Furstenberg boundary and C^*-simplicity

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The mathematical legacy of Uffe Haagerup

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Main Problem of C^*-simplicity

A discrete group G is C^*-simple if its reduced grp C^*-alg $\mathrm{C}^*_{\mathrm{r}}G$ is simple.
Recall $\lambda\colon G \curvearrowright \ell_2 G$ the left regular repn, and $\mathrm{C}^*_{\mathrm{r}}G = \overline{\lambda(\mathbb{C}G)} \subset \mathbb{B}(\ell_2 G)$.

A Well-Known Fact

Let $N \lhd G$ be a normal subgroup. Then the quotient $G \to G/N$ extends to a $*$-homomorphism from $\mathrm{C}^*_{\mathrm{r}}G$ onto $\mathrm{C}^*_{\mathrm{r}}(G/N)$ iff N is amenable.

$R(G)$: amenable radical, the largest amenable normal subgroup of G.
Hence,

$$G \text{ is } \mathrm{C}^*\text{-simple} \Longrightarrow R(G) = \mathbf{1}.$$

Is the converse also true?

Motivated by Kadison's problem if $\mathrm{C}^*_{\mathrm{r}}F_d$ is simple and projectionless,

Theorem (Powers 1975/1968)

The reduced free group C^*-algebra $\mathrm{C}^*_{\mathrm{r}}F_d$, $d \geq 2$, is simple.

F_d has Powers Averaging Property $\rightsquigarrow$ $\mathrm{C}^*_{\mathrm{r}}F_d$ is simple and monotracial.
PAP: $\exists\, E_n \Subset G$ s.t. $\lim_n \left\| \frac{1}{|E_n|} \sum_{t \in E_n} \lambda(tgt^{-1}) \right\| = 0$ for $\forall\, g \neq 1$.

Characterization of C^*-simple groups

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$\Rightarrow \tau_\lambda(a)1 \in \overline{\mathrm{conv}}\{\lambda(t)a\lambda(t^{-1}) : t \in G\}$ for every $a \in \mathrm{C}^*_\mathrm{r} G$

$\Rightarrow \mathrm{C}^*_\mathrm{r} G$ is simple and monotracial.

A striking "if and only if" characterization of C^*-simplicity in terms of the Furstenberg boundary has been found by Kalantar and Kennedy.

Theorem (Kalantar–Kennedy 2014)

$$G \text{ is } \mathrm{C}^*\text{-simple} \iff G \curvearrowright \partial_\mathrm{F} G \text{ is (topologically) free}$$

This lead to a very satisfactory solution to the C^*-simplicity problem.

Theorem (Breuillard–Kalantar–Kennedy–O. 2014)

$\mathrm{C}^*_\mathrm{r} G$ is monotracial $\Leftrightarrow R(G) = \mathbf{1} \Leftrightarrow G$ has no non-trivial IRS

Theorem (Haagerup, Kennedy 2015 (based on Kalantar–Kennedy))

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Here a URS (uniformly recurrent subgroups) is a minimal component of $G \curvearrowright \mathbf{Sub}(G) \subset \{0,1\}^G$ (the Chabauty compactum). In particular, G is C^*-simple iff every minimal compact $G \curvearrowright X$ with amenable stabilizers is topologically free, i.e., $G_x^\circ := \{g \in G : g = \mathrm{id}$ on a nbhd of $x\}$ is trivial.

Examples of C^*-simple groups

The results of BKKO, Haagerup, and Kennedy give very simple proof of C^*-simplicity (PAP) for all known examples: (assume $R(G) = \mathbf{1}$)

- Free groups (Powers 1975)

$$\vdots$$

- Acylindrically hyperbolic groups (Dahmani–Guirardel–Osin 2011).
- Linear groups (Bekka–Cowling–de la Harpe 1994, Poznansky 2008).
- Groups with nontrivial ℓ_2-Betti numbers (Peterson–Thom 2011).
- Free Burnside groups, etc. (Olshanskii–Osin 2014).

$$\vdots$$

A counterexample to "$R(G) = \mathbf{1} \Rightarrow \mathrm{C}^*$-simple" ?

