Functional Analysis Proof of Gromov's Polynomial Growth Theorem

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Introduction

Growth of a group

G finitely generated group, $G = \langle S \rangle$
S finite symmetric (i.e., $g \in S \Leftrightarrow g^{-1} \in S$) generating subset, $e \in S$
$\rightsquigarrow$ word metric $|x|_S := \min\{n : x \in S^n\}$ and $d_S(x,y) := |x^{-1}y|_S$

Definition

G has polynomial growth if $\exists d > 0$ s.t. $\limsup_n |S^n|/n^d < \infty$.
weak polynomial growth if $\exists d > 0$ s.t. $\liminf_n |S^n|/n^d < \infty$.

Note:
- independent of the choice of S
- $H \leq G$ finite index $\Rightarrow$ H and G have the same growth type

$\because$ the growth type ($|S|^n \preceq n^d$, exponential growth, etc.) is a QI-invariant.

Definition

A map $f\colon (X, d_X) \to (Y, d_Y)$ is a **quasi-isometry** (QI) if $\exists K, L > 0$ s.t.
$$\frac{1}{K} d_X(x,y) - L \leq d_Y(f(x), f(y)) \leq K d_X(x,y) + L \quad \text{and} \quad Y \subset_L f(X).$$

Homework: $H \leq_{\mathrm{f.i.}} G$ and $G = \langle S \rangle$, $H = \langle T \rangle$ $\Rightarrow$ $(G, d_S) \simeq_{\mathrm{QI}} (H, d_T)$
$\Rightarrow$ G and H has the same growth type.

Introduction

Theorem (Milnor 1968)

M complete Riem mfld with non-negative Ricci curvature
Then $\forall$ f.g. subgroup of $\pi_1(M)$ has PG.

Theorem (Milnor–Wolf 1968)

Virtually nilpotent groups (i.e. $\exists$ finite-index nilp subgroups) have PG.
Moreover $\forall$ f.g. v.solvable group is either v.nilp or exponential growth.
In fact, $\exists d \in \mathbb{N}$ s.t. $|S^n| \sim n^d$ (Bass–Guivarch).

Theorem (Tits Alternative 1972)

$G \leq \mathrm{GL}(n, F)$ f.g. linear grp $\Rightarrow$ Either G v.solv or $F_2 \leq G$ ($\rightsquigarrow$ exp growth)

Corollary: Every f.g. linear group with wPG is v.nilp.

Theorem (Gromov 1981 (van den Dries–Wilkie 1984))

Every f.g. group with wPG is v.nilp.

Gromov's Theorem

Theorem (Gromov 1981 (van den Dries–Wilkie 1984))

Every f.g. group with wPG is v.nilp.

A cornerstone result of Geometric Group Theory: a geometric condition yields an algebraic result.

Proof: Geometric.
An ultralimit of $(G, \frac{1}{K(n)} d_S)_{n=1}^{\infty}$ is a metric group, which can be arranged to be locally compact under the wPG assumption (bounded doubling).
$\rightsquigarrow$ One can apply the solution to Hilbert's 5th problem by Montgomery, Zippin, and Yamabe, and reduce the problem to a problem on a Lie group.

Other proofs: Kleiner 2007, Analytic "Elementary but Hard"

⋮ Shalom–Tao 2009, Hrushovski 2009, Breuillard–Green–Tao 2011

A new proof (2015): Functional Analytic "Soft and Simple"

The first (or the last) steps of the proof. Algebraic parts.

Recall that G is nilpotent if the lower (or upper) central series terminates:

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e\},$$

where $G_{i+1} = [G_i, G]$, i.e., $G_i/G_{i+1} = \mathcal{Z}(G/G_{i+1})$.

The first (or the last) step of the proof

- Proof is done once we know any infinite G with wPG virtually surjects onto $\mathbb{Z}$, i.e, there is a finite index subgrp $H \leq_{\text{f.i.}} G$ s.t. $H \twoheadrightarrow \mathbb{Z}$.

Proof à la Milnor–Wolf.

