II_1 factors with at most one Cartan subalgebra

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Joint work with Sorin POPA

RIMS, September 2007

Research supported by NSF

What do we classify?

Γ	countable discrete group
(X, μ)	standard probability measure space
$\Gamma \curvearrowright (X, \mu)$	measure preserving action

We only consider either

- $(X, \mu) \cong ([0, 1], \text{Lebesgue})$ and
  $\Gamma \curvearrowright X$ is *essentially-free* i.e. $\mu(\{x : sx = x\}) = 0\ \forall s \in \Gamma \setminus \{1\}$;

or

- $X = \{\text{pt}\}$.

In passing, recall that $\Gamma \curvearrowright X$ is *ergodic* if

$$A \subset X \text{ and } \Gamma A = A \Rightarrow \mu(A) = 0, 1.$$

How do we classify?

GA		OE		vN
$\Gamma \curvearrowright (X, \mu)$	→	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	→	$\mathrm{vN}(X \rtimes \Gamma)$

GP		vN
Γ	→	$\mathrm{vN}(\Gamma)$

To what extent do vN/OE remember OE/GA/GP?

Group measure space constructions

$\Gamma \curvearrowright (X,\mu)$ p.m.p. $\longleftrightarrow$ $\sigma\colon \Gamma \curvearrowright L^\infty(X,\mu)$, $\sigma_s(f)(x) = f(s^{-1}x)$, $\int \sigma_s(f)\,d\mu = \int f\,d\mu$

We encode the information of $\Gamma \curvearrowright X$ into a single $*$-algebra

$$\mathcal{A}(X \rtimes \Gamma) := \{\sum_{s\in\Gamma}^{\text{finite}} f_s\, s : f_s \in L^\infty(X,\mu)\},$$

which is generated by the group algebra $\mathbb{C}\Gamma$ and the function algebra $L^\infty(X)$ with the relationship

$$s\, f\, s^{-1} = \sigma_s(f).$$

Group measure space constructions

Hence, $(\sum_s f_s\, s)(\sum_t g_t\, t) = \sum_{s,t} f_s \sigma_s(g_t)\, st.$

$\mathcal{A}(X \rtimes \Gamma)$ is a pre-Hilbertian algebra:

$$\|\sum_s f_s\, s\|_2 = \sqrt{\sum_s \|f_s\|_2^2}.$$

Denote $\mathcal{H} = \overline{\mathcal{A}(X \rtimes \Gamma)}^{\|\ \|_2} \cong L^2(X,\mu) \otimes_2 \ell_2(\Gamma)$ and define

$$\mathrm{vN}(X \rtimes \Gamma) := \text{WOT-cl}\big(\mathcal{A}(X \rtimes \Gamma)\big) \subset \mathbb{B}(\mathcal{H}).$$

$\mathrm{vN}(X \rtimes \Gamma)$ is often written as $L^\infty(X) \rtimes \Gamma$ and has a finite trace τ, given by $\tau(x) = \langle x\,1, 1\rangle$. (It follows $\tau(xy) = \tau(yx)$.)

Group measure space constructions

$$\begin{aligned} L^\infty(X) &\subset \mathrm{vN}(X \rtimes \Gamma) \\ &= \mathrm{vN}\big((u \otimes \lambda)(\Gamma), L^\infty(X) \otimes \mathbb{C}1\big) \\ &\subset \mathbb{B}(L^2(X,\mu) \otimes \ell_2(\Gamma)) \end{aligned}$$

and $L^\infty(X)$ is a *Cartan subalgebra* of $\mathrm{vN}(X \rtimes \Gamma)$.

Definition

A von Neumann subalgebra $A \subset M$ is called a *Cartan subalgebra* if it is a maximal abelian subalgebra such that the normalizer

$$\mathcal{N}(A) = \{u \in M : \text{unitary} \quad uAu^* = A\}$$

generates M.

Orbit Equivalence Relation

Theorem (Singer, Dye, Krieger, Feldman-Moore 1977)

Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be ess-free p.m.p. actions, and

$$\theta\colon (X,\mu) \to (Y,\nu)$$

be an isomorphism. Then, the isomorphism

$$\theta^*\colon L^\infty(Y,\nu) \ni f \mapsto f\circ\theta \in L^\infty(X,\mu)$$

extends to a $*$-isomorphism

$$\pi\colon \mathrm{vN}(Y \rtimes \Lambda) \to \mathrm{vN}(X \rtimes \Gamma)$$

if and only if θ preserves the orbit equivalence relation:

$$\theta(\Gamma x) = \Lambda\theta(x) \quad \text{for } \mu\text{-a.e. } x.$$

Lack of rigidity

GA	→	OE	→	vN
$\Gamma \curvearrowright (X,\mu)$		$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$		$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Connes 1974, Ornstein-Weiss, C-Feldman-W 1981)