Haagerup–Olesen (2014) suggested Thompson's group T, by showing that T is **not** C^*-simple **if** Thompson's group F is amenable. Thompson's group T is a certain subgroup of piecewise-linear homeomorphisms of $S^1 = \mathbb{R}/\mathbb{Z}$, which is non-amenable and simple. Thompson's group F is the stabilizer subgroup at $0 \in \mathbb{R}/\mathbb{Z}$, and whether F is amenable or not is a very (in)famous open problem (that Uffe was crazy about for these years).

Counterexamples to C^*-simplicity

Haagerup–Olesen (2014): Thompson's group T is a certain simple subgroup of piecewise-linear homeomorphisms of $S^1 = \mathbb{R}/\mathbb{Z}$, and Thompson's group F is the stabilizer subgroup at $0 \in \mathbb{R}/\mathbb{Z}$.
Consider $\pi\colon T \curvearrowright \ell_2 T/F$. It is continuous on $\mathrm{C}^*_{\mathrm{r}} T$ iff F is amenable.
Since $T/F \cong T0 = \mathbb{Z}[\frac{1}{2}] \cap [0,1)$, it is easy to find $a, b \in T \setminus \mathbf{1}$ such that $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$ on T/F, which guarantees $\pi((1-a)(1-b)) = 0$.

Hence if F is amenable, then T is not C^*-simple.

More recently Matte Bon and Le Boudec (2016) have classified the URS's of T and proved that T is not C^*-simple iff F is amenable.

Theorem (Le Boudec 2015)

Aside from $F \leq T$, there is a natural $H \leq G$ as above with H amenable. In particular, there are simple groups G which are not C^*-simple.

$G \leq \mathbf{Aut}(\mathrm{Tree})$ a variant of the Burger–Mozes grp, H a vertex stabilizer.

Topological Boundary Theory.

Furstenberg 1963 $\sim$ 1973
Hamana 1979 $\sim$ 1985

A compact G-space X is minimal if $\overline{Gx} = X$ for every $x \in X$.

It is a G-boundary if $\overline{\mathrm{conv}}\, G\mu = \mathrm{Prob}(G)$ for every $\mu \in \mathrm{Prob}(G)$.

$\iff$ [$\forall Y$ minimal cpt G-space, $\forall \phi\colon C(X) \to C(Y)$ G-ucp map
$\to \phi$ is an isometric $*$-homomorphism]

Example

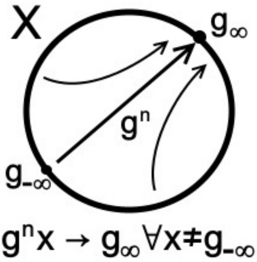
- Trivial G-space $\{\mathrm{pt}\}$. If G is amenable, there is no other G-boundary.
- G: a (non-elementary) Gromov hyperbolic group and $X = \partial G$.

Furstenberg: $\exists!$ largest G-boundary $\partial_{\mathrm{F}} G$.
Ubiquitousness $\forall$cpt cvx G-space K $\exists \partial_{\mathrm{F}} G \to K$.

Hamana: $\exists!$ smallest unital G-inj C^*-alg $C(\partial_{\mathrm{H}} G)$.
Rigidity ↳ $\forall$ G-ucp map on $C(\partial_{\mathrm{H}} G)$ is id.

- Point stabilizers are amenable.
- $\ker(G \curvearrowright \partial_{\mathrm{F}} G) = R(G)$.

Kalantar–Kennedy 2014: $\partial_{\mathrm{F}} G = \partial_{\mathrm{H}} G$.