Let G be a f.g. group with wPG of degree d. We want to show G is v.nilp.
WMA $\exists q\colon G \twoheadrightarrow \mathbb{Z}$. Then $N := \ker q$ is f.g. of wPG of degree $\leq d-1$.
Sketch of the proof: $G = \langle t, s_1, \ldots, s_m \rangle$, $q(t) = 1$ and $q(s_i) = 0$.

$S_l := \{t^k s_i^{\pm} t^{-k} : i = 1, \ldots, m,\ k \in \mathbb{Z},\ |k| \leq l\} \cup \{e\} \ \rightsquigarrow\ N = \langle \bigcup_l S_l \rangle$

Observe that $B_l := (S_l)^l \subset S^{(2l+1)l}$ has polynomial growth (of deg $\leq 2d$).
If $\exists x \in S_{l+1} \setminus (B_l)^2$, then $xB_l \sqcup B_l \subset B_{l+1}$ and so $|B_{l+1}| \geq 2|B_l|$.
$\rightsquigarrow \exists l_0$ s.t. $S_{l_0+1} \subset (B_{l_0})^2 \subset \langle S_{l_0} \rangle$, which implies $\langle S_{l_0} \rangle = \langle \bigcup_l S_l \rangle = N$.
Moreover, $(S_{l_0} \cup \{t^{\pm}\})^{2n} \supset \bigsqcup_{|k| \leq n} t^k (S_{l_0})^n$ yields $n^d \succeq n|(S_{l_0})^n|$. □

Thus, by induction hypothesis, WMA N is nilp and $G = \langle N, t \rangle \cong N \rtimes_t \mathbb{Z}$.
We claim $\exists K \in \mathbb{N}$ s.t. the f.i. subgrp $\langle N, t^K \rangle$ is nilp.
Idea of the proof: Assume for simplicity N is f.g. abelian, $N = \mathbb{Z}^m \times F$.
$\rightsquigarrow \exists K_1$ s.t. $[F, t^{K_1}] = \{e\} \rightsquigarrow \mathrm{Ad}_{t^{K_1}} \sim A \in \mathrm{GL}_m(\mathbb{Z})$ with eigenvalues roots of unity ($\because \mathbb{Z}^m \rtimes_A \mathbb{Z}$ wPG,...) $\rightsquigarrow \exists K_2$ s.t. A^{K_2} unipotent, $K := K_1 K_2$. □

The second (or the last) step of the proof

- If G has a finite-dim (unitary) repn $G \curvearrowright^{\pi} \mathcal{H}$ with infinite image $\pi(G)$, then $\exists H \leq_{\mathrm{f.i.}} G$ s.t. $H \twoheadrightarrow \mathbb{Z}$.

This follows from Tits Alternative, but here's an elementary proof.

Proof by Shalom.

Suppose G has wPG of degree d and $G \subset \mathcal{U}(\mathcal{H})$, $\dim \mathcal{H} < \infty$. Note that $\|1-[g,h]\| = \|gh-hg\| = \|(1-g)(1-h)-(1-h)(1-g)\| \leq 2\|1-g\|\|1-h\|$.
Take $\varepsilon > 0$ small enough. One has $\langle \{g \in G : \|1-g\| < \varepsilon\} \rangle \leq_{\mathrm{f.i.}} G$.
WMA $G = \langle S \rangle$, $S \subset \{g \in G : \|1-g\| < \varepsilon\}$ and $G \subset \mathcal{U}(\mathcal{H})$ irreducible.
We claim $\dim \mathcal{H} = 1$. S'pose not: $\exists g_0 \in G \setminus \mathbb{C}1$ s.t. $\varepsilon_0 := \|1-g_0\| < \varepsilon$.
$\cdots$ $\exists s_k \in S$ s.t. $g_k := [g_{k-1}, s_k] \neq 1 \rightsquigarrow g_k \notin \mathbb{C}1$ ($\because \det g_k = 1$ and $g_k \approx 1$)
$g_0, g_1, \ldots$ are s.t. $\varepsilon_k := \|1-g_k\| < 2\varepsilon\varepsilon_{k-1}$ and $|g_k|_S \leq e^k$.
$g_0^{k_0} g_1^{k_1} \cdots g_m^{k_m}$, $m \in \mathbb{N}$, $|k_i| \leq (10\varepsilon)^{-1}$, are mutually distinct.
$\because$ Given k_l and k'_l, put $l := \min\{l : k_l \neq k'_l\}$. Then $\|g_l^{k_l} - g_l^{k'_l}\| \geq \varepsilon_l$ and

