I - 1

Topics : Non-amenable groups (countable & discrete) Γ

$\mathbb{F}_r$, hyperbolic grps, $SL(n,\mathbb{Z})$, lattices of connected ss Lie grps

Plan : 1, 2 Topologically amenable actions $\Gamma \curvearrowright X$ & exactness

3 Haagerup property (a-T-menability)

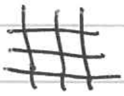
4 ~ weak amenability, property T, ...

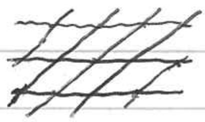
$\mathcal{S} \subseteq \Gamma$ symmetric finite generating subset $\rightsquigarrow \Gamma = \bigcup_n \mathcal{S}^n$

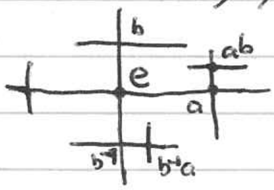
$|x| := \min\{n : \exists s_1, \dots, s_n \in \mathcal{S} \text{ s.t. } x = s_1 \cdots s_n\}$, $|e| := 0$

$d(x,y) = |x^{-1}y|$ $\quad \Gamma \curvearrowright (\Gamma, d)$ by left multiplication

Cayley graph of $(\Gamma, \mathcal{S})$: vertices $= \Gamma$ edges $= \{\{x,y\} : d(x,y) = 1\}$

Example $\quad \mathbb{Z}^2, \{(\pm 1, 0), (0, \pm 1)\}$ 

$\mathbb{Z}^2 \quad \{(\pm 1, 0), \pm(1,1)\}$ 

$\mathbb{F}_r, \{s_1^{\pm}, \dots, s_r^{\pm}\}$ 

$\mathbb{F}_2 = \langle a, b \rangle$

tree of degree $2r$

$\mathcal{S}, \mathcal{S}'$ s.f.g. subsets of Γ

$\longrightarrow \exists K$ s.t. $\mathcal{S} \subseteq \mathcal{S}'^K$ & $\mathcal{S}' \subseteq \mathcal{S}^K$

$\longrightarrow (\Gamma, d_{\mathcal{S}}) \xrightarrow{id} (\Gamma, d_{\mathcal{S}'})$ is a biLipshitz isomorphism

$$\mathrm{Lip}(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)} \underset{\text{graph metric}}{=\!=} \sup\{d(f(x), f(y)) : d(x,y) =$$

$\rightsquigarrow$ Any biLipshitz invariant of the metric sp $(\Gamma, d_{\mathcal{S}})$ is an invariant of Γ.

$f : (X, d) \to (Y, d)$ is a quasi isometry (large scale biLipshit

$\overset{\text{def}}{\Longleftrightarrow} \exists K, L \geq 0$ s.t. $\circ$ $K^{-1} d(x,y) - L \leq d(f(x), f(y)) \leq K d(x,y) + L$

$\circ$ $\forall y \in Y\ \exists x \in X$ s.t. $d(f(x), y) \leq L$

f is a coarse embedding

$\overset{\text{def}}{\Longleftrightarrow} \exists \rho_{\pm} : [0, +\infty) \to [0, +\infty)$ $\quad \rho_- \nearrow +\infty$

s.t. $\rho_-(d(x,y)) \leq d(f(x), f(y)) \leq \rho_+(d(x,y))$

$(\rho_+(r) = \mathrm{Lip}(f)\, r$ if (X, d) is a graph.$)$

I-2.

Def Γ discrete X cpt $\Gamma \curvearrowright X$

$\mathrm{Prob}\,\Gamma = \{\xi \in \ell_1\Gamma : \xi \geq 0\ \|\xi\|_1 = 1\}$

$\Gamma \curvearrowright X$ is (topologically) amenable

$\overset{\text{def}}{\iff} \forall E \Subset \Gamma\ \forall \varepsilon > 0\ \exists \mu : X \to \mathrm{Prob}\,\Gamma$ conti

s.t. $\sup_{\substack{x \in X \\ s \in E}} \|\mu_{sx} - s\mu_x\| < \varepsilon$

modulo perturbation

$\exists F \Subset \Gamma$ s.t. $\mathrm{supp}\,\mu_x \subset F$ for all $x \in X$

& $x \mapsto \mu_x(t)$ conti

When Γ countable

$\iff \exists \mu^{(n)} : X \to \mathrm{Prob}\,\Gamma$ Borel

s.t. $\|\mu^{(n)}_{sx} - s\mu^{(n)}_x\| \xrightarrow[n\to\infty]{} 0\ \forall s \forall x$

Rem
- $\Gamma \curvearrowright X$ amenable & $\exists\ \Gamma$-inv prob measure on X e.g. $X = \{pt\}$
  $\iff \Gamma$ amenable
- $\Gamma \curvearrowright X$ amenable & $\Lambda \leq \Gamma \Rightarrow \Lambda \curvearrowright X$ amenable
- $\Gamma \curvearrowright X$, $\Gamma \curvearrowright Y$, $\exists f : Y \to X$ Γ-equiv
  If $\Gamma \curvearrowright X$ amenable $\Rightarrow \Gamma \curvearrowright Y$ amenable
- Definition generalizes to loc cpt grps & loc cpt spaces, and more generally loc cpt groupoids
  ($\Gamma \curvearrowright X$ amenable $\iff$ the translation groupoid $X \rtimes \Gamma$ is amenable)

Thm $\Gamma \curvearrowright X$ Consider

(1) $\Gamma \curvearrowright X$ amenable

(2) $\Gamma \curvearrowright (X, \mu)$ Zimmer amenable for $\forall \mu$ quasi-inv

(3) $C(X) \rtimes \Gamma$ nuclear C^*-alg

(4) $\forall$ covariant rep $\pi : C(X) \to \mathbb{B}(\mathcal{H})$, $u : \Gamma \to \mathcal{U}(\mathbb{B}(\mathcal{H}))$

$u_s \pi(f) u_s^* = \pi(s \cdot f)$, $(s\cdot f)(x) = f(s^{-1}x)$

e.g. Koopman repn on $L^2(X, \mu)$,

one has $\pi \precsim \lambda$

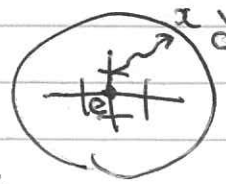
Then $(1) \iff (2) \iff (3) \Rightarrow (4)$

