Kazhdan's property (T) and semidefinite programming

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- N. Ozawa; About the Connes Embedding Conjecture.
  Jpn. J. Math., **8** (2013), 147–183. `arXiv:1212.1700`
- N. Ozawa; Noncommutative real algebraic geometry of Kazhdan's property (T).
  J. Inst. Math. Jussieu, **15** (2016), 85–90. `arxiv:1312.5431`
- M. Kaluba, P. W. Nowak, and N. Ozawa; Aut($\mathbf{F}_5$) has property (T).
  Math. Ann., **375** (2019), 1169–1191. `arXiv:1712.07167`
- M. Kaluba, D. Kielak, and P. W. Nowak; On property (T) for Aut($\mathbf{F}_n$) and $\mathrm{SL}_n(\mathbb{Z})$.
  Ann. of Math. (2), to appear. `arXiv:1812.03456`
- M. Nitsche; Computer proofs for Property (T), and SDP duality.
  Preprint. `arXiv:2009.05134`

The problem and the answer

Problem (. . ., popularized in Lubotzky's book 1994, . . .)

Does $\mathrm{Aut}(\mathbf{F}_n)$ have Kazhdan's property (T) **?**

- $\mathrm{Aut}(\mathbf{F}_n)$ is the noncommutative analogue of $\mathrm{GL}_n(\mathbb{Z})$.
  $\mathbf{F}_n \twoheadrightarrow \mathbb{Z}^n$ abelianization $\rightsquigarrow$ $\mathrm{Aut}(\mathbf{F}_n) \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$.
- Property (T) inherits to finite-index subgroups and quotient groups.
  Any property (T) group that is abelian (amenable) is finite.
  $\rightsquigarrow$ Any f.i. subgroup with property (T) has finite abelianization.
- $\mathrm{GL}_n(\mathbb{Z})$ has property (T) iff $n \geq 3$. $\rightsquigarrow$ $\mathrm{Aut}(\mathbf{F}_2)$ fails property (T).
- $\mathrm{Aut}(\mathbf{F}_3)$ also fails property (T) (McCool 1989).

Thm (KNO '17 for $n = 5$, KKN '18 for $n > 5$, Nitsche '20 for $n = 4$)

$\mathrm{Aut}(\mathbf{F}_n)$ has Kazhdan's property (T) for $n \geq 4$.

- This is proved by a computer (but **it's rigorous!**).
- Prior works by Netzer–Thom, Fujiwara–Kabaya, and Kaluba–Nowak.

Some reaction

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$\mathrm{Aut}(\mathbf{F}_n)$ has Kazhdan's property (T) for $n \geq 4$.

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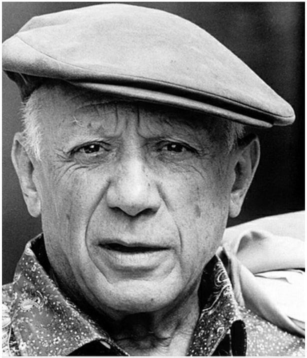


Revista Vea y Lea, January 1962

"But they (= computers) are useless.
They can only give you answers."
Pablo Picasso, 1968.

Kazhdan's property (T)

Theorem (Kazhdan 1967)

Any simple Lie group G of real rank ≥ 2 (e.g., $G = \mathrm{SL}_n(\mathbb{R})$, $n \geq 3$) and its lattice Γ (e.g., $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$) have property (T).

$\rightsquigarrow$ Γ is finitely generated and has finite abelianization.

Definition (for a discrete group Γ)

Γ has (T) $\overset{\text{def}}{\iff} \exists S \subset \Gamma$ finite $\exists \kappa > 0$ s.t. $\forall (\pi, \mathcal{H})$ unitary rep'n and $\forall v \in \mathcal{H}$

$$d(v, \mathcal{H}^\Gamma) \leq \kappa^{-1} \max_{s \in S} \|v - \pi_s v\|.$$

$\iff$ Γ is f.g. & $\forall S \subset \Gamma$ generating $\exists \kappa = \kappa(\Gamma, S) > 0$ s.t. $\cdots$

The optimal $\kappa(\Gamma, S)$ is called the Kazhdan constant for (Γ, S).