$$\|g_{l+1}^{k_{l+1}} \cdots g_m^{k_m} - g_{l+1}^{k'_{l+1}} \cdots g_{m'}^{k'_{m'}}\| \leq \sum_{k>l} \varepsilon_k \cdot \tfrac{1}{10\varepsilon} < \tfrac{1}{2}\varepsilon_l.$$

$\rightsquigarrow |\mathrm{Ball}_S(\frac{1}{10\varepsilon} m e^m)| \geq (\frac{1}{10\varepsilon})^m \rightsquigarrow |\mathrm{Ball}_S(n)| \succeq (\frac{1}{10\varepsilon})^{\frac{1}{2}\log n} = n^{\frac{1}{2}\log(\frac{1}{10\varepsilon})}$. ↯ $\square$

Digest of the first day lecture

G finitely generated group, $G = \langle S \rangle$

S finite symmetric (i.e., $g \in S \Leftrightarrow g^{-1} \in S$) generating subset, $e \in S$

$\rightsquigarrow$ word metric $|x|_S := \min\{n : x \in S^n\}$ and $d_S(x,y) := |x^{-1}y|_S$

G has weak polynomial growth if $\exists d > 0$ s.t. $\liminf_n |S^n|/n^d < \infty$.

Theorem (Gromov 1981 (van den Dries–Wilkie 1984))

Every f.g. group with wPG is virtually nilpotent.

- Proof is done once we know any infinite G with wPG virtually surjects onto $\mathbb{Z}$, i.e, there is a finite index subgrp $H \leq_{\mathrm{f.i.}} G$ s.t. $q\colon H \twoheadrightarrow \mathbb{Z}$.

$\because$ $\ker q$ is f.g. and has wPG of degree $\leq d-1$. $\rightsquigarrow$ Induction.

- If G has a finite-dim (unitary) repn $G \curvearrowright^{\pi} \mathcal{H}$ with infinite image $\pi(G)$, then $\exists H \leq_{\mathrm{f.i.}} G$ s.t. $q\colon H \twoheadrightarrow \mathbb{Z}$.

$\because$ Tits Alternative or an elementary proof by Shalom.

Day 2: **How to obtain a non-trivial finite-dim repn?**

Reduced Cohomology and Finite-Dimensional Representation from Random Walks

Harmonic 1-cocycles

Fix μ a fin-supp symm prob measure on G s.t. $G = \langle \operatorname{supp} \mu \rangle$ & $\mu(e) > 0$.
$(\pi, \mathcal{H})$ a unitary repn, given (not necessarily fin-dim).

$b\colon G \to \mathcal{H}$ 1-cocycle $\overset{\text{def}}{\Leftrightarrow}$ $b(gx) = b(g) + \pi_g b(x)$ for $\forall g, x \in G$

e.g., 1-coboundary $b_v(g) = v - \pi_g v$, where $v \in \mathcal{H}$

μ-harmonic $\overset{\text{def}}{\Leftrightarrow}$ $\sum_x b(gx)\mu(x) = b(g)$ for $\forall g \in G$ (or just $g = e$)

$\rightsquigarrow$ $\|b(x)\| \leq |x|_S \max_{s \in S} \|b(s)\|$ and $0 = b(e) = b(x^{-1}) + \pi_{x^{-1}} b(x)$ for $\forall x$

$\|b(x^{-1}y)\| = \|b(x^{-1}) + \pi_{x^{-1}} b(y)\| = \|b(x) - b(y)\|$

b is a 1-cocycle iff $\rho_g\colon v \mapsto \pi_g v + b(g)$ is an affine isometric action on $\mathcal{H}$.