$\lrcorner$

I-3

Examples

$\mathbb{F}_r \curvearrowright \partial\mathbb{F}_r$



$x \in \partial\mathbb{F}_r$

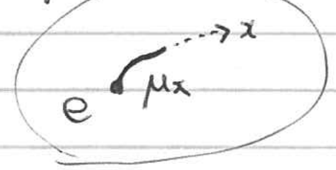
$\partial\mathbb{F}_r$ = infinite reduced words

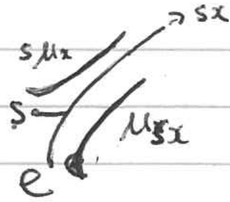
$\underset{e}{\overset{\shortparallel}{x(0)}}, x(1), \cdots \in \mathbb{F}_r$

$d(x(m), x(n)) = |m-n|$

Take $N \in \mathbb{N}$ and

$\mu_x := \frac{1}{N}\sum_{k=0}^{N-1} \delta_{x(k)} \in \mathrm{Prob}\,\mathbb{F}_r$





$\|\mu_{sx} - s\mu_x\| \leq \frac{2|S|}{N}$ $\lrcorner$

Generalizes to hyperbolic grps.

Thm G loc cpt grp $H \leq G$ closed amenable subgrp s.t. $X = G/H$ cpt

$\Gamma \leq G$ discrete (e.g. G conn ss Lie grp $G = KAN$, $H = AN$)

$\Longrightarrow \Gamma \curvearrowright X$ amenable

Pf 1st proof : $\Gamma \curvearrowright G/H \leftrightsquigarrow \Gamma\backslash G \curvearrowleft H$ amenable

2nd proof : Take $\sigma : X \to G$ Borel section s.t. $\sigma(X)$ rel cpt

Given $E \Subset \Gamma$, $\varepsilon > 0$

$\tilde{E} := \{\sigma(sx)^{-1} s\sigma(x) : s \in E, x \in X\} \subseteq H$ rel cpt

$\rightsquigarrow \exists\, \nu \in \mathrm{Prob}\,H$ $(\tilde{E}, \varepsilon)$-invariant

$\mu_x := \sigma(x)\nu \in \mathrm{Prob}\,G$ (Borel map)

$\|\mu_{sx} - s\mu_x\| = \|\nu - \sigma(sx)^{-1}s\sigma(x)\nu\| < \varepsilon$ for $s \in E$, $x \in X$.

$\exists\, \mathrm{Prob}\,G \to \mathrm{Prob}\,\Gamma$ Γ-equiv ☒

Def Γ boundary amenable $\overset{\text{def}}{\Longleftrightarrow} \exists X$ cpt $\Gamma \curvearrowright X$ amenable

$\Longleftrightarrow \Gamma \curvearrowright \beta\Gamma$ amenable

(☺ $\Gamma \ni s \mapsto sx_0 \in X$ extends to a conti Γ-equiv map $\beta\Gamma \to X$)

Γ b.a. $\overset{\text{Higson}}{\Longrightarrow}$ Γ satisfies the injectivity of BC conj

(& Novikov conj)

$\Downarrow$ $\Uparrow$

Γ has property A $\Longrightarrow$ $\Gamma \hookrightarrow \mathcal{H}$ coarse

II - 1

<u>Def</u> Γ boundary amenable $\overset{\text{def}}{\iff} \exists X$ cpt $\Gamma \curvearrowright X$ amenable
$\iff \Gamma \curvearrowright \beta\Gamma$ amenable

Thm For Γ TFAE

(1) Γ b.a.

(2) Γ has property A : For some/any $p \in [1, \infty)$ the following holds
$\forall E \Subset \Gamma\ \forall \varepsilon > 0\ \exists F \Subset \Gamma\ \exists \xi : \Gamma \to \ell_p\Gamma$
s.t.
- $\|\xi_x\| = 1$
- $\|\xi_x - \xi_y\| < \varepsilon$ for $(x,y) \in \mathrm{Tube}(E) := \{(x,y) : x^{-1}y \in E\}$
- $\mathrm{supp}\ \xi_x \subseteq xF$

(3) $\forall E \Subset \Gamma\ \forall \varepsilon > 0\ \exists F \Subset \Gamma\ \exists \theta : \Gamma \times \Gamma \to \mathbb{C}$
s.t.
- θ is pos definite
- $|\theta(x,y) - 1| < \varepsilon$ for $(x,y) \in \mathrm{Tube}(E)$
- $\mathrm{supp}\ \theta \subseteq \mathrm{Tube}(F)$

(4) Γ is exact i.e. $C^*_r\Gamma$ is an exact C^*-alg

(5) $C^*(\ell_\infty\Gamma, \lambda(\Gamma)) \subseteq \mathbb{B}(\ell_2\Gamma)$ is nuclear

<u>Pf</u>

(1) $\iff$ (2 : $p=1$) $\Gamma \curvearrowright \beta\Gamma$ amenable $\iff \exists F \Subset \Gamma\ \exists \mu : \Gamma \to \mathrm{Prob}\,F$
s.t. $\|\mu_{sx} - s\mu_x\| < \varepsilon$ for $s \in E$, $x \in \Gamma$

$$\xi_x := x\mu_{x^{-1}}$$

(2 : $p=2$) $\iff$ (3) $\quad \theta(x,y) = \langle \xi_y, \xi_x \rangle$

<u>Def</u> $\mathcal{G}$ a family of subgroups of Γ
$\Gamma \curvearrowright X$ amenable rel to $\mathcal{G}$
$\overset{\text{def}}{\iff} \forall E \Subset \Gamma\ \forall \varepsilon > 0\ \exists \mu : X \to \mathrm{Prob}\left(\coprod_{\Lambda \in \mathcal{G}} \Gamma/\Lambda\right)$ conti
s.t. $\sup_{\substack{s \in E \\ x \in X}} \|\mu_{sx} - s\mu_x\| < \varepsilon$

<u>Thm</u> $\Gamma \curvearrowright X$ amenable rel to $\mathcal{G}$ & every $\Lambda \in \mathcal{G}$ is exact $\Rightarrow \Gamma$ exact

<u>Cor</u> $\Lambda \trianglelefteq \Gamma$, Λ & Γ/Λ exact $\Rightarrow \Gamma$ exact

<u>Cor</u> $\Gamma \curvearrowright$ Tree $\quad \Gamma$ exact $\iff$ every vertex stabilizer is exact

$\exists \mu^{(n)} : T \to \mathrm{Prob}\,T$ s.t. $\|\mu^{(n)}_{sx} - s\mu^{(n)}_x\| \le \dfrac{2d(so, o)}{n}$.

