Full factors and co-amenable inclusions

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Property Γ and Popa's conjecture

Throughout the talk, M denotes a factor of any type with separable predual.
A central sequence means $(u_n)_n$ in $\mathcal{U}(M)$ s.t. $\forall x\ [u_n, x] \to 0$ in SOT.
It is trivial if $\exists \gamma_n \in \mathbb{C}$ s.t. $u_n - \gamma_n 1 \to 0$ in SOT.

Definition/Theorem (Murray–von Neumann, Connes, Marrakchi)

M has property Γ if ∃ non-trivial central sequences.
M does not have property $\Gamma \iff M' \cap M^\omega = \mathbb{C}1$.
$\iff M$ is full.

Conjecture (Popa 1986)

For a co-amenable II_1 subfactor $N \subset M$,

$$N \text{ has } \Gamma \Rightarrow M \text{ has } \Gamma$$

Theorem (Popa 1983, Bedos 1990, Bisch 1990)

True when $M = N \rtimes \Gamma$ with Γ amenable and $\Gamma \curvearrowright N$ free.
Moreover, $M' \cap N^\omega \neq \mathbb{C}1$. ⚠ Not nec. true even for finite index subfactors.

Partial converse by Marrakchi (2018).

Popa's conjecture (contrapositive)

Popa's Conjecture (Contrapositive)

For a co-amenable II_1 subfactor $N \subset M$,

$$M \text{ is full} \Rightarrow N \text{ is } \textit{full}$$

Theorem (Pimsner–Popa 1986)

True when $N \subset M$ has finite index.
Moreover, $N' \cap M^\omega = N' \cap M$.

Proof.

By the basic construction, it suffices to show: N full $\Rightarrow$ M full.
$E \colon M \to N$ satisfies the Pimsner–Popa inequality $E \geq \gamma^{-1}\mathrm{id}_M$.

$$\begin{array}{cccc} E^\omega \colon & M^\omega & \to & N^\omega \\ & \cup & & \cup \\ & N' \cap M^\omega & \to & N' \cap N^\omega = \mathbb{C}1 \end{array}$$

E^ω satisfies the same inequality.

$\rightsquigarrow \dim N' \cap M^\omega < \infty$ and $N' \cap M^\omega = N' \cap M$.

Hence M is full. $\square$

Solution to Popa's conjecture for any type

Popa's Conjecture (Contrapositive)

For a co-amenable II_1 subfactor $N \subset M$,

$$M \text{ is full} \Rightarrow N \text{ is } \textit{full}$$

Theorem (Bannon–Marrakchi–O 2019)

Let M be full and $N \subset M$ be a co-amenable subalgebra with a **f.n.c.e.** Then $\exists p \in N' \cap M$ non-zero projection s.t.

$$p(N' \cap M^\omega)p = \mathbb{C}p.$$

In particular, p is atomic in $N' \cap M$ and pN is a full factor.

Corollary

Let $G \curvearrowright M$ be a free action of a compact group G on a full factor M. Then $M^G \subset M$ is co-amenable, $(M^G)' \cap M^\omega = \mathbb{C}1$ (i.e., minimal), and M^G and $M \rtimes G$ are full factors.

This generalizes Tomatsu's result (2018).

Fullness via bimodules

${}_M\mathcal{H}_M$ an M-M bimodule, $\lambda_{\mathcal{H}} \times \rho_{\mathcal{H}} \colon M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \to \mathbb{B}(\mathcal{H})$

$$
\begin{aligned}
{}_M\mathcal{H}_M \preceq {}_M\mathcal{K}_M \overset{\text{def}}{\Leftrightarrow} & \forall F \Subset M \ \forall \xi \in \mathcal{H} \ \forall \varepsilon > 0 \ \exists \eta_1, \ldots, \eta_k \in \mathcal{K} \\
& \text{s.t. } \langle x\xi y, \xi \rangle \approx_\varepsilon \textstyle\sum_i \langle x\eta_i y, \eta_i \rangle \ \forall x, y \in F \\
\Leftrightarrow & \ \mathrm{C}^*(\lambda_{\mathcal{K}}(M), \rho_{\mathcal{K}}(M^{\mathrm{op}})) \to \mathrm{C}^*(\lambda_{\mathcal{H}}(M), \rho_{\mathcal{H}}(M^{\mathrm{op}})) \text{ cts}
\end{aligned}
$$

Observation

$$M \text{ full} \iff \forall {}_M\mathcal{H}_M \sim {}_M L^2 M_M \text{ one has } \mathcal{H} \supset L^2 M$$

This is reminiscent of property (T): $\forall {}_M\mathcal{H}_M \succeq {}_M L^2 M_M$ one has $\mathcal{H} \supset L^2 M$.

Proof.

