Kazhdan's property (T) for $\mathrm{Aut}(\mathbf{F}_n)$ and $\mathrm{EL}_n(\mathcal{R})$

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Kazhdan's property (T)

Theorem (Kazhdan 1967)

Any simple Lie group G of real rank ≥ 2 (e.g., $G = \mathrm{SL}_n(\mathbb{R})$, $n \geq 3$) and its lattice Γ (e.g., $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$) have **property (T)**.

$\rightsquigarrow$ Γ is finitely generated and has finite abelianization.

Throughout this talk, $\Gamma = \langle S \rangle$ is a finitely generated group.

Definition (for discrete groups)

Γ has (T) $\overset{\text{def}}{\iff} \exists \kappa = \kappa(\Gamma, S) > 0$ s.t. $\forall (\pi, \mathcal{H})$ unitary rep'n and $\forall v \in \mathcal{H}$

$$d(v, \mathcal{H}^\Gamma) \leq \kappa^{-1} \max_{s \in S} \|v - \pi(s)v\|,$$

i.e., an almost invariant vector v is close to an invariant vector $\mathrm{Proj}_{\mathcal{H}^\Gamma}(v)$.

- Property (T) inherits to finite-index subgroups and quotient groups.
- $\mathbb{Z}$ (or any infinite amenable group) does not have property (T).
  $\because$ $\frac{1}{\sqrt{2k+1}} 1_{[-k,k]} \in \ell^2(\mathbb{Z})$ is asymp. $\mathbb{Z}$-invariant, but $\ell^2(\mathbb{Z})^{\mathbb{Z}} = \{0\}$.

$\rightsquigarrow$ Any f.i. subgroup of a property (T) group has finite abelianization.

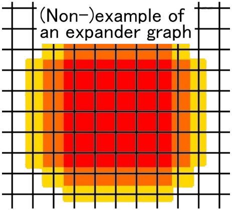
An application of property (T): Expander graphs

Definition

A finite connected graph X is an ε-**expander** if for $\forall A \subset X$ (vertices)

$$|\partial A| \geq \varepsilon |A|(1 - \frac{|A|}{|X|}).$$

- For $\mathcal{N}_k(A) := \{x \in X : \mathrm{d}(x, A) \leq k\}$, $|\mathcal{N}_k(A)| \geq (1 + \frac{\varepsilon}{2})^k |A|$ until it reaches $\frac{1}{2}|X|$. After that $|\mathcal{N}_k(A)^c|$ decreases by a factor $1 + \frac{\varepsilon}{2}$.
- Random walk on X has mixing time $O(\log |X|)$.
- Want large ε-expanders with degree and ε fixed.



Explicit construction of expanders (Margulis 1973)

$\Gamma = \langle S \rangle$ and $N \triangleleft \Gamma$ a finite index normal subgroup

$\rightsquigarrow X = \mathrm{Cayley}(\Gamma/N, S)$, where Edges $= \{\{x, xs\} : x \in \Gamma/N,\ s \in S\}$,

is a $\kappa(\Gamma, S)^2$-expander.

E.g., $\Gamma = \mathrm{SL}(3, \mathbb{Z})$, $S = \{I + E_{ij} : i \neq j\}$, and $X_q = \mathrm{SL}(3, \mathbb{Z}/q\mathbb{Z})$, $q \in \mathbb{N}$.

? What if $S_p = \{I + pE_{ij} : i \neq j\}$ and $X_{p,q} = \mathrm{SL}(3, \mathbb{Z}/q\mathbb{Z})$, $q \perp p$?

Some examples of property (T) groups

- $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, (Kazhdan 1967), but not $\mathrm{SL}_2(\mathbb{Z})$.
  - $\mathrm{EL}_n(\mathcal{R}) = \langle e_{ij}(r) : i \neq j, r \in \mathcal{R} \rangle \subset \mathrm{GL}_n(\mathcal{R})$, $n \geq 3$, where $\mathcal{R}$ finitely generated ring and $e_{ij}(r) := I_n + rE_{ij}$ (Shalom & Vaserstein, Ershov–Jaikin-Zapirain 2006–08).
  - $\mathrm{Aut}(\mathbf{F}_n)$, $n \geq 4$. (Kaluba–Nowak–O., K–Kielak–N., Nitsche 17–20).