$\rightsquigarrow$ b is a coboundary $\Leftrightarrow$ ρ has a fixed point $\Leftrightarrow$ b is bounded

$$Z^1(G, \pi) := \{\text{1-cocycles}\} \supset \{\text{1-coboundaries}\} =: B^1(G, \pi),$$

Z^1 is a Hilbert space w.r.t. $\|b\|_{L^2(\mu)} := (\sum_x \|b(x)\|^2 \mu(x))^{1/2}$.

$Z^1(G, \pi) = \overline{B^1(G, \pi)} \oplus B^1(G, \pi)^{\perp}$ and

$\overline{H^1}(G, \pi) := Z^1(G, \pi)/\overline{B^1(G, \pi)} \cong B^1(G, \pi)^{\perp} = \{\text{harmonic cocycles}\}$.

$\because \sum_x \langle b(x), v - \pi_x v \rangle \mu(x) = 2\langle \sum_x b(x)\mu(x), v \rangle = 0\ \forall v$ $\Leftrightarrow$ harmonic.

Shalom's property H_{FD}

Theorem H (Mok 95, Korevaar–Schoen 97, Shalom 99)

G a f.g. infinite grp of wPG (or amenable or non-(T))
Then, $\exists(\pi, \mathcal{H}, b)$ non-zero μ-harmonic 1-cocycle.

$b(gx) = b(g) + \pi_g b(x) \ \rightsquigarrow\ \overline{\mathrm{span}}\, b(G)$ is $\pi(G)$-invariant.
If $\mathcal{K}$ is a $\pi(G)$-invariant subspace, then $P_{\mathcal{K}} b$ is a (harmonic) cocycle.

Observation (Shalom): If G is v.nilp, then it has property H_{FD}.

H_{FD}: Any $(\pi, \mathcal{H})$ with $\overline{H^1}(G, \pi) \neq 0$ has a non-zero finite-dim subrepn.
Equivalently, any harmonic 1-cocycle has a finite-dim summand.

Shalom's Idea (2004): Prove "wPG $\Rightarrow H_{\mathrm{FD}}$" w/o using Gromov's Thm.
$\rightsquigarrow$ A new proof of Gromov's Thm.

$\because$
- By Theorem H and H_{FD}, $\exists(\pi, \mathcal{H}, b)$ s.t. $\pi\colon G \to \mathcal{U}(\mathcal{H})$ f.d. repn and $b\colon G \to \mathcal{H}$ non-zero harmonic cocycle (unbdd).
- If $|\pi(G)| = \infty$, then we are done.
- If $|\pi(G)| < \infty$, then b is an unbdd additive hom from $\ker \pi$ into $\mathcal{H}$.

We are left to prove Theorem H ($\rightarrow$ Day 3) and H_{FD} for wPG grps.

Proof of H_{FD}

A f.g. group G with wPG has Shalom's property H_{FD}:
Any harmonic 1-cocycle $b\colon G \to \mathcal{H}$ with π no non-zero f.d. subrepn is zero.

We want to show $\langle b(g), v\rangle = 0$ for $\forall\, g \in S$ and $v \in \mathcal{H}$.

$$\langle b(g), v\rangle = \textstyle\sum_x \langle b(gx) - b(x), v\rangle \mu^{*n}(x)$$
$$= \textstyle\sum_x \underbrace{\langle b(x), v\rangle}_{(1)} \underbrace{(g\cdot\mu^{*n} - \mu^{*n})(x)}_{(2)} \qquad (\spadesuit)$$

Lemma (1)

Let $(\pi, \mathcal{H})$ weakly mixing (i.e., no non-zero f.d. subrepn) and b harmonic.
Then, $\frac{1}{n}\sum_x |\langle b(x), v\rangle|^2 \mu^{*n}(x) \to 0.$

Note:
$$\textstyle\sum \|b(x)\|^2\mu^{*n}(x) = \sum \|b(x^{-1}y)\|^2\mu^{*n-1}(x^{-1})\mu(y)$$
$$= \textstyle\sum \|b(x) - b(y)\|^2\mu^{*n-1}(x)\mu(y)$$
$$= \textstyle\sum \|b(x)\|^2\mu^{*n-1}(x) + \|b\|^2_{L^2(\mu)} = n\|b\|^2_{L^2(\mu)}.$$

Some functional analysis (after Shalom, Chifan–Sinclair)

Lemma (1)

$(\pi,\mathcal{H})$ weakly mixing and b harmonic $\Rightarrow$ $\frac{1}{n}\sum_x |\langle b(x),v\rangle|^2\mu^{*n}(x)\to 0$.

