Noncommutative real algebraic geometry of Kazhdan's property (T)

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Noncommutative Real Algebraic Geometry

NCRAG: equations and inequalities in an algebra over $\mathbb{R}$ (or $\mathbb{C}$,...).

Hilbert's 17th Pb: $f \in \mathbb{R}[x_1,\dots,x_d]$, $f \geq 0$ on $\mathbb{R}^d$
(E. Artin 1927) $\implies f = \sum_i g_i^2$ for some $g_1,\dots,g_n \in \mathbb{R}(x_1,\dots,x_d)$.

$\mathcal{A}$ a unital $*$-algebra together with $\mathcal{A}^+$. E.g., $\mathbb{M}_n(\mathbb{R})$, $\mathbb{B}(\mathcal{H})$, $\mathbb{R}[\Gamma]$, ...

$$\begin{array}{l} \mathcal{A}^{\mathrm{h}} = \{x \in \mathcal{A} : x^* = x\} \\ \cup \\ \mathcal{A}^+ = \Sigma^2\mathcal{A} := \{\sum_i x_i^* x_i\} \end{array} \qquad \left(\begin{array}{l}\text{or any other cone } \mathcal{A}^+ \text{ in } \mathcal{A}^{\mathrm{h}} \text{ s.t. } 1 \in \mathcal{A}^+ \\ \text{and } x^*ax \in \mathcal{A}^+ \text{ for } \forall\, a \in \mathcal{A}^+\ \forall x \in \mathcal{A}.\end{array}\right)$$

$\|x\| := \inf\{R \geq 0 : x^*x \leq R^2 1\} \in [0,\infty]$. $\rightsquigarrow$ This is a C^*-norm.

We call $\mathcal{A}$ a semi-pre-C^*-algebra if all elements are bounded.
$\mathrm{C}^*(\mathcal{A})$: the univ env C^*-alg for the positive $*$-rep's on Hilbert spaces.
$\rightsquigarrow \mathrm{C}^*(\mathbb{R}[\Gamma]) = \mathrm{C}^*(\Gamma)$ the full group C^*-algebra.

Theorem (Hahn–Banach + Gelfand–Naimark–Segal)

The "inclusion" $\iota\colon \mathcal{A} \hookrightarrow \mathrm{C}^*(\mathcal{A})$ is isometric and $\iota(a) \geq 0 \iff a \in \overline{\mathcal{A}^+}$.

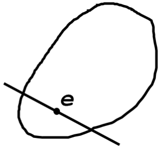
Topology on real vector spaces

$\mathcal{A}$: a semi-pre-C^*-algebra with $\mathcal{A}^+ = \Sigma^2\mathcal{A} = \{\sum_i x_i^* x_i\}$.

Theorem (HB+GNS)

The "inclusion" $\iota\colon \mathcal{A} \hookrightarrow \mathrm{C}^*(\mathcal{A})$ is isometric and $\iota(a) \geq 0 \iff a \in \overline{\mathcal{A}^+}$.

For every $\mathbb{R}$ vector space V, we consider the finest locally convex topology.
For a cone $V^+ \subset V$,
$e \in \mathrm{int}\, V^+$ if $\forall v \in V\ \exists R > 0$ s.t. $v + Re \in V^+$, and
$\overline{V^+} = \{v \in V : v + \epsilon e \in V^+ \text{ for } \forall \epsilon > 0\}$.
(archimedean closure of V^+.)



$\mathcal{A}$ is a semi-pre-C^*-algebra $\iff 1 \in \mathrm{int}\, \mathcal{A}^+$
$\rightsquigarrow \overline{\mathcal{A}^+} = \{a \in \mathcal{A}^{\mathrm{h}} : a + \epsilon 1 \in \mathcal{A}^+ \text{ for } \forall \epsilon > 0\}$.

Problem: Is $\Sigma^2\mathbb{R}[\Gamma]$ (or $\Sigma^2\mathbb{C}[\Gamma]$) closed?

- YES if $\Gamma = \mathbb{Z}$ (Fejér 1916), F_d (Schmüdgen 80s), $\mathbb{Z}^2$ (Scheiderer 06).
- NO if $\Gamma \supset \mathbb{Z}^3$ (Scheiderer 00).
- How about hyperbolic groups? $F_d \times F_d$?

Kazhdan's property (T)

Γ has (T) $\overset{\mathrm{d}}{\Longleftrightarrow}$ $\forall$ orth rep $(\pi, \mathcal{H})$ and $\forall v \in \mathcal{H}$, if v is almost Γ-invariant, then v is close to a Γ-invariant vector.