    $\mathbf{F}_n \twoheadrightarrow \mathbb{Z}^n$ abelianization $\rightsquigarrow$ $\mathrm{Aut}(\mathbf{F}_n) \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$.

    $\rightsquigarrow$ $\mathrm{Aut}(\mathbf{F}_2)$ does not have (T). Neither $\mathrm{Aut}(\mathbf{F}_3)$ (McCool 1989).

⚠ The proof is heavily computer-assisted.

Product Replacement Algorithm (Celler et al. 95, Lubotzky–Pak 01)

$\mathrm{Aut}^+(\mathbf{F}_n) = \langle R_{i,j}, L_{i,j} \rangle \leq_{\text{index } 2} \mathrm{Aut}(\mathbf{F}_n)$, where $\mathbf{F}_n = \langle g_1, \ldots, g_n \rangle$ and

$$R_{i,j} \colon (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_i g_j, g_{i+1}, \ldots, g_n),$$
$$L_{i,j} \colon (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_j g_i, g_{i+1} \ldots, g_n).$$

PRA is a practical algorithm to obtain "random" elements in a given finite group Λ of rank $< n$ via the PRA random walk

$$\mathrm{Aut}^+(\mathbf{F}_n) \curvearrowright \{(h_1, \ldots, h_n) \in \Lambda^n : \Lambda = \langle h_1, \ldots, h_n \rangle\}.$$

Noncommutative real algebraic geometry of property (T)

Hilbert's 17th Pb: $f \in \mathbb{R}(x_1, \ldots, x_d)$, $f \geq 0$ on $\mathbb{R}^d$
(E. Artin 1927) $\implies f = \sum_i g_i^2$ for some $g_1, \ldots, g_k \in \mathbb{R}(x_1, \ldots, x_d)$.

$\mathbb{R}[\Gamma]$ real group algebra with the involution $(\sum_t \alpha_t t)^* = \sum_t \alpha_t t^{-1}$.

$$\Sigma^2\mathbb{R}[\Gamma] := \{\sum_i f_i^* f_i\} = \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_\Gamma^+\}$$ **positive cone**

Here $\mathbb{M}_\Gamma^+$ finitely supported positive semidefinite matrices.

- $\mathbb{B}(\mathcal{H})^+ := \{A = A^* : \langle Av, v\rangle \geq 0\ \forall v \in \mathcal{H}\} = \Sigma^2\mathbb{B}(\mathcal{H})$ psd operators.
- $\forall(\pi, \mathcal{H})$ unitary rep'n, $\pi(\sum_i f_i^* f_i) = \sum_i \pi(f_i)^* \pi(f_i) \geq 0$ in $\mathbb{B}(\mathcal{H})$.
- $C^*[\Gamma]$ the universal enveloping C^*-algebra of $\mathbb{R}[\Gamma]$.

Laplacian (non normalized)**:** For $\Gamma = \langle S\rangle$,

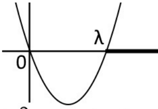
$$\Delta := \sum_{s\in S}(1-s)^*(1-s) = 2|S| - \sum_{s\in S}(s + s^{-1}) \in \Sigma^2\mathbb{R}[\Gamma].$$

Then, $\langle \pi(\Delta)v, v\rangle = \sum_{s\in S} \|v - \pi(s)v\|^2$ and

Γ has (T) $\iff \exists\lambda > 0 \quad \forall(\pi, \mathcal{H}) \quad \mathrm{Sp}(\pi(\Delta)) \subset \{0\} \cup [\lambda, \infty)$

$\iff \exists\lambda > 0$ such that $\Delta^2 - \lambda\Delta \geq 0$ in $C^*[\Gamma]$

$\rightsquigarrow \kappa(\Gamma, S) \geq \sqrt{\lambda/|S|}$



$t^2 - \lambda t \geq 0$

Algebraic characterization of property (T)

Let $\Gamma = \langle S \rangle$.