Note that $|\langle b(x),v\rangle|^2 = \langle b(x)\otimes\bar{b}(x), v\otimes\bar{v}\rangle_{\mathcal{H}\otimes\bar{\mathcal{H}}}$.

$$\begin{aligned}\textstyle\sum_x(b(x)\otimes\bar{b}(x))\mu^{*n}(x) &= \textstyle\sum_{x,y}(b(xy)\otimes\bar{b}(xy))\mu^{*n-1}(x)\mu(y)\\ &= \textstyle\sum_{x,y}(b(x)+\pi_x b(y))\otimes(\bar{b}(x)+\bar{\pi}_x\bar{b}(y))\mu^{*n-1}(x)\mu(y)\\ &= \textstyle\sum_x(b(x)\otimes\bar{b}(x))\mu^{*n-1}(x)+T^{n-1}w\end{aligned}$$

where $T:=\sum_g(\pi_g\otimes\bar{\pi}_g)\mu(g)$ and $w:=\sum_y(b(y)\otimes\bar{b}(y))\mu(y)\in\mathcal{H}\otimes\bar{\mathcal{H}}$

$$= (1+T+\cdots+T^{n-1})w.$$

T is a self-adjoint contraction on $\mathcal{H}\otimes\bar{\mathcal{H}}$.

π w.mixing $\rightsquigarrow$ $\pi(G)'\cap\mathbb{K}(\mathcal{H})=\mathbf{0}$ $\rightsquigarrow$ no nonzero $(\pi\otimes\bar{\pi})(G)$-inv vector

$\Big(\because$ Under $\mathcal{H}\otimes\bar{\mathcal{H}}\cong\mathcal{S}_2(\mathcal{H})$, a $(\pi\otimes\bar{\pi})(G)$-invariant vector corresponds to a Hilbert–Schmidt operator which commutes with $\pi(G)$.$\Big)$

$\rightsquigarrow$ 1 is not an eigenvalue of T ($\because$ $\mathcal{H}$ is strictly convex).

$\frac{1}{n}\sum_x(b(x)\otimes\bar{b}(x))\mu^{*n}(x)=\frac{1}{n}(1+T+\cdots+T^{n-1})w\to 0$ by LDCT. □

Entropy (after Erschler–Karlsson) and QED for H_{FD}

For p prob measure, $H(p) := -\sum_x p(x)\log p(x) \geq 0$. Shannon entropy

$p \mapsto H(p)$ is concave $\because (-t\log t)'' = (-1/t) < 0$.

$$\delta(p,q) := H(\tfrac{p+q}{2}) - \tfrac{1}{2}(H(p)+H(q)) \geq \tfrac{1}{8}\sum_x \frac{|p(x)-q(x)|^2}{p(x)+q(x)}.$$

Thus for $\forall f \geq 0$ one has

$$\sum_x f(x)|p(x)-q(x)| \leq \Big(8\delta(p,q)\sum_x f(x)^2(p(x)+q(x))\Big)^{1/2}. \qquad (2)$$

Why entropy?

- Can estimate $\spadesuit := \sum_x \langle b(x), v)\rangle (g\cdot\mu^{*n} - \mu^{*n})(x)$.
- Convenient to the telescoping argument.

$H(p) = \sum_x p(x)\log(1/p(x)) \leq \log|\operatorname{supp} p|$ by concavity of log.

$\rightsquigarrow\ H(\mu^{*n}) \leq \log|\operatorname{supp}\mu^{*n}| = \log|(\operatorname{supp}\mu)^n| \preceq d\log n$ (w.r.t. $\liminf_n$)

$\mu * \nu = \sum_g \mu(g) g\cdot\nu$ and $H(\mu*\nu) \geq \sum_g \mu(g)H(g\cdot\nu) = H(\nu)$.

