2019. 10. 29. ①

~~Introduction~~

Γ countable gp

$\lambda : \Gamma \curvearrowright \ell_2\Gamma$ left regular repr $\lambda_g \delta_t = \delta_{gt}$, $\lambda(f)\xi = f * \xi$

$C^*_\lambda\Gamma = \overline{\lambda(\mathbb{C}\Gamma)}^{\|\cdot\|} \subseteq B(\ell_2\Gamma)$ reduced C^*-alg

$\Lambda \leq \Gamma \implies C^*_\lambda\Lambda \subseteq C^*_\lambda\Gamma$ naturally

but $C^*_\lambda\Gamma \to C^*_\lambda\Gamma/N$ well-def (i.e. cts) iff $N \trianglelefteq \Gamma$ amenable

amenability : Γ amen $\overset{\text{def}}{\iff} \exists \varphi \in S(\ell_\infty\Gamma)$ left Γ-inv

$\iff 1_\Gamma : \Gamma \to \mathbb{C}$ unit character cts on $C^*_\lambda\Gamma$

- finite, abelian, subgp, extension, inductive limits
- F_2 not

$\tau : \Gamma \to \mathbb{C}$ trace (character) if
- $\tau(xy) = \tau(yx)$ or equivalently conj inv
- pos semidef
- $\tau(e) = 1$ (normalization)

e.g. 1_Γ ; $\tau_\lambda(g) = \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases} \overset{\text{GNS}}{\rightsquigarrow} \lambda : \Gamma \curvearrowright \ell_2\Gamma$

Prob When $C^*_\lambda\Gamma$ simple (said to be C^*-simple)

$\text{Ramen}\,\Gamma :=$ the largest amenable normal subgps of Γ

amenable radical

C^*-simple $\implies \text{Ramen} = 1$ Converse ?

Thm Powers 1967/75 $C^*_\lambda F_d$ simple & unique trace

Sketch of Pf Kesten : $g_1, \dots, g_n$ free ($n \geq 2$) $\implies \|\sum_{i=1}^n \lambda(g_i)\| = 2\sqrt{n-1}$

$\rightsquigarrow$ F_d has "Powers Averaging Property"

$\exists \mu_n \in \text{Prob}\,\Gamma\ \ \forall g \neq e\ \ \|\sum \mu_n(t)\lambda(tgt^{-1})\| \to 0$

$\Phi_\mu = \sum_t \mu(t)\,\mathrm{Ad}_t$

$\rightsquigarrow \Phi_{\mu_n}(\cdot) \to \tau_\lambda(\cdot)1$ on $C^*_\lambda\Gamma$

$\rightsquigarrow C^*_\lambda\Gamma$ simple & unique trace.

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LeBoudec ($\neq$)

Thm KK (Kennedy, Haagerup) BKKO TFAE
(i) Γ C^*-simple
(ii) PAP : $\exists \mu_n$ $\Phi_{\mu_n} \to \tau_\lambda$ on $C^*_\lambda\Gamma$
(iii) $\Gamma \curvearrowright \partial_F\Gamma$ (topo) free
(iii') $\exists$ ~~topo~~ free boundary action
(iv) No nontrivial amenable URS

Thm BKKO, $\frac{K}{H}$ Bader-Duchesne-Lecureux TFAE
(i) $C^*_\lambda\Gamma$ unique trace $\tau_\lambda(g)$
(ii) $\forall g \neq e$ $\exists \mu_n$ $\Phi_{\mu_n}(g) \to 0$"
(iii) $\Gamma \curvearrowright \partial_F\Gamma$ faithful
(iv) No nontrivial amenable IRS
(v) $\mathrm{Ramen}\,\Gamma = \mathbb{1}$

Topological Boundary Theory (Furstenberg)