Amenable **vN** *and* **OE** *are unique modulo center.*

Theorem (Connes-Jones 1982)

OE ⟶ **vN** *is not one-to-one,*
i.e. $\exists$ *a* II_1*-factor with non-conjugate Cartan subalgebras.*

Theorem (Dykema 1993)

$\mathrm{vN}(\Gamma_1 * \Gamma_2) \cong \mathrm{vN}(\mathbb{F}_2)$ *for any infinite amenable groups* Γ_1 *and* Γ_2.

Lack of rigidity

GA: $\Gamma \curvearrowright (X, \mu)$ → **OE**: $L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$ → **vN**: $\mathrm{vN}(X \rtimes \Gamma)$

GP: Γ → **vN**: $\mathrm{vN}(\Gamma)$

Theorem (Connes 1975)

$\exists$ *a* II_1*-factor which is not* $*$*-isomorphic to its complex conjugate.*

Theorem (Voiculescu 1994)

$\mathrm{vN}(\mathbb{F}_r)$ *does not have a Cartan subalgebra.*

Rigidity

GA		OE		vN
$\Gamma \curvearrowright (X,\mu)$	→	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	→	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Furman 1999, (Monod-Shalom,) Popa, Kida, Ioana)

Some **OE** *fully remembers* **GA**.

Theorem (Oz-Popa)

Some **vN** *fully remembers* **OE**, *i.e.*
$\exists$ *a (non-amenable)* II_1*-factor with a unique Cartan subalgebra,*
unique up to unitary conjugacy.

Open problems

GA: $\Gamma \curvearrowright (X, \mu)$ → **OE**: $L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$ → **vN**: $\mathrm{vN}(X \rtimes \Gamma)$

GP: Γ → **vN**: $\mathrm{vN}(\Gamma)$

Problem

- Is there **vN** which fully remembers **GA**?
- Is there **vN** which fully remembers **GP**?
- $\mathrm{vN}(\mathbb{F}_2) \not\cong \mathrm{vN}(\mathbb{F}_3)$?

OE to Cocycle (after Zimmer)

Suppose $(\Gamma \curvearrowright X) \cong_{\mathrm{OE}} (\Lambda \curvearrowright Y)$, i.e. $\exists\ \theta\colon X \overset{\sim}{\to} Y$ such that

$$\theta(\Gamma x) = \Lambda\theta(x) \qquad \text{for } \mu\text{-a.e. } x.$$

Define $\alpha\colon X \times \Gamma \to \Lambda$ by

$$\theta(x) = \alpha(x,s)\theta(s^{-1}x).$$

Then, α satisfies the cocycle identity:

$$\alpha(x,s)\alpha(s^{-1}x,t) = \alpha(x,st).$$

Cocycles α and β are *equivalent* if $\exists\ \phi\colon X \to \Lambda$ such that

$$\beta(x,s) = \phi(x)\alpha(x,s)\phi(s^{-1}x)^{-1}.$$

Cocycle Superrigidity

One can recover **GA** from **OE** if one has

Theorem (Cocycle Superrigidity)

With some assumption on $\Gamma \curvearrowright X$ (and not on Λ), any cocycle

$$\alpha\colon \Gamma \times X \to \Lambda$$

is equivalent to a cocycle β which is independent on $x \in X$.

Examples

- Γ higher rank lattice + Λ simple Lie group (Zimmer)
- Γ Kazhdan (T) / product + $\Gamma \curvearrowright X$ malleable (Popa)
- Γ Kazhdan (T) + $\Gamma \curvearrowright X$ profinite (Ioana)

Popa's formulation

$\Gamma \curvearrowright X$ ⟷ $\sigma\colon \Gamma \curvearrowright L^\infty(X)$

$\alpha\colon X \times \Gamma \to \Lambda$ ⟷ $\alpha\colon \Gamma \to L^\infty(X, \mathrm{vN}(\Lambda)) \cong L^\infty(X) \bar{\otimes} N$, $\alpha_s(x) = \alpha(x, s)$

$\alpha(x,s)\alpha(s^{-1}x, t) = \alpha(x, st)$ ⟷ $\alpha_s\,\sigma_s(\alpha_t) = \alpha_{st}$