- Property (T) inherits to finite-index subgroups and quotient groups.
- $\mathbb{Z}$ (or any infinite abelian group) does not have property (T).
  $\because \frac{1}{\sqrt{2k+1}} 1_{[-k,k]} \in \ell^2(\mathbb{Z})$ is asymp. $\mathbb{Z}$-invariant, but $\ell^2(\mathbb{Z})^{\mathbb{Z}} = \{0\}$.

An application of property (T): Expander graphs

Explicit construction of expanders (Margulis 1973)

$\Gamma = \langle S \rangle$, X a finite set, and $\Gamma \curvearrowright X$ transitively

$\rightsquigarrow$ Schreier graph: Vertices $= X$ and Edges $= \{\{x, sx\} : x \in X,\ s \in S\}$ is a $(|S|, \frac{\kappa(\Gamma,S)^2}{2})$-expander. Namely, for $\forall A \subset X$ one has

$$|\partial A| \geq \frac{\kappa(\Gamma,S)^2}{2}|A|(1 - \frac{|A|}{|X|}).$$

$\rightsquigarrow$ Random walk on X has mixing time $O(\log|X|)$.

Product Replacement Algorithm (Celler et al., Lubotzky–Pak 2001)

$\mathrm{Aut}^+(\mathbf{F}_n) = \langle R^{\pm}_{i,j}, L^{\pm}_{i,j} \rangle \leq_{\text{index } 2} \mathrm{Aut}(\mathbf{F}_n)$, where $\mathbf{F}_n = \langle g_1, \ldots, g_n \rangle$ and

$$R^{\pm}_{i,j} \colon (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_i g_j^{\pm}, g_{i+1}, \ldots, g_n),$$
$$L^{\pm}_{i,j} \colon (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_j^{\pm} g_i, g_{i+1} \ldots, g_n).$$

PRA is a practical algorithm to obtain "random" elements in a given finite group Λ of rank $< n$ via the PRA random walk

$$\mathrm{Aut}^+(\mathbf{F}_n) \curvearrowright \{(h_1, \ldots, h_n) \in \Lambda^n : \Lambda = \langle h_1, \ldots, h_n \rangle\}.$$

Noncommutative real algebraic geometry of property (T)

Hilbert's 17th Pb: $f \in \mathbb{R}(x_1, \dots, x_d)$, $f \geq 0$ on $\mathbb{R}^d$
(E. Artin 1927) $\Longrightarrow f = \sum_i g_i^2$ for some $g_1, \dots, g_k \in \mathbb{R}(x_1, \dots, x_d)$.

$\mathbb{R}[\Gamma]$ real group algebra with the involution $(\sum_t \alpha_t t)^* = \sum_t \alpha_t t^{-1}$.

$$\Sigma^2\mathbb{R}[\Gamma] := \{\sum_i f_i^* f_i\} = \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_\Gamma^+\} \text{ positive cone}$$

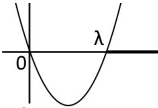
Here $\mathbb{M}_\Gamma^+$ finitely supported positive semidefinite matrices.

- $\mathbb{B}(\mathcal{H})^+ := \{A = A^* : \langle Av, v\rangle \geq 0\ \forall v \in \mathcal{H}\} = \Sigma^2\mathbb{B}(\mathcal{H})$ psd operators.
- $\forall(\pi, \mathcal{H})$ unitary rep'n, $\pi(\sum_i f_i^* f_i) = \sum_i \pi(f_i)^*\pi(f_i) \geq 0$ in $\mathbb{B}(\mathcal{H})$.
- $C^*[\Gamma]$ the universal enveloping C^*-algebra of $\mathbb{R}[\Gamma]$.