Thm GHW $\quad$ All linear groups are exact.

II - 2

Thm Γ f.g. property A (or (Γ, d) property A metric sp) $\Rightarrow \Gamma \hookrightarrow \ell_p$ coarsely

Pf

For each n $\exists \zeta^{(n)} : \Gamma \to \ell_p$

s.t. $\begin{cases} \circ \ \|\zeta^{(n)}_x\| = 1 \\ \circ \ \|\zeta^{(n)}_x - \zeta^{(n)}_y\| < \frac{1}{2^n} \text{ if } d(x,y) \le n \\ \circ \ \exists R_n \quad R_n \nearrow \quad \text{s.t. } \operatorname{supp} \zeta^{(n)}_x \subseteq B(x, R_n) \end{cases}$

$\rightsquigarrow \zeta_x := \bigoplus_n (\zeta^{(n)}_x - \zeta^{(n)}_e) \in \ell_p(\mathbb{N}; \ell_p\Gamma)$

$\|\zeta_x - \zeta_y\|^p \ge \max\{n : d(x,y) \ge 2R_n\}$ ▨

The converse is almost true:

Thm (Schönberg) $\rho : X \times X \to \mathbb{R}$ symmetric zeros on diagonal. TFAE

(1) $\exists f : X \to \mathcal{H}$ s.t. $\rho(x,y) = \|f(x) - f(y)\|^2$

(2) ρ is conditionally negative definite

$\forall \alpha_x \in \mathbb{R}$ (finitely many) $\sum \alpha_x = 0 \Rightarrow \sum_{x,y} \alpha_x \alpha_y \rho(x,y) \le 0$

(3) $e^{-t\rho}$ is pos def for $\forall t \ge 0$

Pf

(1) $\Rightarrow$ (2) $\quad \sum_{x,y} \alpha_x \alpha_y \|f(x) - f(y)\|^2 = -2\|\sum \alpha_x f(x)\|^2 \le 0$

If $f : \Gamma \hookrightarrow \mathcal{H}$ coarse $\Rightarrow \Theta_t(x,y) := e^{-t\|f(x) - f(y)\|^2}$ pos def

For each $t > 0$ & $\varepsilon > 0$

$\{(x,y) : \Theta_t(x,y) \ge \varepsilon\} \subseteq \mathrm{Tube}(F)$

$\Theta_t \to 1$ as $t \searrow 0$.

Poincaré inequality: $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R})$ $A \ge 0$ $A\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$

(Exercise)

eigenvalues $\lambda_0 = 0 \le \lambda_1 \le \cdots$

For $\forall x_1, \cdots, x_n \in \mathcal{H}$ one has $-\sum_{i,j} a_{ij} \|x_i - x_j\|^2 \le \frac{\lambda_1}{n} \sum_{i,j} \|x_i - x_j\|^2$

Def X finite connected graph, d-regular

$\Delta_X \in \mathbb{B}(\ell_2 X)$ $\quad \Delta_X(x,y) := \begin{cases} d & \text{if } x = y \\ -\#\text{ of edges between } x \text{ \& } y \end{cases}$

$\Delta_X \ge 0 \quad \lambda_0(\Delta_X) = 0 < \lambda_1(\Delta_X) \le \cdots$

(X_n) a seq of f.c. d-reg graphs $|X_n| \to \infty$

is called a seq of expanders if $\inf_n \lambda_1(\Delta_{X_n}) > 0$.

II-3

Cor A seq of expanders does not coarsely embed into a Hilbert sp

☺ $f : X \to \mathcal{H}$ 1-Lip

Then $$\frac{1}{|X|} \sum_{(x,y) \in \mathrm{Edge}} \|f(x) - f(y)\|^2 \geq \frac{\lambda_1}{|X|^2} \sum_{(x,y) \in X^2} \|f(x) - f(y)\|^2$$

⇝ Average distance between $f(x)$ & $f(y)$ is

d = degree

at most d/λ_1 (a constant).

While the ball of radius R in X can contain at most d^R elements.

⇝ If $|X|$ is large, many pair (x,y) are such that $d(x,y) > R$ but $d(f(x), f(y)) < 100\, d/\lambda_1$. ☺

Thm? (Gromov) $\exists$ f.g. grp whose Cayley graph contains an uncollapsed image of expander seq.

$$\left((X_n)_n \text{ expanders},\ f_n : X_n \to \Gamma \qquad \limsup_n \max_{s \in \Gamma} \frac{|f_n^{-1}(s)|}{|X_n|} = 0 \right) \longleftrightarrow \frac{1}{|X_n|^2} \sum_{X_n^2} \|f(x) - f(y)\| \to +\infty$$

⟶ does not coarsely embed into a Hilb. sp

⟶ is not exact.

III-1

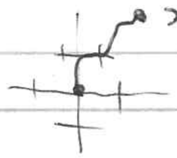
Hilbert compression constant $C_{\mathcal{H}}(X)$ quasi isom invariant
(Guentner–Kaminker) $\alpha \in [0,1]$

(*) $C_{\mathcal{H}}(X) \geq \alpha$ if $\exists f : X \hookrightarrow \mathcal{H}$ $\exists K, L \geq 0$
s.t. $K^{-1} d(x,y)^{\alpha} - L \leq d(f(x), f(y)) \leq K\, d(x,y) + L$

$C_{\mathcal{H}}(X) = \sup\{\alpha : C_{\mathcal{H}}(X) \geq \alpha\}$

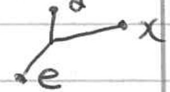
NB: (*) need not be true for $\alpha = C_{\mathcal{H}}(X)$.

Ex $\mathbb{F}_r$ (works for any tree)



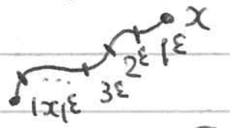
$f(x) = 1_{[e,x]} \in \ell_2 \,\mathrm{Edges}$
char function of the path $[e,x]$

$\| f(x) - f(y) \|^2 = d(x,y)$



$\rightsquigarrow C_{\mathcal{H}}(\mathbb{F}_r) \geq 1/2$

Better construction: Take $0 < \varepsilon < 1/2$.