Since $\sigma_s(f) = s\,f\,s^{-1}$ in $\mathrm{vN}(X \rtimes \Gamma)$,

$$\Gamma \ni s \mapsto \alpha_s\, s \in \mathrm{vN}(X \rtimes \Gamma) \bar{\otimes} N$$

is a group homomorphism which extends to an inclusion

$$\Theta\colon \mathrm{vN}(\Gamma) \hookrightarrow \mathrm{vN}(X \rtimes \Gamma) \bar{\otimes} N.$$

Profinite Action

Definition

An ergodic action $\Gamma \curvearrowright X$ is *profinite* if $X = \varprojlim \Gamma/\Gamma_n$ for some finite index subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$;
or equivalently $\exists A_1 \subset A_2 \subset \cdots \subset L^\infty(X)$ finite dimensional Γ-invariant vN-subalgebras with dense union. $(A_n = \ell_\infty(\Gamma/\Gamma_n).)$

$$\mathrm{vN}(X \rtimes \Gamma) = \Big(\bigcup \mathrm{vN}((\Gamma/\Gamma_n) \rtimes \Gamma)\Big)'' \cong \Big(\bigcup \mathbb{M}_{[\Gamma:\Gamma_n]}(\mathrm{vN}(\Gamma_n))\Big)''.$$

What's behind Ioana's Cocycle Superrigidity

$$\Theta\colon \mathrm{vN}(\Gamma) \hookrightarrow \mathrm{vN}(X \rtimes \Gamma) \mathbin{\bar\otimes} N = \Big(\bigcup \big(\mathrm{vN}((\Gamma/\Gamma_n) \rtimes \Gamma) \mathbin{\bar\otimes} N\big)\Big)''$$

$\longrightarrow$ Because of the Kazhdan property (T), for a large n, $\Theta(\mathrm{vN}(\Gamma))$ is almost contained in $\mathrm{vN}((\Gamma/\Gamma_n) \rtimes \Gamma) \mathbin{\bar\otimes} N$.

Complete Metric Approximation Property

Definition

A group Γ has the CMAP if $\exists\ f_n$ such that

- $f_n\colon \Gamma \to \mathbb{C}$ finitely supported,
- $f_n \to 1$ pointwise,
- $\|m_{f_n}\|_{\mathrm{cb}} \leq 1$.

The multiplier $m_f\colon \mathrm{vN}(\Gamma) \to \mathrm{vN}(\Gamma)$ is defined by $m_f(s) = f(s)\,s$.

Besides amenable groups, the following groups have the CMAP.

Theorem (De Cannière-Haagerup 1985, Cowling-Haagerup 1989)

Free groups $\mathbb{F}_r$ have the CMAP.
Discrete subgroups of $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$ have the CMAP.

Groups with CMAP

Theorem A (Oz-Popa)

Suppose Γ CMAP and $\exists\ \Lambda \triangleleft \Gamma$ infinite normal amenable subgroup. Then, Λ has an invariant mean which is $\mathrm{Ad}(\Gamma)$-invariant. In particular, Γ is inner-amenable.

Proof (Assuming Λ is abelian).

Recall $\mathrm{vN}(\Lambda) \cong L^\infty(\widehat{\Lambda})$ via the Fourier transform $\ell_2(\Lambda) \cong L^2(\widehat{\Lambda})$.
Let $\tau_0 \colon C(\widehat{\Lambda}) \to \mathbb{C}$ be the evaluation at the trivial character 1.
$f : \Lambda \to \mathbb{C}$ fin. supp. $\Rightarrow \tau_0 \circ m_f \cong \widehat{f} \in L^1(\widehat{\Lambda})$ and $\|\widehat{f}\|_1 = \|m_f\|_{\mathrm{cb}}$.
Take (f_n) as in Definition. Then $\forall s \quad \|m_{f_n} - m_{f_n} \circ \mathrm{Ad}_s\|_{\mathrm{cb}} \to 0$.
Hence, if $\ell_2(\Lambda) \ni \xi_n \overset{\text{Fourier}}{\longleftrightarrow} |\widehat{f_n|_\Lambda}|^{1/2} \in L^2(\widehat{\Lambda})$, then $|\xi|^2 \in \ell_1(\Lambda)$ is approximately Λ-invariant and approximately $\mathrm{Ad}(\Gamma)$-invariant. □

Groups with CMAP

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Corollary

The lamplighter group

$$(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_r = \left(\bigoplus_{\mathbb{F}_r} (\mathbb{Z}/2\mathbb{Z})\right) \rtimes \mathbb{F}_r$$

does not have the CMAP.

Theorem (de Cornulier-Stalder-Valette)

The lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_r$ has the Haagerup property.

von Neumann algebra with CMAP

Definition

A finite vN-algebra M has the CMAP if $\exists\ \phi_n$ such that

- $\phi_n \colon M \to M$ finite rank,
- $\phi_n \to \mathrm{id}_M$ pointwise-ultraweak,
- $\|\phi_n\|_{\mathrm{cb}} \leq 1$.