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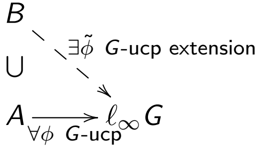
$\ell_\infty G$ is G-injective:



$\because\ \phi \longleftrightarrow \mathrm{ev}_1 \circ \phi \in S(A)$

Tracial states on $\mathrm{C}^*_{\mathrm{r}} G$

Corollary (Breuillard–Kalantar–Kennedy–O, Kennedy)

Every tracial state on $\mathrm{C}^*_{\mathrm{r}} G$ is zero outside of $R(G)$. In particular if $R(G) = \mathbf{1}$, then τ_λ is the unique tracial state. If moreover $\partial_{\mathrm{H}} G$ is (topo) free, then τ_λ is the unique G-ucp map from $\mathrm{C}^*_{\mathrm{r}} G$ into $C(\partial_{\mathrm{H}} G)$.

Proof.

Observe that a tracial state is a G-ucp map from $\mathrm{C}^*_{\mathrm{r}} G$ into $\mathbb{C}1 \subset C(\partial_{\mathrm{H}} G)$. Let $\phi\colon \mathrm{C}^*_{\mathrm{r}} G \to C(\partial_{\mathrm{H}} G)$ be a G-ucp map. By G-injectivity, it extends to $\tilde{\phi}\colon C(\partial_{\mathrm{F}} G) \rtimes_{\mathrm{r}} G \to C(\partial_{\mathrm{H}} G)$, which is a conditional expectation by rigidity. Thus, for any $t \in G$ and $x \in \partial_{\mathrm{H}} G$ with $tx \neq x$, one has $\phi(\lambda_t)(x) = 0$. (Take $f \in C(\partial_{\mathrm{H}} G)$ s.t. $f(x) = 1$ and $\mathrm{supp}(f) \cap t\,\mathrm{supp}(f) = \emptyset$. Then $\phi(\lambda_t)(x) = \tilde{\phi}(f\lambda_t f)(x) = 0$.) Recall that $\ker(G \curvearrowright \partial_{\mathrm{H}} G) = R(G)$. □

An IRS (invariant random subgroup) is a G-inv prob measure on $\mathbf{Sub}(G)$.

$\rightsquigarrow$ Every amenable IRS μ is contained in $R(G)$.
(Bader–Duchesne–Lécureux 2014)

$\because$ $\phi(g) := \mu(\{H : g \in H\})$ is a tracial state on $\mathrm{C}^*_{\mathrm{r}} G$ (Tucker-Drob 2012).

Closing the Circle

Theorem (Kalantar–Kennedy 2014)

$$G \text{ is } \mathrm{C}^*\text{-simple} \iff G \curvearrowright \partial_{\mathrm{F}} G \text{ is (topo) free}$$

Note that $G \curvearrowright \partial_{\mathrm{F}} G$ is a minimal action with amenable point stabilizers. Hence by Effros–Hahn, Kawamura–Tomiyama, and Archbold–Spielberg,

$$G \curvearrowright \partial_{\mathrm{F}} G \ \text{ topo free} \iff C(\partial_{\mathrm{F}} G) \rtimes_{\mathrm{r}} G \ \text{ simple.}$$

Theorem (Breuillard–Kalantar–Kennedy–O 2014)

$$G \text{ is } \mathrm{C}^*\text{-simple} \iff A \rtimes_{\mathrm{r}} G \text{ is simple for any unital } G\text{-simple } \mathrm{C}^*\text{-alg } A$$

Theorem (Haagerup, Kennedy 2015)

$$G \text{ is } \mathrm{C}^*\text{-simple} \iff G \text{ has Powers Averaging Property}$$

Proof of $\Rightarrow$.

It suffices to show that for every state ψ on $\mathrm{C}^*_{\mathrm{r}} G$, one has $\tau_\lambda \in \overline{\mathrm{conv}}\, G\psi$. By ubiquitousness, there is a G-map $\phi_* \colon \partial_{\mathrm{F}} G \to \overline{\mathrm{conv}}\, G\psi$. This gives rise to a G-ucp map $\phi \colon \mathrm{C}^*_{\mathrm{r}} G \to C(\partial_{\mathrm{F}} G)$, defined by $\phi(a)(x) = \langle \phi_*(x), a \rangle$. But $\phi = \tau_\lambda(\,\cdot\,)1$ by the previous Corollary, and so $\phi_*(\partial_{\mathrm{F}} G) = \{\tau_\lambda\}$. □