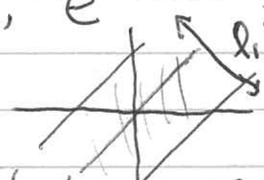
$f_{\varepsilon}(x) :=$  $\in \ell_2 \,\mathrm{Edges}$

Check
- $\sup_{d(x,y)=1} \| f_{\varepsilon}(x) - f_{\varepsilon}(y) \| < +\infty$
- $\| f_{\varepsilon}(x) - f_{\varepsilon}(y) \| \geq \frac{1}{K_{\varepsilon}} d(x,y)^{1+2\varepsilon}$

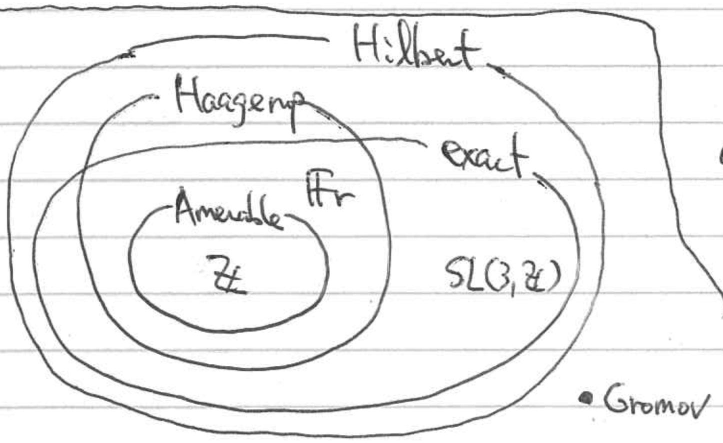
$\rightsquigarrow C_{\mathcal{H}}(\mathbb{F}_r) = 1$.

Thm (Guentner–Kaminker) $C_{\mathcal{H}}(X) > 1/2 \Rightarrow X$ has property A

Pf $f : X \hookrightarrow \mathcal{H}$ $\alpha > 1/2$ $\Rightarrow$ For each t, $e^{-t\|f(x)-f(y)\|^2}$



approximable by pos def kernel of finite width.



These properties are all ME invariant

co	Haagerup	$\Rightarrow$	Hilbert	$\Rightarrow$ Injectivity of BCC
$\Uparrow$	$\Uparrow$		$\Uparrow$	Strong Novikov
finite	Amenable	$\Rightarrow$	exact	

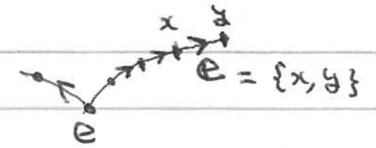
group equiv $\rightarrow$ non equivariant (metric space)

III-2

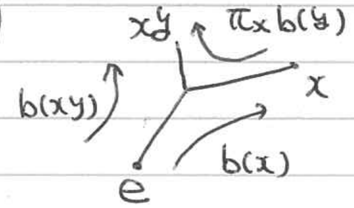
The first embedding $b : \mathbb{F}_r \hookrightarrow \ell_2 \mathrm{Edges}$ group structure

$b(x) = 1_{[e,x]}$

has the property that $\|b(x) - b(y)\|^2 = d(x,y) = |x^{-1}y|$ depends only on $x^{-1}y$.

Orientation on edges 

$\pi : \mathbb{F}_r \curvearrowright \ell_2 \mathrm{Edges}$, $\pi_s \delta_e = \pm \delta_{se}$ $\pm$ depends on orientation

Then $\boxed{b(xy) = b(x) + \pi_x b(y)}$ 

$b : \Gamma \to \mathcal{H}$ satisfying this identity is called a cocycle.

$$\|b(x) - b(y)\| = \|-\pi_x b(x^{-1}y)\| = \|b(x^{-1}y)\|$$
$$y = x(x^{-1}y)$$

Exercise: $b : \Gamma \to \mathcal{H}$ $\|b(x) - b(y)\| = f(x^{-1}y)$, $b(e) = 0$

$\mathcal{H} = \overline{\mathrm{span}}\{b(x) : x \in \Gamma\}$

$\Rightarrow$ $\pi_s : b(x) \mapsto b(sx) - b(s)$ extends to a unitary repn $\pi : \Gamma \curvearrowright \mathcal{H}$.

- $\theta_s : \mathcal{H} \ni \xi \mapsto \pi_s \xi + b(s) \in \mathcal{H}$ affine isometric action of Γ on $\mathcal{H}$
- b is a inner cocycle $\overset{\text{def}}{\iff}$ $\exists \xi \in \mathcal{H}$ s.t. $b(x) = \xi - \pi_x \xi$
  $\underset{\text{Fact}}{\iff}$ b is bdd

<u>Def</u> Γ countable grp

Γ has Haagerup property (a-T-menable)

if $\exists b : \Gamma \to \mathcal{H}$ a <u>proper cocycle</u>

($\forall R > 0$ $\{x \in \Gamma : \|b(x)\| \le R\}$ is finite.)

<u>Thm</u>(H) $\mathbb{F}_r$ has (H).

<u>Def</u> Γ countable has Kazhdan property (T)

if $\forall$ cocycle $b : \Gamma \to \mathcal{H}$ is bdd

$\longrightarrow$ (H) $\cap$ (T) = finite groups

III-3

$b : \Gamma \to \mathcal{H}$ cocycle $\leftrightarrow$ $e^{-t\|b(x)\|^2}$ pos def on Γ for $t \geq 0$

Thm For Γ TFAE

(1) Γ has (H)

(2) $\exists\, \varphi_n$ pos ~~type~~ def on Γ

s.t. · $\varphi_n \in c_0$

· $\varphi_n \to 1$

(3) $\exists\, (\pi, \mathcal{H})$ unitary rep with c_0 coefficients

$\exists\, \xi_n \in \mathcal{H}$ $\|\xi_n\| = 1$

s.t. $\|\pi_x \xi_n - \xi_n\| \to 0$ $\forall x$

(i) Γ has (T)

(ii) φ_n pos def on Γ

If $\varphi_n \to 1$ pointwise

$\Rightarrow \varphi_n \to 1$ uniformly on Γ

(iii) $\forall\, (\pi, \mathcal{H})$ unitary

If $\exists\, \xi_n$ $\|\xi_n\| = 1$

$\|\pi_x \xi_n - \xi_n\| \to 0$ $\forall x$

$\Rightarrow \exists\, \xi \neq 0$ s.t. $\pi_x \xi = \xi$ $\forall x$

$\because$ (1) $\Rightarrow$ (2) : $e^{-t\|b(x)\|^2}$

(2) $\Rightarrow$ (3) : GNS

(3) $\Rightarrow$ (1) : $b(x) := \bigoplus_n (\xi_n - \pi_x \xi_n) \in \bigoplus \mathcal{H}$