$\mathbb{R}[\Gamma]$ real group algebra with the involution $(\sum_t \alpha_t t)^* = \sum_t \alpha_t t^{-1}$.

$$\Sigma^2\mathbb{R}[\Gamma] := \{\sum_i f_i^* f_i\} = \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_\Gamma^+\}$$

Here $\mathbb{M}_\Gamma^+$ finitely supported positive semidefinite matrices.

$$\Delta := \sum_{s\in S}(1-s)^*(1-s) \in \Sigma^2\mathbb{R}[\Gamma].$$

$\mathrm{C}^*[\Gamma]$ the universal enveloping C^*-algebra of $\mathbb{R}[\Gamma]$.

Then,

$$\Gamma \text{ has (T)} \iff \exists\lambda > 0 \text{ such that } \Delta^2 - \lambda\Delta \geq 0 \text{ in } \mathrm{C}^*[\Gamma]$$

Theorem (O 2013)

$$\Gamma \text{ has (T)} \iff \exists\lambda > 0 \text{ such that } \Delta^2 - \lambda\Delta \succeq 0 \text{ in } \mathbb{R}[\Gamma]$$

Stability (Netzer–Thom): It suffices if $\exists\lambda > 0\ \exists\Theta \in \Sigma^2\mathbb{R}[\Gamma]$ such that

$$\|\Delta^2 - \lambda\Delta - \Theta\|_1 \ll \lambda,$$

i.e., an almost solution to the inequality is close to an honest solution.

Semidefinite Programming (SDP)

Γ has (T) $\iff \exists\lambda > 0$ such that $\Delta^2 - \lambda\Delta \in \Sigma^2\mathbb{R}[\Gamma]$

$$\iff \exists E \Subset \Gamma\ \exists\lambda > 0 \text{ s.t. } \Delta^2 - \lambda\Delta \in \{\sum_{x,y} P_{x,y}x^{-1}y : P \in \mathbb{M}_E^+\}$$

By fixing a finite subset $E \Subset \Gamma$, we arrive at the SDP:

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maximize      λ
subject to    Δ² − λΔ = Σ_{x,y∈E} P_{x,y} x⁻¹ y,   P ∈ M_E^+
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$$\begin{aligned} &\texttt{maximize} && \lambda \\ &\texttt{subject to} && \Delta^2 - \lambda\Delta = \textstyle\sum_{x,y\in E} P_{x,y}x^{-1}y, \quad P \in \mathbb{M}_E^+ \end{aligned}$$

- Due to computer capacity limitation, we almost always take

$$E := \mathrm{Ball}(2) = \{e\} \cup S \cup S^2 = \operatorname{supp}\Delta \cup \operatorname{supp}\Delta^2.$$

$\rightsquigarrow$ Size of SDP: dimension $|E|^2$ and constraints $|E^{-1}E| = |\mathrm{Ball}(4)|$.

Certification Procedure:

Suppose (λ_0, P_0) is a hypothetical solution obtained by a computer.
Find $P_0 \approx Q^{\mathrm{T}}Q$ (with $Q\mathbf{1} = 0$) and calculate **with guaranteed accuracy**

$$\|\Delta^2 - \lambda_0\Delta - \textstyle\sum_{x,y}(Q^{\mathrm{T}}Q)_{x,y}(1-x)^*(1-y)\|_1 \ll \lambda_0.$$

- Solving SDP is computationally hard, but certifying (T) is relatively easy.

Results

Γ has (T) $\iff \exists E \Subset \Gamma\ \exists \lambda > 0$ s.t. $\Delta^2 - \lambda\Delta \in \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_E^+\}$

Results of SDP for $E = \mathrm{Ball}(2)$.

- $\mathrm{SL}_n(\mathbb{Z})$ with $S = \{e_{ij} : i \neq j\}$: $\lambda_3 > 0.27$, $\lambda_4 > 1.3$, $\lambda_5 > 2.6$. (Netzer–Thom 2014, Fujiwara–Kabaya 2017, Kaluba–Nowak 2017)
- No response for $\mathrm{SL}_6(\mathbb{Z})$.