Examples

- Γ has CMAP $\Leftrightarrow$ $\mathrm{vN}(\Gamma)$ has CMAP (Haagerup)
- CMAP inherits to a vN-subalgebra (assuming finiteness).
- Γ has CMAP and $\Gamma \curvearrowright X$ profinite $\Rightarrow$ $\mathrm{vN}(X \rtimes \Gamma)$ has CMAP. (Note: $\mathrm{vN}(X \rtimes \Gamma)$ can be non-(Γ).)

Upgrading Theorem **A**

Use $\mu\colon P \otimes \bar{P} \ni \sum a_k \otimes \bar{b}_k \mapsto \tau(\sum a_k b_k^*) \in \mathbb{C}$ instead of τ_0.

Theorem A+ (Oz-Popa)

Suppose that M has CMAP and P is an amenable vN-subalgebra. Then, $\exists\ \eta_n \in L^2(P \bar{\otimes} \bar{P})_+$ such that

- $\|\eta_n - (u \otimes \bar{u})\eta_n\|_2 \to 0$ *for every* $u \in \mathcal{U}(P)$;
- $\|\eta_n - \mathrm{Ad}(u \otimes \bar{u})\eta_n\|_2 \to 0$ *for every* $u \in \mathcal{N}(P)$;
- $\langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle \eta_n, (1 \otimes \bar{x})\eta_n \rangle$ *for every* $x \in M$.

We say $P \subset M$ is *weakly profinite* if the above conclusion holds.

If $M = P \rtimes \Gamma$ and $\exists\ P_1 \subset P_2 \subset \cdots \subset P$ finite dim. Γ-invariant vN-subalgebras with dense union, then $P \subset M$ is weakly profinite with $\eta_n = \mu_n^{1/2} \in L^2(P_n \bar{\otimes} \bar{P}_n)_+$.

Main Results

Theorem B (Oz-Popa)

Suppose that $M = Q \rtimes \mathbb{F}_r$ and that $P \subset M$ is weakly profinite. Then, either one of the following occurs

- *a nonzero corner of P is unitarily conjugated into Q;*
- *$\mathcal{N}(P)''$ is amenable relative to Q.*

Corollary

- $P \subset \mathrm{vN}(\mathbb{F}_r)$ *diffuse amenable* $\Rightarrow \mathcal{N}(P)''$ *amenable.*
- Q *CMAP* $\Rightarrow Q \bar{\otimes} \mathrm{vN}(\mathbb{F}_r)$ *has no Cartan subalgebra.*
- $\mathbb{F}_r \curvearrowright X$ *profinite* $\Rightarrow$ $L^\infty(X) \subset \mathrm{vN}(X \rtimes \mathbb{F}_r)$ *is the unique Cartan subalgebra.*

Proof in the case of $P \subset \mathrm{vN}(\mathbb{F}_r)$ diffuse amenable

Let $a_1, \ldots, a_r \in M = \mathrm{vN}(\mathbb{F}_r)$ be the standard unitary generators, and $M_1 = \langle b_1, \ldots, b_r \rangle$ be a copy of $\mathrm{vN}(\mathbb{F}_r)$.
For $t \in \mathbb{R}$, define a $*$-homomorphism $\alpha_t \colon M \to M * M_1$ by

$$\alpha_t(a_k) = a_k \exp(t \log b_k).$$

Observe that $E_M \circ \alpha_t$ is the Haagerup multiplier on M associated with $\mathbb{F}_r \ni s \mapsto \gamma_t^{|s|} \in \mathbb{R}$, where $\gamma_t = \tau(\exp(t \log b_k)) = \frac{\sin(\pi t)}{\pi t}$.
For a given finite subset $\mathfrak{F} \subset \mathcal{N}(P)$, choose $t > 0$ small enough so that $\alpha = \alpha_t$ satisfies $\alpha(u) \approx u$ for all $u \in \mathfrak{F}$.
Since $\eta_n \in L^2(P \bar{\otimes} \bar{P})$ are "almost supported on diagonal," $((E_M^\perp \circ \alpha) \otimes 1)\eta_n$ is a non-null sequence, almost $\mathrm{Ad}(\mathfrak{F})$-invariant.
But $L^2(M * M_1) \ominus L^2(M) \cong \bigoplus L^2(M) \bar{\otimes} L^2(M)$ as an M-bimodule, this implies amenability of $\mathcal{N}(P)''$. □