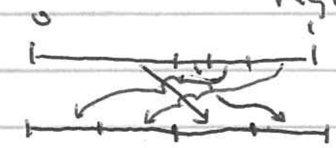
IV - 1

Example : Thompson's group V

$g \in V$ bijection on $[0,1]$

except for finitely many diadic rational pts, it is affine with positive diadic rational slopes

right conti

g  standard diadic partition

— obtained by bisecting an interval several times

 good  no good

$g \longleftrightarrow (P_1, P_2, \sigma)$ P_i standard diadic partition, σ permutation

Note: $\exists$ reduced diagram

$\left(P = \{0, r_1, \cdots, r_{k+1}, 1\} \quad (r_i, r_{i+1}) = (a/2^\ell, a+1/2^\ell) \right)$

$F \subseteq V$ $F = \{ g \in V : \text{continuous} \} \subseteq PL^+[0,1]$ piecewise linear homeo's

Thm (Farley) V has (H)

(V acts properly on a CAT(0) cube cplx $\longrightarrow$ (H))

Pf.

$V_{1/2} := \{ g \in V : g|_{[0,1/2)} = id_{[0,1/2)} \} \subseteq V$

$V/V_{1/2} \ni gV_{1/2} \longleftrightarrow g|_{[0,1/2)}$ $h \sim (Q_1, Q_2, \sigma)$, $Q_1 \cap (0, 1/2) = \phi$

$X := \{ hV_{1/2} : \cancel{h \text{ affine on } [0,1/2)} \} \subseteq V/V_{1/2}$

Claim ① $b(g) := 1_X - 1_{gX}$ finitely supported for $\forall g \in V$

② $b : V \to \ell_2 V/V_{1/2}$ proper (cocycle with $\pi : V \curvearrowright \ell_2 V/V_{1/2}$ ☺ inner cocycle into ℓ_∞)

$g \in V$ $g \sim (P_1, P_2, \sigma)$ given

What is $X \cap gX$?

For $hV_{1/2} \in X$ one has $ghV_{1/2} \in X \iff h((0,1/2)) \cap P_1 = \phi$

$X_{P_i} := \{ hV_{1/2} : h((0,1/2)) \cap P_i = \phi \}$

$g(P_1) = P_2 \longrightarrow gX_{P_1} = X_{P_2}$

IV-2

$$b(g) = 1_X - 1_{gX} = 1_{X-X_{P_2}} - 1_{g(X-X_{P_1})}$$

$X - X_P \longrightarrow P - \{0, 1/2, 1\}$ bijection

$h V_{1/2} \longmapsto$ midpoint of $h((0,1/2))$

$$\|b(g)\|^2 = |X - X_{P_1}| + |X - X_{P_2}|$$

$g \neq e$ $\quad = |P_1| - 3 + |P_2| - 3$ proper

$\square$

Arzhantseva - Guba - Sapir $\quad C_{\mathcal{H}}(F) = 1/2$

Other example : $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{F}_r$ has (H) (Cornulier - Stalder - Valette)

also acts properly on CAT(0) cube cplx

Weak amenability

Fejér : $M_{\varphi_n} : C(\mathbb{T}) \ni f \sim \sum_k \hat{f}(k) z^k \longmapsto \sum_k \varphi_n(k) \hat{f}(k) z^k \in C(\mathbb{T})$

trigonometric polyn

$\varphi_n(k) = (1 - \frac{|k|}{n}) \vee 0 \qquad M_{\varphi_n} \to \mathrm{id}$

$M_\varphi : C^*_r\Gamma \to C^*_r\Gamma$

$\lambda(f) \mapsto \lambda(\varphi f)$

Def Γ is weakly amenable if

$\exists \varphi_n : \Gamma \to \mathbb{C}$

s.t.
- φ_n finitely supported
- $\varphi_n \to 1$ pointwise
- $C := \limsup \| M_{\varphi_n} : C^*_r\Gamma \to C^*_r\Gamma \|_{cb} < +\infty$

$\Lambda_{cb}(\Gamma) :=$ optimal C

Th Weakly amenable $\Rightarrow$ exact

Thm (Haagerup & his friends)

$\Gamma \leq G$ lattice $\Rightarrow \Lambda_{cb}(\Gamma) = \Lambda_{cb}(G)$

G conn ss Lie grp $\Rightarrow$

$\Lambda_{cb}(G) = 1$ if $G = SO(1,n), SU(1,n)$ Haagerup property

$= 2n-1$ if $G = Sp(1,n)$ (T) but rank 1

$= +\infty$ if rank $G \geq 2$

N − 3

Thm Oz. • hyperbolic groups are weakly amenable

• G w.a. (loc cpt) $N \trianglelefteq G$ closed amenable norma

$\Longrightarrow$ $\exists$ N-inv mean on N which is $\mathrm{Ad}\,G$ - inv

$\longrightarrow$ $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{F}_r$, $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$, $SL(3,\mathbb{Z})$ are not w.a.

Th Guentner - Higson

$\Gamma \curvearrowright$ properly finite dim CAT(0) cube cplx

So it doesn't apply for $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{F}_n$

$\Longrightarrow \Gamma$ w.a.

F, V

V − 1

Fortified Kazhdan's property (after Lafforgue) de la Salle

Thm (Lafforgue) $G = SL(n \geq 3, \mathbb{Q}_p)$ or its cocompact lattice.

$\forall V$ Banach sp with type >1

$\exists F \subseteq G$ cpt, $\exists \varepsilon > 0$ $\quad \forall \pi : G \curvearrowright V$ $\quad \|\pi_g\|$ small exponential growth

If $\pi_g v \approx_\varepsilon v$ for $g \in F$, $\Longrightarrow$ $v \approx v_0 \in V^G$.