Kaluba, Nowak, and I tried $\mathrm{Aut}^+(\mathbf{F}_d)$ with a help of Polish supercomputer.

- $\mathrm{Aut}^+(\mathbf{F}_4)$: ☹ ☹ ☹ No response.
- $\mathrm{Aut}^+(\mathbf{F}_5)$: !☺⅄☺⅄☺! **YES!!!** with $\lambda > 1.2$.

Theorem

$\mathrm{Aut}^+(\mathbf{F}_n)$ has property (T) for

- $n = 5$ (Kaluba–Nowak–O. 2017)
- $n \geq 6$ (Kaluba–Kielak–Nowak 2018, by "stability" explained below)
- $n = 4$ (Nitsche 2020, by a new SDP method)

Property (T) for an infinite series (KKN 2018)

$\Gamma_n := \mathrm{Aut}^+(\mathbf{F}_n)$, $\quad S_n := \{R_{i,j}, L_{i,j} : i \neq j\}$, $\quad \mathrm{E}_n := \{\{i,j\} : i \neq j\}$

Want to show $\Delta_n = \sum_{s \in S_n} (1-s)^*(1-s)$ satisfies $\Delta_n^2 - \lambda_n \Delta_n \succeq 0.$

$$\Delta_n = \sum_{\mathrm{e} \in \mathrm{E}_n} \Delta_{\mathrm{e}},$$

$$\Delta_n^2 = \sum_{\mathrm{e}} \Delta_{\mathrm{e}}^2 + \sum_{\mathrm{e} \sim \mathrm{f}} \Delta_{\mathrm{e}} \Delta_{\mathrm{f}} + \sum_{\mathrm{e} \perp \mathrm{f}} \Delta_{\mathrm{e}} \Delta_{\mathrm{f}}$$

$$=: \ \mathbf{Sq}_n \ + \ \mathbf{Adj}_n \ + \ \mathbf{Op}_n \, .$$

- $\mathbf{Sq}_n$ and $\mathbf{Op}_n$ are positive, but $\mathbf{Adj}_n$ may not.

For $n > m$, let's see what we can tell about Δ_n knowing about Δ_m:

$$\sum_{\sigma \in \mathfrak{S}(n)} \sigma(\Delta_m) = m(m-1) \cdot (n-2)! \cdot \Delta_n$$

$$\sum_{\sigma \in \mathfrak{S}(n)} \sigma(\mathbf{Adj}_m) = m(m-1)(m-2) \cdot (n-3)! \cdot \mathbf{Adj}_n$$

$$\sum_{\sigma \in \mathfrak{S}(n)} \sigma(\mathbf{Op}_m) = m(m-1)(m-2)(m-3) \cdot (n-4)! \cdot \mathbf{Op}_n$$

⚠ $\mathbf{Op}_n$ multiplies faster and overtakes $\mathbf{Adj}_n$.

Trial and error on the computer has confirmed

$$(\heartsuit) \quad \mathbf{Adj}_5 + \alpha\, \mathbf{Op}_5 - \varepsilon \Delta_5 \succeq 0$$

for $\alpha = 2$ and $\varepsilon = 0.13$. It follows that for $n \geq 2\alpha + 3$

$$0 \preceq 60(n-3)!\big(\mathbf{Adj}_n + \tfrac{2\alpha}{n-3}\, \mathbf{Op}_n - \tfrac{n-2}{3} \varepsilon \Delta_n\big) \preceq 60(n-3)!\big(\Delta_n^2 - \tfrac{n-2}{3} \varepsilon \Delta_n\big).$$

Computer taught us the ad hoc inequality $(\heartsuit)\ \mathbf{Adj}_5 + \alpha\, \mathbf{Op}_5 - \varepsilon \Delta_5 \succeq 0$ is not only true but even **easy to prove** if $\alpha > 0$ is large enough.

