principle

• The general classical method of duality in homogenous dynamics aims to establish properties of the Γ-orbits in $H \setminus G$ by using properties of the H-orbits in G/Γ.

• This approach has been applied to solve many problem in homogeneous dynamics from its beginning (Hedlund, Hopf, Furstenberg, Veech, Zimmer, Dani, Margulis, Margulis-Kleinbock.......)

• A general effective form of the duality principle has been developed in joint work with Alex Gorodnik (2012).

The duality principle

• The general classical method of duality in homogenous dynamics aims to establish properties of the Γ-orbits in $H \setminus G$ by using properties of the H-orbits in G/Γ.

• This approach has been applied to solve many problem in homogeneous dynamics from its beginning (Hedlund, Hopf, Furstenberg, Veech, Zimmer, Dani, Margulis, Margulis-Kleinbock.......)

• A general effective form of the duality principle has been developed in joint work with Alex Gorodnik (2012).

• This method allows proving effective mean and pointwise ergodic theorems for the discrete averages supported on Γ-orbit points $Hg\gamma$ in $H \setminus G$, when the elements $\gamma \in \Gamma$ are ordered by a norm, namely when averaging on the lattice points in $B_t \cap \Gamma$.

The method of effective duality

• The quantitative method of duality applies in considerable generality, for all locally compact groups, closed subgroups H, and discrete lattices Γ,

The method of effective duality

• The quantitative method of duality applies in considerable generality, for all locally compact groups, closed subgroups H, and discrete lattices Γ,

• subject only to natural and necessary assumptions about

1. the growth of the sets $H_t = H \cap B_t$ and the lattice points in their vicinity,
2. the spectral theory of H in $L^2_o(\Gamma \setminus G)$, and particularly spectral estimate for the averages supported on H_t,
3. the local behavior of the invariant measure $m_{G/H}$ on the homogeneous space $G/H = V$.

The method of effective duality

• The quantitative method of duality applies in considerable generality, for all locally compact groups, closed subgroups H, and discrete lattices Γ,

• subject only to natural and necessary assumptions about

1. the growth of the sets $H_t = H \cap B_t$ and the lattice points in their vicinity,
2. the spectral theory of H in $L^2_0(\Gamma \setminus G)$, and particularly spectral estimate for the averages supported on H_t,
3. the local behavior of the invariant measure $m_{G/H}$ on the homogeneous space $G/H = V$.

• We will conclude by stating an application of the method of effective duality to the evaluation of best possible exponents for intrinsic Diophantine approximation on homogeneous algebraic varieties.

Diophantine approximation on affine homogeneous varieties

- Let G be an algebraic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$, Γ a lattice subgroup of G, e.g. $G(\mathbb{Z})$ in $G(\mathbb{R})$.

Diophantine approximation on affine homogeneous varieties

- Let G be an algebraic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$, Γ a lattice subgroup of G, e.g. $G(\mathbb{Z})$ in $G(\mathbb{R})$.

- Fix a norm on $\mathbb{R}^n$ and on $M_n(\mathbb{R})$.

Diophantine approximation on affine homogeneous varieties

• Let G be an algebraic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$, Γ a lattice subgroup of G, e.g. $G(\mathbb{Z})$ in $G(\mathbb{R})$.

• Fix a norm on $\mathbb{R}^n$ and on $M_n(\mathbb{R})$.

• Consider an affine subvariety $V \subset \mathbb{R}^n$ which is invariant and homogeneous under the G-action, so that $V = G/H$, H a $\mathbb{Q}$-algebraic subgroup.

Diophantine approximation on affine homogeneous varieties

- Let G be an algebraic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$, Γ a lattice subgroup of G, e.g. $G(\mathbb{Z})$ in $G(\mathbb{R})$.

- Fix a norm on $\mathbb{R}^n$ and on $M_n(\mathbb{R})$.

- Consider an affine subvariety $V \subset \mathbb{R}^n$ which is invariant and homogeneous under the G-action, so that $V = G/H$, H a $\mathbb{Q}$-algebraic subgroup.

- Restrict the norm chosen on $\mathbb{R}^n$ to V.

Diophantine approximation on affine homogeneous varieties

- Let G be an algebraic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$, Γ a lattice subgroup of G, e.g. $G(\mathbb{Z})$ in $G(\mathbb{R})$.

- Fix a norm on $\mathbb{R}^n$ and on $M_n(\mathbb{R})$.

- Consider an affine subvariety $V \subset \mathbb{R}^n$ which is invariant and homogeneous under the G-action, so that $V = G/H$, H a $\mathbb{Q}$-algebraic subgroup.

- Restrict the norm chosen on $\mathbb{R}^n$ to V.

- Assume that Γ is ergodic on V, so that almost every Γ-orbit is dense in V.

Quantifying denseness

- Consider the Diophantine inequality $\left\|\gamma^{-1}x - x_0\right\| < \epsilon$, with $\gamma \in \Gamma$ satisfying the norm bound $\|\gamma\| \leq B\epsilon^{-\zeta}$.