For simplicity $V = \mathcal{H}$ and $G = SL(3,\mathbb{R}) \overset{\pi}{\curvearrowright} \mathcal{H}$ unitary

$K = SO(3) \leq G$ cpt

P_K = proj onto $\mathcal{H}^K = \int_K \pi(k)\, dk$

$\pi(KgK) = \iint_{K^2} \pi(kgk')\, dk\, dk' = P_K \pi(g) P_K$

Thm $\forall \pi : G \curvearrowright \mathcal{H}$ one has $\|\pi(KgK) - P_G\| \to 0$ as $g \to \infty$

(Also, $\exists \Theta \in C_0(G)_+$ s.t. $|\varphi(g)| \leq \Theta(g) \|M_\varphi\|_{cb}$ for $\forall \varphi \in C_c(G)$ $\longrightarrow$ not weakly amenable nor AP)

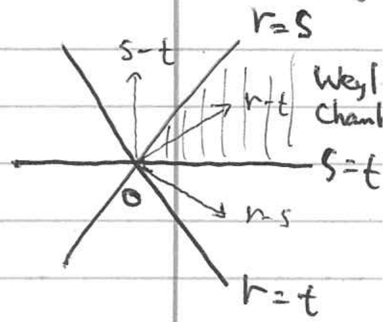
Average over K ~ Average over G

$KgK \in K \backslash G / K$

$\updownarrow \qquad\qquad \wr$ Cartan dec

$K\begin{pmatrix} e^r & & \\ & e^s & \\ & & e^t \end{pmatrix}K \qquad \{(r,s,t) \in \mathbb{R}^3 : \begin{matrix} r+s+t=0 \\ r \geq s \geq t \end{matrix}\}$

singular values



Want to study the geometry of $(K\backslash G/K, d)$

$d(KgK, KhK) := \|\pi(KgK) - \pi(KhK)\|$

We can understand everything on K

Fix $\alpha > 0$ and let $d_\alpha = \begin{pmatrix} e^{2\alpha} & & \\ & e^{-\alpha} & \\ & & e^{-\alpha} \end{pmatrix} \in G$.

d_α commutes with $\mathcal{U} := \begin{pmatrix} 1 & \\ & SO(2) \end{pmatrix} \subseteq K$

$$\begin{array}{ccc} K \ni k & \longmapsto & d_\alpha k d_\alpha \in G \\ \downarrow & & \downarrow \\ \eta_\alpha : \mathcal{U}\backslash K/\mathcal{U} & \longrightarrow & K\backslash G/K \end{array}$$

V-2

$z_\alpha : \mathcal{U}\backslash K/\mathcal{U} \to K\backslash G/K$ $\qquad k \mapsto d_\alpha k d_\alpha$

Identify $\mathcal{U}\backslash K/\mathcal{U}$ with $[-1,1]$

$k=(k_{ij}) \longleftrightarrow k_{11}$

$k_\delta := \begin{pmatrix}\delta & -\sqrt{1-\delta^2} & \\ \sqrt{1-\delta^2} & \delta & \\ & & 1\end{pmatrix}$ representative for $\delta \in \mathcal{U}\backslash K/\mathcal{U}$.

Since $z_\alpha(-\delta)=z_\alpha(\delta)$, we consider $\delta\geq 0$ only.

$$\pi(K z_\alpha(\delta) K) = \pi(K d_\alpha k_\delta d_\alpha K) = \pi(K d_\alpha)\,\pi(\mathcal{U} k_\delta \mathcal{U})\,\pi(d_\alpha K)$$

$$\to \|\pi(Kz_\alpha(\delta)K) - \pi(Kz_\alpha(0)K)\| \leq \|\pi(\mathcal{U} k_\delta \mathcal{U}) - \pi(\mathcal{U} k_0 \mathcal{U})\| \cdots (*)$$

<u>Lem</u> $(*) \leq 4\sqrt{\delta}$.

<u>Pf.</u> $\pi|_K$ decomposes into irreps

$K=SO(3)$ $\quad \forall$ irrep of $K \simeq (\pi_n, \mathcal{H}_n)$

where $\mathcal{H}_n \subseteq L^2(S^2)$ harmonic homogeneous polyn of degree n

$$\to (*) \leq \sup_n \|\pi_n(\mathcal{U} k_\delta\mathcal{U}) - \pi_n(\mathcal{U} k_0 \mathcal{U})\| \cdots (**)$$

For each n, $\pi_n(\mathcal{U})$ is the proj onto $\pi_n(\mathcal{U})$-inv vectors.

(Since $\mathcal{U}\backslash K/\mathcal{U}$ is commutative) $\pi_n(\mathcal{U})$ is rank one proj onto $\mathbb{C} P_n$

p_n Legendre polyn of degree n, $\quad \pi_n(\mathcal{U} k_\delta \mathcal{U}) = p_n(\delta)\, P_{\pi_n(\mathcal{U})}$

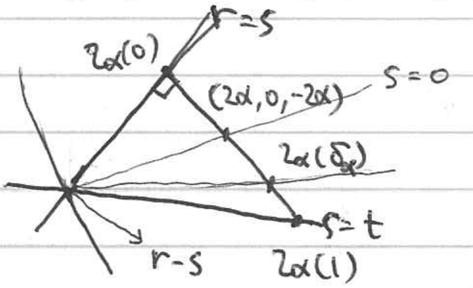
$$\to (**) \leq \sup_n |p_n(\delta) - p_n(0)| \leq 4\sqrt{\delta} \qquad \blacksquare$$

$$z_\alpha(\delta) = \left[\begin{pmatrix} e^{2\alpha} && \\ & e^{-\alpha} & \\ && e^{-\alpha}\end{pmatrix}\begin{pmatrix}\delta & * & \\ * & * & \\ && 1\end{pmatrix}\begin{pmatrix} e^{2\alpha}&&\\&e^{-\alpha}&\\&&e^{-\alpha}\end{pmatrix}\right]_{K\backslash G/K}$$

$$z_\alpha(0) = \begin{pmatrix} 0 & -e^{\alpha} & \\ e^{\alpha} & 0 & \\ && e^{-2\alpha}\end{pmatrix} = (\alpha,\alpha,-2\alpha)$$

$$z_\alpha(1) = \begin{pmatrix} e^{4\alpha} && \\ & e^{-2\alpha} & \\ && e^{-2\alpha}\end{pmatrix} = (4\alpha,-2\alpha,-2\alpha)$$

$$z_\alpha(\delta) = (r,s,t) \quad \text{s.t. } 4\alpha \geq r \geq s \geq \underline{\underline{-2\alpha = t}}.$$



Take $\varepsilon = 1/2$.

Take δ_α s.t. $z_\alpha(\delta_\alpha) = ((2+2\varepsilon)\alpha, -2\varepsilon\alpha, -2\alpha$

$$\|d_\alpha k_{\delta_\alpha} d_\alpha\| = e^{(2+2\varepsilon)\alpha}$$

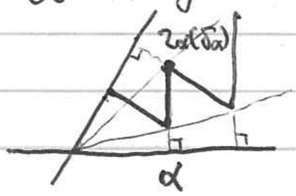
$\vee|$ look at the (1,1) entry

$e^{4\alpha}\delta_\alpha \quad \rightsquigarrow \delta_\alpha \leq e^{-(2-2\varepsilon)\alpha}$.