Laplacian: For $\Gamma = \langle S \rangle$ with $S = S^{-1}$ finite,

$$\Delta := \tfrac{1}{2}\sum_{s\in S}(1-s)^*(1-s) = |S| - \sum_{s \in S} s \in \Sigma^2\mathbb{R}[\Gamma].$$

$$\begin{aligned}
\Gamma \text{ has (T)} &\iff \exists\lambda > 0 \quad \forall(\pi,\mathcal{H}) \quad \mathrm{Sp}(\pi(\Delta)) \subset \{0\} \cup [\lambda, \infty) \\
&\iff \exists\lambda > 0 \quad \forall(\pi,\mathcal{H}) \quad \pi(\Delta^2 - \lambda\Delta) \geq 0 \text{ in } \mathbb{B}(\mathcal{H}) \\
&\iff \exists\lambda > 0 \quad \text{such that} \quad \Delta^2 - \lambda\Delta \geq 0 \text{ in } C^*[\Gamma] \\
&\qquad \rightsquigarrow \kappa(\Gamma, S) \geq \sqrt{2\lambda/|S|}
\end{aligned}$$



$t^2 - \lambda t \geq 0$

Algebraic characterization of property (T)

Let $\Gamma = \langle S \rangle$ with $S = S^{-1}$ finite.

$\mathbb{R}[\Gamma]$ real group algebra with the involution $(\sum_t \alpha_t t)^* = \sum_t \alpha_t t^{-1}$.

$$\Sigma^2\mathbb{R}[\Gamma] := \{\sum_i f_i^* f_i\} = \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_\Gamma^+\}$$

Here $\mathbb{M}_\Gamma^+$ finitely supported positive semidefinite matrices.

$$\Delta := \tfrac{1}{2} \sum_{s \in S} (1-s)^*(1-s) = |S| - \sum_{s \in S} s \in \Sigma^2\mathbb{R}[\Gamma].$$

$\mathrm{C}^*[\Gamma]$ the universal enveloping C^*-algebra of $\mathbb{R}[\Gamma]$.

Then,

$$\Gamma \text{ has (T)} \iff \exists \lambda > 0 \text{ such that } \Delta^2 - \lambda\Delta \geq 0 \text{ in } \mathrm{C}^*[\Gamma]$$

$$\rightsquigarrow \kappa(\Gamma, S) \geq \sqrt{2\lambda/|S|}$$

Theorem (O 2013)

$$\Gamma \text{ has (T)} \iff \exists \lambda > 0 \text{ such that } \Delta^2 - \lambda\Delta \geq 0 \text{ in } \mathbb{R}[\Gamma]$$

Stability (Netzer–Thom): It suffices if $\exists \lambda > 0$ $\exists \Theta \in \Sigma^2\mathbb{R}[\Gamma]$ such that

$$\|\Delta^2 - \lambda\Delta - \Theta\|_1 \ll \lambda.$$

$\because$ Δ is an order unit for $I[\Gamma] := \ker(\mathbb{R}[\Gamma] \to \mathbb{R})$.

Semidefinite Programming (SDP)

Γ has (T) $\iff \exists\lambda > 0$ such that $\Delta^2 - \lambda\Delta \in \Sigma^2\mathbb{R}[\Gamma]$
$\iff \exists E \Subset \Gamma\ \exists\lambda > 0$ s.t. $\Delta^2 - \lambda\Delta \in \{\sum_{x,y} P_{x,y}x^{-1}y : P \in \mathbb{M}_E^+\}$

By fixing a finite subset $E \Subset \Gamma$, we arrive at the SDP:

minimize $-\lambda$
subject to $\Delta^2 - \lambda\Delta = \sum_{x,y\in E} P_{x,y}x^{-1}y, \quad P \in \mathbb{M}_E^+$

- Due to computer capacity limitation, we almost always take

$$E := \mathrm{Ball}(2) = \{e\} \cup S \cup S^2 = \operatorname{supp}\Delta \cup \operatorname{supp}\Delta^2.$$

$\rightsquigarrow$ Size of SDP: dimension $|E|^2$ and constraints $|E^{-1}E| = |\,\mathrm{Ball}(4)|$.