$\rightsquigarrow\ H(\mu*\nu) - H(\nu) \geq 2\min\{\mu(e),\mu(g)\}\,\delta(\nu, g\cdot\nu)$ for $\forall g \in S$

$\rightsquigarrow\ \liminf_n n\,\delta(\mu^{*n}, g\cdot\mu^{*n}) \leq C\liminf_n n\,(H(\mu^{*n+1}) - H(\mu^{*n})) < \infty$

$$|\spadesuit|^2 \leq 8n\delta(\mu^{*n}, g\cdot\mu^{*n})\cdot\tfrac{1}{n}\sum_x|\langle b(x),v)\rangle|^2(g\cdot\mu^{*n}+\mu^{*n})(x) \underset{\liminf}{\to} 0. \quad \square$$

Digest of the second day lecture

G finitely generated group, $G = \langle S \rangle$

S finite symmetric (i.e., $g \in S \Leftrightarrow g^{-1} \in S$) generating subset, $e \in S$

$\rightsquigarrow$ word metric $|x|_S := \min\{n : x \in S^n\}$ and $d_S(x,y) := |x^{-1}y|_S$

G has weak polynomial growth if $\exists d > 0$ s.t. $\liminf_n |S^n|/n^d < \infty$.

Theorem (Gromov 1981 (van den Dries–Wilkie 1984))

Every f.g. group with wPG is virtually nilpotent.

Theorem H (Mok 95, Korevaar–Schoen 97, Shalom 99. To be proved.)

G a f.g. infinite grp of wPG (or amenable or non-(T))
Then, $\exists(\pi, \mathcal{H}, b)$ non-zero harmonic 1-cocycle.

A f.g. group G with wPG has Shalom's property H_{FD}:
Any non-zero harmonic 1-cocycle has a non-zero finite-dim summand.

$\exists$ non-trivial f.d. cocycle $\rightsquigarrow$ $\exists$ a virtual surjection to $\mathbb{Z}$ $\rightsquigarrow$ Gromov's Thm.

Day 3: **Proof of Theorem H and further development**

Review on Amenability

Review on Amenability

Fix μ a fin-supp symm prob measure on G s.t. $G = \langle \operatorname{supp} \mu \rangle$.

A group G is amenable if it satisfies the following equivalent conditions.

- (invariant mean) $\exists \varphi \colon \ell_\infty(G) \to \mathbb{C}$ a left G-invariant state;
- (approximate invariant mean) $\exists \xi_n \in \mathrm{Prob}(G)$ approx G-invariant;
- (Hulanicki) $\exists \xi_n \in \ell_2(G)$ approx G-invariant unit vectors;
- (Kesten) $\lim_n \mu^{*2n}(e)^{1/2n} = \|\lambda(\mu)^n \delta_e\|^{1/n} = \|\lambda(\mu)\| = 1$.

Here $\lambda \colon G \curvearrowright \ell_2 G$ the left reg repn, $\lambda_g \delta_x = \delta_{gx}$, or $\lambda(\mu)\xi = \mu * \xi$.

$(\mu * \nu)(x) := (\sum_g \mu(g)\, g \cdot \nu)(x) = \sum_g \mu(g)\nu(g^{-1}x)$, $\lambda(\mu * \nu) = \lambda(\mu)\lambda(\nu)$.

! μ^{*n} may not be approx G-inv in $\mathrm{Prob}(G)$ (failure of the Liouville prty), although they are always approx G-inv in $\ell_2(G)$ after normalization.

Examples of amenable grps include finite grps, abelian grps, subgrps, quotients, extensions, inductive limits, solvable grps, subexp growth grps ($\because \mu(e)^{*2n} \geq \mu^{*2n}(g)$ for $\forall g$ and $\mu^{*2n}(e) \geq \frac{1}{|\operatorname{supp} \mu^{*2n}|} = \frac{1}{|(\operatorname{supp} \mu)^{2n}|}$).

Grigorchuk (1980/84): $\exists$ an intermediate growth group, $G = \langle S \rangle$ with $\exp(n^{0.5}) \preceq |S^n| \preceq \exp(n^{0.9})$.