Consequence : $d(\gamma_\alpha(\delta_\alpha), \gamma_\alpha(0)) \le 4\sqrt{\delta_\alpha} \le 4e^{-(1-\varepsilon)\alpha}$

Cor $\pi(KgK)$ converges as $g \to \infty$.

Pf. Cauchy seq.



the length of each segment $\le 4e^{-(1-\varepsilon)\alpha}$

α grows linearly. $\square$

Hence $\exists P = \lim \pi(KgK)$

$$\pi(KgK)\,\pi(KhK) = \int_K \pi(Kgkh\,K)\,dk \longrightarrow P \quad \text{as } h \to \infty.$$

$\rightsquigarrow \pi(KgK)\,P = P \quad , \quad P^2 = P$

$$\xi \in P\mathcal{H} \Rightarrow \underset{\pi(K)\pi(g)\xi}{\underset{\parallel}{\pi(KgK)\xi}} = \xi \Rightarrow \pi(g)\xi = \xi$$

$\square$

VI - 1

Simple amenable group (after Juschenko - Monod)

first example finitely generated, infinite

OPEN : finitely presented

X cpt $T \in \mathrm{Homeo}(X)$

minimal $\overset{\text{def}}{\Longleftrightarrow}$ $T^{\mathbb{Z}} x$ is dense for $\forall x \in X$

$\rightsquigarrow$ $C(X) \rtimes \mathbb{Z}$ simple C^*-alg, classification program ongoing

From now on, X Cantor sp (the unique cpt metrizable zero-dim no isolated pts $\to \simeq \{0,1\}^{\mathbb{N}}$)

$C(X) \rtimes \mathbb{Z}$ classified by

Giordano - Putnam - Skau (1995)

Full group $[T] := \{ g \in \mathrm{Homeo}(X) :$ g preserves every T orbit $\}$

$$g x = T^{c_g(x)} x \qquad c_g : X \to \mathbb{Z}$$

Topological full grp $[\![T]\!] := \{ g \in [T] :$ $c_g : X \to \mathbb{Z}$ is conti $\}$

$\Downarrow g \rightsquigarrow X = \bigsqcup X_{g,n}$ partition into finitely many clopen subset

$$g|_{X_{g,n}} = T^n|_{X_{g,n}}$$

Ex 

$\to$ $[\![T]\!]$ a countable group

Thm (GPS) $[\![T]\!]$ (or $[\![T]\!]'$) is a complete invariant for
T minimal
the flip conjugacy of T.

Thm (Matui 2006) $\cdot$ $[\![T]\!]'$ simple
T minimal
$\cdot$ $[\![T]\!]'$ is f.g. $\Leftrightarrow$ T minimal subshift
($\to$ never finitely presented)

$\mathrm{ind} : [\![T]\!] \to \mathbb{Z}$ homomorphism, $\mathrm{ind}\, T = 1$

x fixed $\quad \mathrm{ind}\, g := | T^{\mathbb{N}} x \setminus g(T^{\mathbb{N}} x) | - | g(T^{\mathbb{N}} x) \setminus T^{\mathbb{N}} x |$

(indep of the choice)

$$\begin{array}{ccc} [\![T]\!] & \xrightarrow{\mathrm{ind}} & \mathbb{Z} \\ \cap\!\! \mid & & \wr\!\mid \\ \mathcal{U}(C(X)\rtimes\mathbb{Z}) & \xrightarrow{\mathrm{ind}} & K_1(C(X)\rtimes \mathbb{Z}) \end{array}$$

$\overline{\text{A}\mathbb{T} \text{ alg}}$

$[\![T]\!]_x := \{ g \in [\![T]\!] : g(T^{\mathbb{N}} x) = T^{\mathbb{N}} x \}$

locally finite (AF model for $[\![T]\!]$)

Matui : $\ker \mathrm{ind} = [\![T]\!]_x [\![T]\!]_y$

whenever $T^{\mathbb{Z}} x \neq T^{\mathbb{Z}} y$

$\rightsquigarrow$ $[\![T]\!]$ amenable ?

Thm (Juschenko - Monod) $[\![T]\!]$ is amenable.
T minimal

VI-2

$x_0 \in X$ fixed $\quad [\![T]\!] \hookrightarrow \mathrm{Bijection}(\mathbb{Z})$ $\quad$ injective if $\overline{T^{\mathbb{Z}} x_0} = X$

$\qquad\qquad\qquad\qquad g \qquad\qquad g T^n x_0 = T^{g(n)} x_0$

⇝ $g \in W(\mathbb{Z}) := \{ g : \text{bijection on } \mathbb{Z},\ |g|_w := \sup_n |g(n) - n| < +\infty \}$

Group of wobbling / Piecewise translations (top full grp of $\beta\mathbb{Z}$

($W(\mathbb{Z})$ contains free grps, but not an infinite (T) grp
☺ $g \mapsto 1_{\mathbb{N}} - 1_{g(\mathbb{N})} \in \ell_2\mathbb{Z}$ a cocycle)

<u>Def</u> $\Gamma \leq W(\mathbb{Z})$ is called

<u>recurrent</u> $\overset{\text{def}}{\iff}$ $\forall E \Subset \Gamma\ \forall n\ \exists$ infinitely many $K \in \mathbb{Z}$
s.t. $g \in E$ acts on $[K, K+n)$ in the same way as it acts on $[0, n)$.