Certification Procedure:

Suppose (λ_0, P_0) is a hypothetical solution obtained by a computer.
Find $P_0 \approx Q^{\mathrm{T}}Q$ (with $Q\mathbf{1} = 0$) and calculate with guaranteed accuracy

$$r := \|\Delta^2 - \lambda_0\Delta - \textstyle\sum_{x,y}(Q^{\mathrm{T}}Q)_{x,y}(1-x)^*(1-y)\|_1 \ll \lambda_0.$$

$\rightsquigarrow$ Γ has (T) with $\lambda = \lambda_0 - 2r$ (in the case of $E = \mathrm{Ball}(2)$).

- Solving SDP is computationally hard, but certifying (T) is relatively easy.

Previous implementation results

Netzer–Thom 2014, Fujiwara–Kabaya 2017, Kaluba–Nowak 2017

- $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and $S = \{I \pm E_{i,j} : i \neq j\}$.

n	3	4	5	6	$\cdots$
$\lvert S\rvert = 2n(n-1)$	12	24	40	60	$\cdots$
$\lambda(\Gamma, S) >$	**.27**	1.3	2.6	$\circlearrowright$	$\cdots$

- $\Gamma = \mathrm{Aut}^+(\mathbf{F}_n)$ and $S = \{L_{i,j}^{\pm}, R_{i,j}^{\pm}\}$.

n	3	4	5	6	$\cdots$
$\lvert S\rvert = 4n(n-1)$	24	48	80	120	$\cdots$
$\lambda(\Gamma, S) >$	$\circlearrowright$	$\circlearrowright$	$\circlearrowright$	$\circlearrowright$	$\cdots$

- A few more groups that are known to have property (T).

So, we needed (1) some speed-up of the algorithm and (2) an infinite ladder to climb up the sequence $n = 3, 4, 5, \ldots$.

Speed-up by Invariant SDP

$$\Sigma := \{\sigma \in \mathrm{Aut}(\Gamma) : \sigma(S) = S\}$$
$$\cong \mathfrak{S}(n) \ltimes (\mathbb{Z}/2)^{\oplus n} \text{ for } \Gamma = \mathrm{Aut}^{+}(\mathbf{F}_n).$$

When Σ is large, we can exploit it and arrive at the Σ-invariant SDP:

$$\texttt{minimize} \quad -\lambda$$
$$\texttt{subject to} \quad (\Delta^2 - \lambda\Delta)_t = \sum_{\substack{x,y \in E \\ x^{-1}y = t}} P_{x,y}, \ \forall t \in E^{-1}E/\Sigma, \quad P \in (\mathbb{M}_E^{\Sigma})^{+}$$

- For $n = 5$, one has $\dim \mathbb{M}_{\mathrm{Ball}(2)} = 4641^2$ and $|\,\mathrm{Ball}(4)| = 11\,154\,301$, while $\dim \mathbb{M}^{\Sigma}_{\mathrm{Ball}(2)} = 13\,232$ with 36 blocks and $|\,\mathrm{Ball}(4)/\Sigma| = 7\,229$.

Results (KNO 2017):

- $\mathrm{Aut}^{+}(\mathbf{F}_4)$: ☹ ☹ ☹ No result. $\rightsquigarrow$ Probably no solution in Ball(2).
- $\mathrm{Aut}^{+}(\mathbf{F}_5)$: !☺⅄☺⅄☺! **YES!!!** with $\lambda > 1.2$.

Climbing up the sequence $n = 5, 6, 7, \ldots$ (KKN 2018)

$\Gamma_n := \mathrm{Aut}^+(\mathbf{F}_n)$, $\quad S_n := \{R^{\pm}_{i,j}, L^{\pm}_{i,j} : i \neq j\}$, $\quad \mathrm{E}_n := \{\{i,j\} : i \neq j\}$

Want to show $\Delta_n = \sum_{s \in S_n} 1 - s$ satisfies $\Delta_n^2 - \lambda_n \Delta_n \geq 0$.

$$\Delta_n = \sum_{\mathrm{e} \in \mathrm{E}_n} \Delta_{\mathrm{e}},$$

$$\Delta_n^2 = \sum_{\mathrm{e}} \Delta_{\mathrm{e}}^2 + \sum_{\mathrm{e} \sim \mathrm{f}} \Delta_{\mathrm{e}} \Delta_{\mathrm{f}} + \sum_{\mathrm{e} \perp \mathrm{f}} \Delta_{\mathrm{e}} \Delta_{\mathrm{f}}$$

$$=: \quad \mathbf{Sq}_n \quad + \quad \mathbf{Adj}_n \quad + \quad \mathbf{Op}_n\,.$$

- $\mathbf{Sq}_n$ and $\mathbf{Op}_n$ are positive, but $\mathbf{Adj}_n$ may not.