Quantifying denseness

• Consider the Diophantine inequality $\|\gamma^{-1}x - x_0\| < \epsilon$, with $\gamma \in \Gamma$ satisfying the norm bound $\|\gamma\| \le B\epsilon^{-\zeta}$.

• Define the Diophantine approximation exponent $\kappa(x, x_0)$ as the infimum of $\zeta > 0$ such that the foregoing inequality has a solution with the properties stated.

Quantifying denseness

• Consider the Diophantine inequality $\left\|\gamma^{-1}x - x_0\right\| < \epsilon$, with $\gamma \in \Gamma$ satisfying the norm bound $\|\gamma\| \leq B\epsilon^{-\zeta}$.

• Define the Diophantine approximation exponent $\kappa(x, x_0)$ as the infimum of $\zeta > 0$ such that the foregoing inequality has a solution with the properties stated.

• $\kappa(x, x_0)$ is a $\Gamma \times \Gamma$-invariant function, hence almost surely a constant κ when the action is ergodic. κ depends on G, Γ and V, but not on the norms chosen on $\mathbb{R}^n$ and $M_n(\mathbb{R})$.

Deriving an upper bound for the exponent

- An upper bound on the Diophantine exponent can be derived using

Deriving an upper bound for the exponent

- An upper bound on the Diophantine exponent can be derived using

- spectral estimates in the automorphic representation $L^2(\Gamma \setminus G)$ leading to a quantitative mean ergodic theorem for H,

Deriving an upper bound for the exponent

• An upper bound on the Diophantine exponent can be derived using

• spectral estimates in the automorphic representation $L^2(\Gamma \setminus G)$ leading to a quantitative mean ergodic theorem for H,

• dynamical arguments exploiting the speed of distribution of H-orbits in $\Gamma \setminus G$, in the form of a shrinking target argument.

Deriving an upper bound for the exponent

- An upper bound on the Diophantine exponent can be derived using

- spectral estimates in the automorphic representation $L^2(\Gamma \setminus G)$ leading to a quantitative mean ergodic theorem for H,

- dynamical arguments exploiting the speed of distribution of H-orbits in $\Gamma \setminus G$, in the form of a shrinking target argument.

- A quantitative duality argument translating a rate in the action of H on $\Gamma \setminus G$ to a rate in the action of Γ on G/H

Deriving an upper bound for the exponent

• An upper bound on the Diophantine exponent can be derived using

• spectral estimates in the automorphic representation $L^2(\Gamma \setminus G)$ leading to a quantitative mean ergodic theorem for H,

• dynamical arguments exploiting the speed of distribution of H-orbits in $\Gamma \setminus G$, in the form of a shrinking target argument.

• A quantitative duality argument translating a rate in the action of H on $\Gamma \setminus G$ to a rate in the action of Γ on G/H

• Specifically, consider the intersection of norm balls with the stability group H, namely $H_T = \{h \in H \,;\, \|h\| < T\}$.

Deriving an upper bound for the exponent

- An upper bound on the Diophantine exponent can be derived using

- spectral estimates in the automorphic representation $L^2(\Gamma \setminus G)$ leading to a quantitative mean ergodic theorem for H,

- dynamical arguments exploiting the speed of distribution of H-orbits in $\Gamma \setminus G$, in the form of a shrinking target argument.

- A quantitative duality argument translating a rate in the action of H on $\Gamma \setminus G$ to a rate in the action of Γ on G/H

- Specifically, consider the intersection of norm balls with the stability group H, namely $H_T = \{h \in H\,;\, \|h\| < T\}$.

- Consider the invariant probability measure $m_{G/\Gamma}$ on $Y = G/\Gamma$ and define averaging operators $\pi_Y(\beta_T) : L^2(G/\Gamma) \to L^2(G/\Gamma)$, given by

The quantitative mean ergodic theorem

$$\pi_Y(\beta_T)f(y) = \frac{1}{m_H(H_T)} \int_{h \in H_T} f(h^{-1}y)dm_H(h) \ , \ y \in \Gamma \setminus G \, .$$

The quantitative mean ergodic theorem

$$\pi_Y(\beta_T)f(y) = \frac{1}{m_H(H_T)} \int_{h\in H_T} f(h^{-1}y)dm_H(h)\ ,\ \ y \in \Gamma \setminus G\,.$$

• Assume that the quantitative mean ergodic theorem for the averaging operators $\pi_Y^0(\beta_T)$ holds, namely :

The quantitative mean ergodic theorem

$$\pi_Y(\beta_T)f(y) = \frac{1}{m_H(H_T)} \int_{h \in H_T} f(h^{-1}y)dm_H(h)\ ,\ \ y \in \Gamma \setminus G\,.$$

• Assume that the quantitative mean ergodic theorem for the averaging operators $\pi_Y^0(\beta_T)$ holds, namely :

• there exists $\theta > 0$ such that

$$\|\pi_Y(\beta_T)f - \int_Y fdm\|_{L^2(\Gamma\setminus G)} \leq C(\eta)m_H(H_T)^{-\theta+\eta}\|f\|_{L^2(\Gamma\setminus G)}$$

for every $\eta > 0$, suitable $C(\eta)$, and $t \geq t_\eta$.