$\leftrightarrow$ x_0 is recurrent

<u>minimal</u> / ubiquitous $\overset{\text{def}}{\iff}$ $\{K \in \mathbb{Z}$ as above$\}$ is syndetic.
(gaps are bounded)

$\leftrightarrow$ T is minimal on $\overline{T^{\mathbb{Z}} x_0}$

<u>Lem</u> $\Gamma \leq W(\mathbb{Z})$ minimal $\Rightarrow$ $\mathrm{Stab}(\mathbb{N}) := \{ g \in \Gamma : g(\mathbb{N}) = \mathbb{N} \}$ is loc finite

☺ $E \Subset \mathrm{Stab}(\mathbb{N})$ given. $n := \max_{g \in E} |g|_w + 1$

$\exists K_i$

Every $g \in E$ preserves each interval $[K_i, K_{i+1})$ → finite ▨
of bdd lengths

Thm (Grigorchuk–Medynets 2012) $\Gamma \leq W(\mathbb{Z})$ recurrent $\Rightarrow$ LEF

Γ is LEF (loc embeddable into finite grps) if $\forall E \Subset \Gamma\ \exists F$ a finite grp
$\exists \pi : E \to F$ injection
s.t. $\pi(st) = \pi(s)\pi(t)$
whenever $s, t, st \in E$

- LEF + f.p. $\Rightarrow$ residually finite

☺ $E \Subset \Gamma$ given
$n := 2 \max_{g \in E} |g|_w + 1$
E acts in the same way on $[0, n)$ & $[K, n)$

Overlap $[0, n)$ & $[K, K+n)$

▨

Cor (Kerr) $[\![T]\!]$ is QD (☺ Amenable + LEF $\Rightarrow$ QD)

VI-3

Lamplighter

S a set $\Gamma \curvearrowright S$

Def $\Gamma \curvearrowright S$ is (von Neumann) amenable

$\overset{\text{def}}{\Leftrightarrow}$ $\exists$ Γ-invariant mean on S

$\Leftrightarrow$ $\exists$ approx Γ-inv unit vector in $\ell_p S$ for some/any $1 \leq p < \infty$

$\vdots$

$\bigoplus_S \mathbb{Z}/2\mathbb{Z} \simeq \mathcal{P}_f(S)$ finite subsets $E \Delta F$

$\Gamma \curvearrowright \mathcal{P}_f(S)$, $\Gamma \ltimes \mathcal{P}_f(S) \curvearrowright \mathcal{P}_f(S)$ "affine action"

Problem : When $\Gamma \ltimes \mathcal{P}_f(S) \curvearrowright \mathcal{P}_f(S)$ amenable ?

Necessary : $\Gamma \curvearrowright S$ amenable

not sufficient (see Juschenko – de la Salle)

Thm $W(\mathbb{Z}^d) \ltimes \mathcal{P}_f(\mathbb{Z}^d) \curvearrowright \mathcal{P}_f(\mathbb{Z}^d)$ amenable for $d = 1$ (JM)

$d = 2$ (JS)

OPEN for $d \geqslant 3$.

Proof of amenability of $\Gamma \leq W(\mathbb{Z})$ minimal

$$\Gamma \leq W(\mathbb{Z}) \hookrightarrow W(\mathbb{Z}) \ltimes \mathcal{P}_f(\mathbb{Z}) \curvearrowright \mathcal{P}_f(\mathbb{Z}) \text{ amenable}$$

$$g \longmapsto \mathbb{N}\, g\, \mathbb{N} \quad (= g \cdot (g(\mathbb{N}) \Delta \mathbb{N}))$$

$$(W(\mathbb{Z}) \ltimes \mathcal{P}_f(\mathbb{Z}) \leq W(\mathbb{Z}) \ltimes \mathcal{P}(\mathbb{Z}))$$

Exercise : $\Gamma \curvearrowright S$ amenable + every stabilizer of $x \in S$ amenable

$\Longrightarrow \Gamma$ amenable

$E \in \mathcal{P}_f(\mathbb{Z})$ given.

$$\mathrm{Stab}(E) = \{ g : (\mathbb{N} g \mathbb{N}) \cdot E = E \}$$
$$= \{ g : g(E \Delta \mathbb{N}) = E \Delta \mathbb{N} \} \text{ loc finite} \quad \blacksquare$$

VI-4

When $\Gamma \ltimes \mathcal{P}_f(S) \curvearrowright \mathcal{P}_f(S)$ amenable

Assume for simplicity $S = \Gamma x_0$

Suffices to find $\xi \in \ell_2 \mathcal{P}_f(S)$

s.t. $\cdot\ x_0 \xi = \xi$

$\cdot\ g\xi \approx \xi$ for $g \in E \Subset \Gamma$

Take $w : S \to [0,1]$ finitely supported, $w(x_0) = 1$

and define $\xi \in \ell_2 \mathcal{P}_f(S)$ by

$$\xi(B) = \prod_{x \in B} w(x) \quad , \quad \xi(\phi) = 1$$

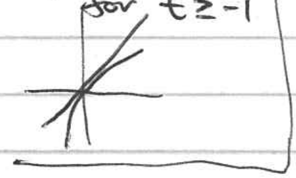
$$\rightsquigarrow \|\xi\|^2 = \sum_B \prod_{x \in B} w(x)^2 = \prod_{x \in S} (1 + w(x)^2)$$

Want to show $\langle g\xi, \xi\rangle / \|\xi\|^2 \approx 1$ for $g \in E$.

$$0 \le \log \frac{\|\xi\|^2}{\langle g\xi, \xi\rangle} = \log \frac{\prod (1 + w(x)^2)}{\prod_{x \in S} (1 + w(x) w(gx))}$$

$$\frac{1+s}{1+t} \le 1 + \frac{s-t}{1+t} \le 1 + s - t$$

$\log(1+t) \le t$ for $t \ge -1$



$$\le \sum_{x \in S} \log\big(1 + w(x)^2 - w(x) w(gx)\big)$$

$$\le \sum_{x \in S} \big(w(x)^2 - w(x) w(gx)\big)$$

$$= \frac{1}{2} \|w - gw\|_2^2 .$$

Conclusion: If $\forall E \Subset \Gamma\ \forall \varepsilon > 0\ \exists w : S \to [0,1]$

(JS)

s.t. $\begin{cases} \cdot\ \|w\|_2 < +\infty \\ \cdot\ w(x_0) = 1 \\ \cdot\ \|w - gw\|_2 < \varepsilon \text{ for } g \in E \end{cases}$

not $\varepsilon \|w\|_2$!

$\Rightarrow \Gamma \ltimes \mathcal{P}_f(S) \curvearrowright \mathcal{P}_f(S)$ amenable

Specialized to a graph of bdd degree

$W(S) \ltimes \mathcal{P}_f(S) \curvearrowright \mathcal{P}_f(S)$ amenable if (not necessary only if)

$\forall \varepsilon > 0\ \exists w \in \ell_2 S$ s.t. $\begin{cases} \cdot\ w(x_0) = 1 \\ \cdot\ \sum_{(x,y) \in \text{Edge}} |w(x) - w(y)|^2 < \varepsilon \end{cases}$

True for $\mathbb{Z}^d$ $d = 1, 2$ but not for $d \ge 3$.