$\Longleftrightarrow$ $\exists S \subset \Gamma$ finite, $\exists K > 0$ such that

$$d(v, \mathcal{H}^\Gamma) \leq K \max_{g \in S} \|v - \pi(g)v\|.$$

$\rightsquigarrow$ $\Gamma = \langle S \rangle$ and K^{-1} is called the **Kazhdan constant** for (Γ, S).

Non-examples: infinite amenable groups, groups that act on a tree.

Example (Kazhdan 1967)

$\mathrm{SL}(3,\mathbb{Z})$ and $S = \{\begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \ldots\}$, or any lattice of a s.c. Lie group of rank ≥ 2.

$\underset{\text{Margulis 73}}{\rightsquigarrow}$ $X_m := \mathrm{Cayley}(\mathrm{SL}(3,\mathbb{Z}/m\mathbb{Z}), S)$ form a sequence of expanders.

i.e., $\forall A \subset V(X_m)$ one has $|\partial A| \geq K^{-1}|A|(1 - \frac{|A|}{|V(X_m)|})$.

Explicit estimate of the Kazhdan constant: Burger 91 and Shalom 99.

OPEN PROBLEM: $\mathrm{Aut}(F_d)$, $d \geq 4$ $\rightsquigarrow$ Product Replacement Algorithm

Laplacian

Let $\Gamma = \langle S \rangle$ with S finite and symmetric.

$$\Delta := \frac{1}{2|S|}\sum_{g\in S}(1-g)^*(1-g) = 1 - \frac{1}{|S|}\sum_{g\in S} g \in \mathbb{R}[\Gamma].$$

Since $\langle \pi(\Delta)v, v\rangle = \frac{1}{2|S|}\sum \|v - \pi(g)v\|^2$,
Hence,

v is Γ-invariant iff $\pi(\Delta)v = 0$.
v is almost Γ-inv iff $\pi(\Delta)v \approx 0$.

Γ has (T) $\iff \exists\kappa > 0$ s.t. $\mathrm{Sp}(\pi(\Delta)) \subset \{0\} \cup [\kappa, \infty)$ for $\forall(\pi, \mathcal{H})$

$\iff \exists\kappa > 0$ s.t. $\Delta^2 - \kappa\Delta \geq 0$ in $\mathrm{C}^*(\Gamma)$.

$\iff \exists\kappa > 0$ s.t. $\forall\epsilon > 0$ $\Delta^2 - \kappa\Delta + \epsilon 1 \in \Sigma^2\mathbb{R}[\Gamma]$. $\lightning$ **No Good!** $\lightning$

Theorem (Oz. 2013)

Γ has (T) iff $\exists\kappa > 0$ s.t. $\Delta^2 - \kappa\Delta \in \Sigma^2\mathbb{R}[\Gamma]$ (or $\Sigma^2\mathbb{Q}[\Gamma]$).

Note that ($\Leftarrow$) is trivial. Explicit formula for $(\mathrm{SL}(3,\mathbb{Z}), S)$ has been found by Netzer and Thom with **Matlab** in May 2014.

$$\Delta^2 - \kappa\Delta = \sum_{i=1}^{n} \xi_i^*\xi_i \text{ with } \kappa = \frac{1}{120},\ n = 80, \text{ and } \operatorname{supp}\xi_i \subset (S \cup \{1\})^2.$$

$\rightsquigarrow$ Kazhdan const $\geq 1/16$. (Previous record by a human $\sim 1/700$.)

Algorithm that detects (T)

Theorem (Oz. 2013)

Γ has (T) iff $\exists\kappa > 0$ s.t. $\Delta^2 - \kappa\Delta = \sum_{i=1}^{n} \xi_i^* \xi_i$ for some $\xi_1, \ldots, \xi_n \in \mathbb{R}[\Gamma]$.

Corollary (Shalom 00)

Every (T) group is a quotient of finitely presented (T) group.

Corollary

There is a "good" algorithm that detects (T). $\rightsquigarrow$ Implementation?

Input: $\Gamma = \langle S \mid \mathcal{R} \rangle$. $\quad \rightsquigarrow Q \colon F_S \twoheadrightarrow \Gamma$.

Step 1: Take $R > 0$ and $\sim$ on $B_R = B_R(F_S)$ s.t. $x \sim y \Rightarrow Q(x) = Q(y)$.