For $n > m$,

$$\sum_{\sigma \in \mathfrak{S}(n)} \sigma(\Delta_m) = m(m-1) \cdot (n-2)! \cdot \Delta_n$$

$$\sum_{\sigma \in \mathfrak{S}(n)} \sigma(\mathbf{Adj}_m) = m(m-1)(m-2) \cdot (n-3)! \cdot \mathbf{Adj}_n$$

$$\sum_{\sigma \in \mathfrak{S}(n)} \sigma(\mathbf{Op}_m) = m(m-1)(m-2)(m-3) \cdot (n-4)! \cdot \mathbf{Op}_n$$

Trial and error on the computer has confirmed

$$\mathbf{Adj}_5 + \alpha\, \mathbf{Op}_5 - \varepsilon \Delta_5 \geq 0$$

with $\alpha = 2$ and $\varepsilon = 0.13$. It follows that

$$0 \leq 60(n-3)!\left(\mathbf{Adj}_n + \tfrac{2\alpha}{n-3}\,\mathbf{Op}_n - \tfrac{n-2}{3}\varepsilon\Delta_n\right) \leq 60(n-3)!\left(\Delta_n^2 - \tfrac{n-2}{3}\varepsilon\Delta_n\right),$$

provided $2\alpha/(n-3) \leq 1$. $\quad \rightsquigarrow \quad \kappa(\mathrm{Aut}^+(\mathbf{F}_n), S_n) \geq \sqrt{2\lambda_n/|S_n|} \geq \sqrt{\varepsilon/6n}$

Simplifying SDP by duality (Nitsche 2020)

Recall $\Delta = |S| - \sum_{s \in S} s \in I[\Gamma] = \ker(\mathbb{R}[\Gamma] \to \mathbb{R})$.
If Γ does not have (T), $(\mathbb{R}\Delta^2 - \Delta) \cap \Sigma^2 I[\Gamma] = \emptyset$, then by the HB theorem, $\exists$ a positive linear functional φ on $I[\Gamma]$ with $\varphi(\Delta) = 1$ and $\varphi(\Delta^2) = 0$.
To prove Γ has (T), it suffices to show $-\Delta \in \Sigma^2 I[\Gamma] + \mathbb{R}\Delta^2 + \ker \varphi$.

$\rightsquigarrow$ If one finds many elements in $\ker \varphi$, it makes SDP easier.

$\langle f, g \rangle := \varphi(f^* g)$ makes $I[\Gamma]$ a (pre-)Hilbert space $\mathcal{H}$ on which Γ acts unitary.
One has $\|1 - x\|^2 = \varphi(2 - x + x^*) = 2\varphi(1 - x)$ and $\|\Delta\|^2 = \varphi(\Delta^2) = 0$.

$\rightsquigarrow$ This amounts to that the 1-cocycle $x \mapsto 1 - x \in \mathcal{H}$ is **harmonic**.

Observation (Nitsche):
Assume that Γ has finite abelianization ($\rightsquigarrow \mathcal{H}^\Gamma = \mathbf{0}$).
For any $t \in \Gamma$ with $tSt^{-1} = S$, one has $x(1 - t) \in \ker \varphi$ for $\forall x$.

$\because \Delta(1 - t) = (1 - t)\Delta = 0$ in $\mathcal{H}$, which implies $1 - t = 0$ in $\mathcal{H}$.

E.g., $t = L_{i,j} L_{j,i}^{-1} R_{i,j} = (g_i \mapsto g_j;\ g_j \mapsto g_i^{-1})$ in $\mathrm{Aut}(\mathbf{F}_n)$.

Theorem (Nitsche 2020)

$\mathrm{Aut}^+(\mathbf{F}_4)$ has property (T).