Existence of harmonic cocycles

Existence of a harmonic 1-cocycle

Theorem (Mok 95, Korevaar–Schoen 97, Shalom 99)

G a f.g. infinite grp of wPG or more generally amenable (or non-(T))
Then, $\exists(\pi, \mathcal{H}, b)$ non-zero μ-harmonic 1-cocycle.

Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. $\lim_{\mathcal{U}} \colon \ell_\infty(\mathbb{N}) \to \mathbb{C}$ non-principal character
$\mathcal{H}$ Hilb space $\rightsquigarrow$ $\mathcal{H}^{\mathcal{U}} := \ell_\infty(\mathbb{N};\mathcal{H})/\{(v_n)_n : \lim_{\mathcal{U}} \|v_n\| = 0\}$ ultrapower
$\langle [v'_n]_n, [v_n]_n \rangle_{\mathcal{H}^{\mathcal{U}}} := \lim_{\mathcal{U}} \langle v'_n, v_n \rangle_{\mathcal{H}}$, $\quad \pi^{\mathcal{U}}_g [v_n]_n := [\pi_g v_n]_n$ ultrapower repn

To avoid the parity problem, we will assume $\mu^{*1/2}$ exists.

$\|\lambda(\mu)^{n/2}\delta_e\|^2 = \mu^{*n}(e) \to 0$ but $\|\lambda(\mu)^{n/2}\delta_e\|^{2/n} = \mu^{*n}(e)^{1/n} \to 1$.
$b_n(g) := \lambda(\mu^{*n/2} - g \cdot \mu^{*n/2})\delta_e = \mu^{*n/2} - g \cdot \mu^{*n/2}$ (omit writing λ).
$\gamma(n) := \|b_n\|^2_{L^2(\mu)} = \sum_g \|b_n(g)\|^2 \mu(g) = 2(\mu^{*n}(e) - \mu^{*n+1}(e))$.
$b(g) := [\gamma(n)^{-1/2} b_n(g)]_n \in (\ell_2 G)^{\mathcal{U}}$ $\rightsquigarrow$ b is normalized, i.e., $\|b\|_{L^2(\mu)} = 1$.
$\|\sum_x b(x)\mu(x)\|^2 = \lim_{\mathcal{U}} \gamma(n)^{-1} \|\mu^{*n/2} - \mu^{*n/2+1}\|^2 = \lim_{\mathcal{U}} \frac{\gamma(n)-\gamma(n+1)}{2\gamma(n)} \underset{\text{Lem}}{=} 0.$
Hence b is a normalized μ-harmonic 1-cocycle into $(\ell_2 G)^{\mathcal{U}}$. $\square$

Existence of a harmonic 1-cocycle: Proof continues

Recall that G is amenable iff $\frac{\sum_g \mu(g)\|\mu^{*n/2} - g\cdot\mu^{*n/2}\|^2}{2\|\mu^{*n/2}\|^2} = \frac{\mu^{*n}(e) - \mu^{*n+1}(e)}{\mu^{*n}(e)} \to 0$.

Lemma (A refinement of Avez's Lemma)

For $\gamma(n) = 2(\mu^{*n}(e) - \mu^{*n+1}(e))$, one has $\lim_{n\to\infty} \frac{\gamma(n+1)}{\gamma(n)} = 1$.

Proof. Recall that $\exists \mu^{*1/2}$, $\mu^{*n}(e) \to 0$, and $\mu^{*n}(e)^{1/n} \to 1$.

$\gamma(n) = 2\langle \lambda(\mu)^n (1-\lambda(\mu))\delta_e, \delta_e\rangle$ decreasing ($\because \lambda(\mu) = \lambda(\mu^{*1/2})^2 \geq 0$).
$\delta(n) := \gamma(2n) + \gamma(2n+1) = 2(\mu^{*2n}(e) - \mu^{*2(n+1)}(e))$ also decreasing.
$\delta(n+1)^2 = (\sum_g \langle \mu^{*n} - g\cdot\mu^{*n}, \mu^{*n+2} - g\cdot\mu^{*n+2}\rangle \mu^{*2}(g))^2 \leq \delta(n)\delta(n+2)$.
$\rightsquigarrow \delta(n+1)/\delta(n) \leq \delta(n+2)/\delta(n+1) \nearrow \exists \delta \leq 1$.
Thus $\gamma(n) \leq C\delta^{n/2}$ and so $2\mu^{*n}(e) = \sum_{k=n}^{\infty} \gamma(k) \leq C'\delta^{n/2} \rightsquigarrow \delta = 1$.
$\rightsquigarrow \lim_n \gamma(n+1)/\gamma(n) = 1$. □