Step 2: Use Semi-Definite Program to compute

$$\sup\{\kappa \geq 0 : \exists (a_{x,y})_{x,y} \in \mathbb{M}_{B_R}(\mathbb{R})^+ \text{ s.t. } \Delta^2 - \kappa\Delta = \sum a_{x,y} x^{-1} y \bmod \sim\}.$$

If this value is > 0, then Γ **has (T)**. Else repeat with a larger R... $\circlearrowleft$

⚠ The problem Is(T)? is semidecidable, but not decidable (Silberman), i.e., there is no a priori estimate of R that works.

Proof of Theorem

$\omega\colon \mathbb{R}[\Gamma] \to \mathbb{R}$ the unit character, $\omega(\sum a_g g) = \sum a_g$.

$I[\Gamma] = \ker\omega$, $\quad \Delta = \frac{1}{2|S|}\sum_{g\in S}(1-g)^*(1-g) \in \Sigma^2\mathbb{R}[\Gamma] \cap I[\Gamma] = \Sigma^2 I[\Gamma]$.

Lemma. $\Delta \in \operatorname{int}\Sigma^2 I[\Gamma]$ in $I[\Gamma]^{\mathrm{h}}$.

Proof.

Lemma claims that for $\forall f \in I[\Gamma]^{\mathrm{h}}$ $\exists R > 0$ s.t. $f \leq R\Delta$ (write $f \precsim \Delta$).
Since $I[\Gamma]^{\mathrm{h}}$ is spanned by $2 - x - x^{-1} = (1-x)^*(1-x)$,
it suffices to show $(1-x)^*(1-x) \precsim \Delta$ for all $x \in \Gamma$.
This is true for $x \in S$. Moreover, if true for $x$and y, then true for xy.

$$\begin{aligned}(1-xy)^*(1-xy) &= (1-x+x(1-y))^*(\text{———})\\ &\leq 2(1-x)^*(1-x) + 2(1-y)^*(1-y) \precsim \Delta. \qquad \square\end{aligned}$$

Consequently, $\overline{\Sigma^2 I[\Gamma]} = \{f \in I[\Gamma]^{\mathrm{h}} : f + \epsilon\Delta \in \Sigma^2 I[\Gamma] \text{ for } \forall\epsilon > 0\}$.

$$\begin{aligned}\Gamma \text{ has (T)} \iff \exists\kappa > 0 \text{ s.t. } \Delta^2 - \kappa\Delta &\in \overline{\Sigma^2\mathbb{R}[\Gamma]} \cap I[\Gamma] \stackrel{?}{=} \overline{\Sigma^2 I[\Gamma]}\\ \implies \Delta^2 - (\kappa-\epsilon)\Delta &\in \Sigma^2 I[\Gamma] \text{ for } \epsilon \in (0,\kappa) \qquad \square\end{aligned}$$

Proof of Theorem, completed

Γ has (T) $\iff \exists\kappa > 0$ s.t. $\Delta^2 - \kappa\Delta \in \overline{\Sigma^2 R[\Gamma]} \cap I[\Gamma] \overset{?}{=} \overline{\Sigma^2 I[\Gamma]}$

$\implies \Delta^2 - (\kappa - \epsilon)\Delta \in \Sigma^2 I[\Gamma]$ for $\epsilon \in (0, \kappa)$ $\square$

Lemma. $\overline{\Sigma^2 R[\Gamma]} \cap I[\Gamma] = \overline{\Sigma^2 I[\Gamma]}$.

Proof.

Since $\overline{\Sigma^2 R[\Gamma]} \cap I[\Gamma]$ is closed and contains $\Sigma^2 I[\Gamma]$, the inclusion $\supset$ is clear.

For the converse inclusion $\subset$, let $f \in \overline{\Sigma^2 R[\Gamma]} \cap I[\Gamma]$ be given.

By HB, it suffices to show $\phi(f) \geq 0$ for $\forall\phi\colon I[\Gamma] \to \mathbb{R}$ positive.

The function $\psi(g) = \phi(1 - g)$ is conditionally negative type on Γ,

i.e., for $\forall\xi \in I[\Gamma]$, one has

$$\psi(\xi^*\xi) = \sum_{x,y} \xi_x \xi_y \psi(x^{-1}y) = \sum_{x,y} \xi_x \xi_y \phi(1 - x^{-1}y) = -\phi(\xi^*\xi) \leq 0.$$

By Schoenberg's theorem, $\phi_t(x) = \exp(-t\psi(x))$ is positive type on Γ.

Hence ϕ_t is positive on $\mathbb{R}[\Gamma]$ and $\phi(f) = \lim_{t\downarrow 0} t^{-1}\phi_t(f) \geq 0$. $\square$

Thank You for Your Attention!