$(\Rightarrow)$ $\quad P_{\hat{1}} \in \mathbb{K}(L^2 M) \underset{\text{Marrakchi 2018}}{\subset} \mathrm{C}^*(\lambda_{L^2 M}, \rho_{L^2 M}) \cong \mathrm{C}^*(\lambda_{\mathcal{H}}, \rho_{\mathcal{H}})$

$(\Leftarrow)$ $\quad$ If $\sigma \in \overline{\mathrm{Inn}}(M) \setminus \mathrm{Inn}(M)$, then ${}_M L^2 M_{\sigma(M)} \sim {}_M L^2 M_M$ but $\not\supset$. $\square$

Co-amenability (Popa, Anantharaman-Delaroche)

$\Lambda \leq \Gamma$ co-amenable $\overset{\text{def}}{\Leftrightarrow}$ Γ/Λ admits Γ-invariant mean
$\Leftrightarrow \quad \dots$
$\Leftrightarrow L\Lambda \leq L\Gamma$ co-amenable

Recall M amenable if (Connes 1976)

- injective: $\exists$ cond. exp. $M \subset \mathbb{B}(L^2M) \xrightarrow{E} M$
- semi-discrete: ${}_M L^2 M_M \preceq {}_M L^2 M \bar{\otimes} L^2 M_M$

$N \subset M$ is co-injective $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ cond. exp. $M' \subset N' \xrightarrow{E} M'$
$\Leftrightarrow \exists E : \langle M, e_N \rangle \to M$ in the presence of a f.n.c.e.

co-semi-discrete $\overset{\text{def}}{\Leftrightarrow}$ ${}_M L^2 M_M \preceq {}_M L^2 M \bar{\otimes}_N L^2 M_M \ (= L^2\langle M, e_N\rangle)$

Theorem (Popa, A-D, Pisier, Haagerup, BMO 2019)

co-injectivity $\Leftrightarrow$ co-semi-discreteness

Moreover for $\forall N \subset M$, $\exists$ cond. exp. $\Leftrightarrow {}_N L^2 N_N \preceq {}_N L^2 M_N$.

Recall Takesaki's theorem (1972): $\exists$ normal c.e. $\Leftrightarrow {}_N L^2 N_N \subset {}_N L^2 M_N$.

Corollary

$N \subset M$ co-amenable and $N \subset P \subset M \Rightarrow P \subset M$ co-amenable

⚠ $N \subset P$ may not! (Monod–Popa 2003)

Proof of Theorem in case of type II_1 modulo a lemma

Theorem (BMO 2019)

Let M be full and $N \subset M$ be a co-amenable subalgebra with a f.n.c.e.
Then $\exists p \in N' \cap M$ non-zero projection s.t.

$$p(N' \cap M^\omega)p = \mathbb{C}p.$$

Let M be a full factor of type II_1 and $N \subset M$ be a co-amenable subalgebra.

$${}_M L^2 M_M \preceq {}_M L^2 M \mathbin{\bar{\otimes}_N} L^2 M_M \qquad \text{by co-amenability}$$

If $\succeq$ also, then $\subset$ by fullness.

$$\underset{\text{intertwining bimodule}}{\rightsquigarrow} \exists p \in N' \cap M \text{ s.t. } pN \subset pMp \text{ finite index irr. subfactor}$$

$$\underset{\text{Pimsner–Popa}}{\rightsquigarrow} p(N' \cap M^\omega)p = \mathbb{C}p \qquad \square$$

⚠ Unfortunately, it isn't clear when $\succeq$ holds.

Put $P := (N' \cap M^\omega)' \cap M$. Then $N \subset P \subset M$ and $N' \cap M^\omega = P' \cap M^\omega$.

Lemma (BMO 2019)

For $P \subset M$ s.t. $P = (P' \cap M^\omega)' \cap M$, one has ${}_M L^2 M \mathbin{\bar{\otimes}_P} L^2 M_M \preceq {}_M L^2 M_M$.

Proof of Lemma

For simplicity, assume M is a factor of type II_1.

Lemma (BMO 2019)

For $P \subset M$ s.t. $P = (P' \cap M^\omega)' \cap M$, one has ${}_M L^2 M \,\bar{\otimes}_P\, L^2 M_M \preceq {}_M L^2 M_M$.

Proof.

It suffices to show $\forall F \Subset M$ $\forall \varepsilon > 0$ $\exists u_1, \ldots, u_k \in \mathcal{U}(M)$ s.t.

$$E_P(x) \approx_\varepsilon \tfrac{1}{k} \textstyle\sum_i u_i x u_i^* \text{ for } \forall x \in F$$

which amounts to

$$\langle x(1 \otimes_P 1)y, 1 \otimes_P 1 \rangle_{L^2 M \bar{\otimes}_P L^2 M} = \tau(E_P(x)y) \approx_\varepsilon \tfrac{1}{k} \textstyle\sum_i \langle x\hat{u}_i y, \hat{u}_i \rangle_{L^2 M}.$$

Claim. $\overline{\mathrm{conv}}^{\|\cdot\|_2}\{uxu^* : u \in \mathcal{U}(P' \cap M^\omega)\} \ni z = (z(n))_n$ with min. norm
Then $z(n)$ is ω-convergent and $z = \lim_\omega z(n) = E_P(x)$.

$\because$ Observe that since $z \in M^\omega \cap (P' \cap M^\omega)'$, it is left to show $z \in M$.
If $z(n)$ not convergent, one can arrange $\frac{z(n_l) + z(n_l')}{2}$ has smaller 2-norm.
Hence, $z(n)$ is convergent and $z = \lim_\omega z(n) \in M$. □