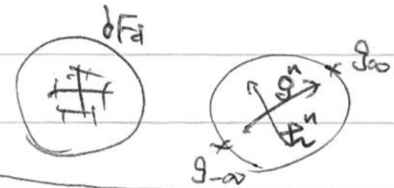
$\Gamma \curvearrowright X^{cpt}$ Γ-boundary $\overset{def}{\iff}$ $\forall \mu \in \mathrm{Prob}X$ $\overline{\Gamma\mu} \supseteq \{\delta_x : x \in X\}$ ($= X$)

i.e. X is the unique minimal Γ-component of $\mathrm{Prob}\,X$

$\circledast$ $\Gamma \curvearrowright Y^{cpt}$ minimal, $f : Y \xrightarrow[\Gamma]{} \mathrm{Prob}\,X$ ($\leftrightarrow$ $C(X) \xrightarrow{\Phi} C(Y)$ unital positive Γ-map)
$\implies f(Y) \supseteq X$ and such a map is unique (if exists)

Ex 0 $\Gamma \curvearrowright \{\cdot\}$

Ex 1 $F_d \curvearrowright \partial F_d$



$\forall g \in F_d \setminus \mathbb{1}$ acts hyperbolically
$\exists g_{-\infty}, g_\infty \in \partial F_d$
$g^n x \to g_\infty$ $\forall x \in \partial F_d \setminus \{g_{-\infty}\}$

Ex 2 (S. Glasner) K minimal cpt convex Γ-sp
$\implies \overline{\mathrm{ext}\,K}$ is a Γ-boundary.

Th (Furstenberg) $\exists!$ largest Γ-boundary $\partial_F\Gamma$ "Furstenberg boundary"
In fact $\forall K \subseteq S(\ell_\infty\Gamma) = \mathrm{Prob}(\beta\Gamma)$ minimal cpt cvx Γ-sp
$\partial_F\Gamma \cong \overline{\mathrm{ext}\,K}$

Pf $\Gamma \curvearrowright X_{\mathrm{bdy}}$ $x_0 \in X$ $\rightsquigarrow$ $C(X) \xhookrightarrow{\text{orbit map}} \ell_\infty\Gamma \xrightarrow{\mathrm{ex}} C(\overline{\mathrm{ext}\,K})$
$\overset{\circledast}{\rightsquigarrow}$ $\overline{\mathrm{ext}\,K} \to X$ unique $\square$

Cor $\forall x \in \partial_F\Gamma$ Γ_x amen
$\mathrm{Ker}(\Gamma \curvearrowright \partial_F\Gamma) = \mathrm{Ramen}\,\Gamma$

(:) $\exists$ Γ_x-inv mean on Γ
(:) . $\mathrm{Ker}\,\Gamma \curvearrowright \partial_F\Gamma$ amenable normal
. $\exists \mu \in \mathrm{Prob}(\partial_F\Gamma)^{\mathrm{Ramen}\,\Gamma}$
$\overline{\Gamma\mu} \supseteq \partial_F\Gamma$
~~Fix~~ $\supseteq$ $\mathrm{Ramen}\,\Gamma$ $\square$

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IRS & URS

$\mathrm{Sub}\,\Gamma \subseteq \{0,1\}^\Gamma$ cpt Chabauty sp

$\Gamma \curvearrowright \mathrm{Sub}\,\Gamma$ by conjugation

$\Gamma \curvearrowright X^{\mathrm{cpt}}$ $\Gamma_x := \{g \in \Gamma : gx = x\}$ stabilizer

$\Gamma_x^0 := \{g \in \Gamma : g = \mathrm{id}$ on a nbhd of $x\} \trianglelefteq \Gamma_x$

$\Gamma \curvearrowright X$ top. free $\overset{\mathrm{def}}{\iff}$ $\Gamma_x^0 = \mathbb{1}$ $\forall x$

Exercise: $x \mapsto \Gamma_x \in \mathrm{Sub}\,\Gamma$ is cts at x iff $\Gamma_x = \Gamma_x^0$

$X_0 := \{x : \Gamma_x = \Gamma_x^0\}$ pt of continuity

$= (\bigcup_g \partial \mathrm{Fix}\, g)^c$ dense G_δ in X

Def An IRS is a Γ-inv prob measure on $\mathrm{Sub}\,\Gamma$

Abert - Y. Glasner - Virag

Ex 1 $N \trianglelefteq \Gamma$ δ_N

Ex 2 $\Gamma \curvearrowright (X,\mu)$ pmp stabilizer IRS

(Fact $\forall$ IRS arises in this way)