Bounds for the Diophantine exponent

• Assume that $H_T = G_T \cap H$ satisfies a volume growth bounds with rate $a > 0$, namely $T^{a-\eta} \ll m_H(H_T) \ll T^{a+\eta}$.

Bounds for the Diophantine exponent

• Assume that $H_T = G_T \cap H$ satisfies a volume growth bounds with rate $a > 0$, namely $T^{a-\eta} \ll m_H(H_T) \ll T^{a+\eta}$.

Bounds for the Diophantine exponent

- Assume that $H_T = G_T \cap H$ satisfies a volume growth bounds with rate $a > 0$, namely $T^{a-\eta} \ll m_H(H_T) \ll T^{a+\eta}$.

- Let d denote the real dimension of the variety $V = G/H$.

Bounds for the Diophantine exponent

- Assume that $H_T = G_T \cap H$ satisfies a volume growth bounds with rate $a > 0$, namely $T^{a-\eta} \ll m_H(H_T) \ll T^{a+\eta}$.

- Let d denote the real dimension of the variety $V = G/H$.

- Theorem (Ghosh-Gorodnik-N. 2014) Under the assumptions stated above, the Diophantine exponent satisfies the bound $\frac{d}{a} \leq \kappa \leq \frac{1}{2\theta} \cdot \frac{d}{a}$.

Bounds for the Diophantine exponent

• Assume that $H_T = G_T \cap H$ satisfies a volume growth bounds with rate $a > 0$, namely $T^{a-\eta} \ll m_H(H_T) \ll T^{a+\eta}$.

• Let d denote the real dimension of the variety $V = G/H$.

• Theorem (Ghosh-Gorodnik-N. 2014) Under the assumptions stated above, the Diophantine exponent satisfies the bound $\frac{d}{a} \leq \kappa \leq \frac{1}{2\theta} \cdot \frac{d}{a}$.

Bounds for the Diophantine exponent

• Assume that $H_T = G_T \cap H$ satisfies a volume growth bounds with rate $a > 0$, namely $T^{a-\eta} \ll m_H(H_T) \ll T^{a+\eta}$.

• Let d denote the real dimension of the variety $V = G/H$.

• Theorem (Ghosh-Gorodnik-N. 2014) Under the assumptions stated above, the Diophantine exponent satisfies the bound $\frac{d}{a} \le \kappa \le \frac{1}{2\theta} \cdot \frac{d}{a}$.

• Conclusion : if $2\theta = 1$ then the lower and upper bounds for the Diophantine exponent coincide !

Best possible rate of approximation

- Corollary 1. If the rate of convergence in the mean ergodic theorem for the averaging operators β_T acting on $L_0^2(\Gamma \setminus G)$, is as fast as the inverse of the square root of the volume of H_T, then the rate of Diophantine approximation of Γ-orbits on the variety $V = G/H$ is best possible, and the Diophantine exponent is given by $\kappa = \frac{d}{a}$ (which is an a-priori pigeon-hole bound).

Best possible rate of approximation

- Corollary 1. If the rate of convergence in the mean ergodic theorem for the averaging operators β_T acting on $L^2_0(\Gamma \setminus G)$, is as fast as the inverse of the square root of the volume of H_T, then the rate of Diophantine approximation of Γ-orbits on the variety $V = G/H$ is best possible, and the Diophantine exponent is given by $\kappa = \frac{d}{a}$ (which is an a-priori pigeon-hole bound).

- Corollary 2. If the stability group H is semi simple and non-compact, and the restriction of the automorphic representation $\pi^0_{G/\Gamma}$ to H is a tempered representation of H, then the Diophantine exponent of the irreducible lattice Γ of G in its action on G/H is best possible, and is given by $\kappa = \frac{d}{a}$.

Best possible rate of approximation

- Corollary 1. If the rate of convergence in the mean ergodic theorem for the averaging operators β_T acting on $L^2_0(\Gamma \setminus G)$, is as fast as the inverse of the square root of the volume of H_T, then the rate of Diophantine approximation of Γ-orbits on the variety $V = G/H$ is best possible, and the Diophantine exponent is given by $\kappa = \frac{d}{a}$ (which is an a-priori pigeon-hole bound).

- Corollary 2. If the stability group H is semi simple and non-compact, and the restriction of the automorphic representation $\pi^0_{G/\Gamma}$ to H is a tempered representation of H, then the Diophantine exponent of the irreducible lattice Γ of G in its action on G/H is best possible, and is given by $\kappa = \frac{d}{a}$.

- Thus "subgroup temperedness" for the triples (G, H, Γ) (when G and H are semisimple) implies the best possible estimate for intrinsic Diophantine approximation by Γ-lattice orbits on the homogeneous variety $H \setminus G$.