We apply the KKN method to $\mathrm{EL}_n(\mathcal{R})$ with the "rng" (="ring"-"i")

$$\mathcal{R} = \mathbb{Z}\langle X_1, \ldots, X_d \rangle \text{ polynomials with zero constant term.}$$

The group $\mathrm{EL}_n(\mathcal{R})$ appears as the parent group for, e.g.,

- $\{\mathrm{EL}_n(p\mathbb{Z}) : p \in \mathbb{N}\}$ which are **not** uniformly (T).
- $\{(\mathrm{SL}_n(\mathbb{Z}), S_{p,q} = \{e_{ij}(p), e_{ij}(q) : i \neq j\}) : p \perp q\}$, uniformly (T) **??**

⚠ $\mathrm{EL}_n(\mathcal{R}) \twoheadrightarrow \mathrm{EL}_n(\mathcal{R}/\mathcal{R}^2) \cong (\mathcal{R}/\mathcal{R}^2)^{\oplus n(n-1)}$ abelian quotient

Motto: $\exists$ (T) type rigidity if the nilp. quotients $\mathrm{EL}_n(\mathcal{R}/\mathcal{R}^k)$ are kept away.

Theorem (O. 2022)

For any f.g. **comm. rng** $\mathcal{R}$ generated by $R_0 \Subset \mathcal{R}$ and for n **large enough**,

$$\Delta := \sum_{r \in R_0} \sum_{i \neq j} (1 - e_{ij}(r))^*(1 - e_{ij}(r)) \quad \text{for } \mathrm{EL}_n(\mathcal{R}) \text{ and}$$
$$\Delta^{(2)} := \sum_{r,s \in R_0} \sum_{i \neq j} (1 - e_{ij}(rs))^*(1 - e_{ij}(rs)) \quad \text{for } \mathrm{EL}_n(\mathcal{R}^2)$$

in $\mathbb{R}[\mathrm{EL}_n(\mathcal{R})]$ satisfy $\Delta^2 \geq \lambda \Delta^{(2)}$ in $C^*[\mathrm{EL}_n(\mathcal{R})]$ for some $\lambda > 0$.

Generalizing property (T) for $\mathrm{EL}_n(\mathcal{R})$ for a rng $\mathcal{R}$, cont'd

Theorem (O. 2022)

For any f.g. **comm. rng** $\mathcal{R}$ generated by $R_0 \Subset \mathcal{R}$ and for n **large enough**,

$$\Delta := \sum_{r\in R_0} \sum_{i\neq j} (1-e_{ij}(r))^*(1-e_{ij}(r)) \quad \text{for } \mathrm{EL}_n(\mathcal{R}) \text{ and}$$

$$\Delta^{(2)} := \sum_{r,s\in R_0} \sum_{i\neq j} (1-e_{ij}(rs))^*(1-e_{ij}(rs)) \quad \text{for } \mathrm{EL}_n(\mathcal{R}^2)$$

in $\mathbb{R}[\mathrm{EL}_n(\mathcal{R})]$ satisfy $\Delta^2 \geq \lambda\Delta^{(2)}$ in $\mathrm{C}^*[\mathrm{EL}_n(\mathcal{R})]$ for some $\lambda > 0$.

Corollary

$\exists n\ \exists \varepsilon > 0$ s.t. $\mathrm{Cayley}(\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}), \{e_{ij}(p) : i \neq j\})$, $p \perp q$, are ε-expanders.

$\Delta^2 \succeq \lambda\Delta^{(2)}$ does not hold in $\mathbb{R}[\mathrm{EL}_n(\mathcal{R})]$ and the proof is silicon-free. Instead it relies on Boca & Zaharescu's work (2005) on the almost Mathieu operators in the rotation C^*-algebras $\mathcal{A}_\theta$ (aka noncomm. tori) that, for the Heisenberg group $H = \langle x, y : z = [x, y] \text{ is central}\rangle$,

$$\Delta_H^2 = \big((1-x)^*(1-x) + (1-y)^*(1-y)\big)^2 \geq \tfrac{1}{4}(1-z)^*(1-z) = \tfrac{1}{4}\Delta_{Z(H)}$$

holds in $\mathrm{C}^*[H]$ (⚠ but never in $\mathbb{R}[H]$; a failure of Hilbert's 17th).