Def An URS is a minimal Γ-component of $\mathrm{Sub}\,\Gamma$

E. Glasner - Weiss

Ex 1 $\{N\}$

Ex 2 $\Gamma \curvearrowright X$ minimal $\underset{\mathrm{Fact}}{\Longrightarrow}$ closure$\{\Gamma_x : x \in X_0\}$ is a URS

~~(Fact $\forall$ URS arises in this way)~~

$\mathcal{S}_\Gamma :=$ the stabilizer URS for $\Gamma \curvearrowright \partial_F \Gamma$ "Furstenberg" URS

(Fact: $\Gamma_x = \Gamma_x^0$ $\forall x \in \partial_F\Gamma$ by Frolik's Th)

$\mathcal{U}, \mathcal{V}$ URS $\mathcal{U} \prec \mathcal{V} \overset{\mathrm{def}}{\iff} \exists \Lambda \in \mathcal{U}\ \exists \Delta \in \mathcal{V}\ \Lambda \leq \Delta$

$\iff \forall \Lambda$ $\exists \Delta$ —

$\iff \forall \Delta \in \mathcal{V}\ \exists \Lambda \in \mathcal{U}$ —

Prop $\mathcal{S}_\Gamma$ is the largest amenable URS

☺ $\mathcal{U}$ amenable URS $\Lambda \in \mathcal{U}$

$\exists \mu \in \mathrm{Prob}(\partial_F \Gamma)^\Lambda$ $g_n \mu \to \delta_x$ $x \in X_0$

WMA $g_n \Lambda g_n^{-1} \to \Lambda' \in \mathcal{U}$ $\leadsto$ $\Lambda' \leq \Gamma_x$ ☒

Weak Containment

π, σ unitary rep of Γ

$\pi \prec \sigma \overset{\text{def}}{\iff} C^*(\sigma) \to C^*(\pi)$ well def (i.e. cts)

S for state

$\iff S(C^*(\pi)) \subseteq S(C^*(\sigma))$ as functions on Γ

When φ GNS faithful on $C^*(\pi)$

$\pi \prec \sigma \iff \varphi \in S(C^*(\sigma))$

$\Lambda \leq \Gamma$ $\quad 1_\Lambda \in S(C^*_\lambda \Gamma) \iff \Lambda$ amen

$\underset{\text{GNS}}{\rightsquigarrow} \lambda_{\Gamma/\Lambda} : \Gamma \curvearrowright \ell_2 \Gamma/\Lambda$

Cor $\lambda_{\Gamma/\Lambda} \prec \lambda_\Gamma \iff \Lambda$ amen

◎ spectral invariants of $\Gamma \curvearrowright X$

ν quasi-inv prob measure $\rightsquigarrow \overset{\text{Koopman rep}}{K^\nu : \Gamma \curvearrowright L^2(X, \nu)}$ $\quad (K_g^\nu \xi)(x) = \xi(g^{-1}x)\sqrt{\frac{d g\nu}{d\nu}(x)}$

$K = \bigoplus_\nu K^\nu$ universal Koopman rep (Note ν upward directed)

$\tilde{K}$ universal covariant repn i.e. $\exists \tilde{\pi} : C(X) \to \mathbb{B}(\mathcal{H}_{\tilde{\pi}})$

$K \prec \tilde{K}$ $\quad \tilde{K}_g \tilde{\pi}(f) \tilde{K}_g^* = \tilde{\pi}(g.f)$

Ex $\Gamma \curvearrowright \{\cdot\} \Rightarrow K = 1_\Gamma$, $\tilde{K} = \pi_{\text{univ}}$

Prop $\lambda_{\Gamma/\Gamma_x} \prec K \quad \forall x \in X_0$

(:) $U_i \to \{x\}$ $\quad \frac{1}{\nu(U_i)} \langle K^\nu(\cdot) 1_{U_i}, 1_{U_i} \rangle \to 1_{\Gamma/\Gamma_x}(\cdot)$ $\square$

Prop (Raum) $\Gamma \curvearrowright X$ Γ-boundary $x \in X_0$

Then $C^*(\lambda_{\Gamma/\Gamma_x})$ simple

Easy → In fact unique simple quotient of $C^*(K)$

Moreover, (♠) $\forall \varphi \in S(C^*(K))$ $\quad \overline{\Gamma\varphi} \ni 1_{\Gamma_x}$ $\quad$ | (♠') $\forall \varphi \in S(C^*(\tilde{K}))$ $\quad \overline{\Gamma\varphi} \ni \psi$ s.t. $\operatorname{supp}\psi \subseteq \Gamma_x$

Pf of (♠) : φ, $F \Subset \Gamma$, $\varepsilon > 0$ given $\quad \varphi$ extended on $\mathbb{B}(L^2(X, \nu))$

((♠') is similar) $\quad \exists U \in \mathcal{N}_x$ s.t. $gU \cap U = \phi$ or $g|_U = \mathrm{id}_U$

depending on $g \in F \setminus \Gamma_x$ or $g \in F \cap \Gamma_x$

$\exists t \quad \varphi(1_{tU}) \underset{\varepsilon}{\approx} 1$

$\rightsquigarrow (t^{-1}\varphi)(\underset{\cap F}{K_g}) \approx (t^{-1}\varphi)(1_U K_g 1_U) \approx 1_{\Gamma_x}(g)$ $\square$

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<u>Cor</u> $\Gamma \curvearrowright X$ Γ-boundary

top. free $\Leftrightarrow$ $\lambda \prec k$

☺ $\Rightarrow$ Already done $\Leftarrow$: $\lambda \prec k \Rightarrow \tau_\lambda \in S(C^*(k))$

$\Rightarrow 1_{\Gamma_x} \in \overline{\Gamma \tau_\lambda} = \{\tau_\lambda\} \ \forall x \in X_0$ ☒

<u>Cor</u> Γ C^*-simple $\Rightarrow$ $\Omega_\Gamma = \mathbb{1}$

In particular $\Gamma \curvearrowright X$ minimal & Γ_x amen $\Rightarrow$ top. free

<u>Pf</u> $\Gamma \curvearrowright \partial_F \Gamma$ Γ_x amen $\to$ $\lambda_{\Gamma/\Gamma_x} \prec \lambda_\Gamma$

$\xrightarrow{C^*\text{-simple}}$ $\lambda_{\Gamma/\Gamma_x} \sim \lambda_\Gamma$

$\quad \wr$

$\quad k$ ☒

- $\Gamma \curvearrowright \partial_F \Gamma$ top. free $\Rightarrow$ PAP

☺ $\lambda \prec k$ $\to$ $\forall \varphi \in S(C^*(\lambda)) \subseteq S(C^*(k_{\Gamma \curvearrowright \partial_F \Gamma}))$

$\to \overline{\Gamma \varphi} \ni \tau_\lambda$

$\underset{\text{separation}}{\overset{\text{HB}}{\rightsquigarrow}}$ PAP ☒

<u>Cor</u> $\forall \tau \in S(C^*_\lambda \Gamma)$ trace $\Rightarrow$ $\operatorname{supp} \tau \subseteq \operatorname{Ramen} \Gamma$

☺ $\tau \in S(C^*(\lambda \otimes k_{\Gamma \curvearrowright \partial_F \Gamma})) \subseteq S(\widetilde{K}_{\Gamma \curvearrowright \partial_F \Gamma})$

$\rightsquigarrow \underset{\{\tau\}}{\overset{\shortparallel}{\overline{\Gamma \tau}}} \ni \psi$ s.t. $\operatorname{supp} \psi \subseteq \Gamma_x$

$\rightsquigarrow \operatorname{supp} \tau \subseteq \bigcap \Gamma_x = \operatorname{Ramen} \Gamma$ ☒

$\rightsquigarrow$ $\forall$ amenable IRS μ is supported on $\operatorname{Ramen} \Gamma$

☺ $\int 1_\Lambda \, d\mu(\Lambda)$ trace on $C^*_\lambda \Gamma$ ⅃

Tucker-Drob

locally compact grp G ?