Thus $b(g) := [\gamma(n)^{-1/2}(\mu^{n/2} - g\cdot\mu^{*n/2})]_n \in (\ell_2 G)^{\mathcal{U}}$ is a nor. μ-harm. coc.

⚠ The 1-cocycle b may depend on the choice of an ultrafilter $\mathcal{U}$.

Is it possible to tell when b is f.d. or has a f.d. summand?

Further applications: Motivations

Theorem (Shalom 2004)

H_{FD} is a QI-invariant among f.g. amenable groups.

Some motivation: Virtual nilpotency is a QI invariant by Gromov's Thm.

Conjecture (Gromov ?): Virtual polycyclicity is a QI invariant.

(Malcev–Mostow Theorem: G is v.polycyc iff it is virtually isomorphic to a (uniform) lattice in a simply connected solvable Lie group.)

Theorem (Shalom 2004)

Some groups have property H_{FD}, e.g.,
$L(F) := \mathbb{Z} \ltimes (\bigoplus_{\mathbb{Z}} F)$, $\mathrm{BS}(1,p) := \{a, t : tat^{-1} = a^p\}$, polycyclic grps,...
and many groups don't, e.g.,
$L(\mathbb{Z}) := \mathbb{Z} \ltimes (\bigoplus_{\mathbb{Z}} \mathbb{Z})$, infinite amenable + no virtual surjection onto $\mathbb{Z}$,...

Grigorchuk's Gap Conjecture: Any f.g. group of super-polynomial growth has growth rate at least $\exp(\sqrt{n})$.

Is it true: Every infinite sub-$\exp(\sqrt{n})$ group has a virtual surjection onto $\mathbb{Z}$?

Further applications of harmonic cocycle methods

X_n Random Walk associated with (G,μ),
i.e., $X_n\colon \prod(G,\mu)^{\mathbb{N}} \ni (s_k)_{k=1}^{\infty} \mapsto s_1\cdots s_n \in G$.

Theorem (Erschler–Oz.)

Let b be a normalized μ-harmonic 1-cocycle. Then,

$$\beta := \lim_{n\to\infty} \frac{1}{2}\sum_x \left| \frac{\|b(x)\|^2}{n} - 1 \right|^2 \mu^{*n}(x) = \lim_{n\to\infty} \frac{1}{2}\,\mathbb{E}\left| \frac{\|b(X_n)\|^2}{n} - 1 \right|^2$$

exists. Moreover, $\beta > 0$ iff b has a non-zero f.d. summand (of dim $\le 1/\beta$).

Corollary (Erschler–Oz.)

If G does not have property H_{FD}, then
- $\liminf_n \|\mu^{*n} - \mu^{*(1+\delta)n}\|_1 = 2$ for every $\delta > 0$.
- $\limsup_n \mathbb{P}(|X_n|_S \le c\sqrt{n}) = 0$ for some $c > 0$.

Proof.

If G fails H_{FD}, then $\exists$ a normalized μ-harmonic w.mixing 1-cocycle b.
By Theorem, $n^{-1/2}\|b(X_n)\| \to 1$ in probability. $\square$

Further applications of harmonic cocycle methods

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This gives a simple proof of property H_{FD} for many (all?) known cases.
E.g., $L(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z} \ltimes (\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z})$ has property H_{FD}.

$\because$ $\left[\begin{array}{l} \mu := \frac{1}{2}(\mu_0 + \mu_1),\ \ \mu_i \text{ standard nbhd RW on } \mathbb{Z} \text{ (resp. } \mathbb{Z}/2\mathbb{Z}\text{).} \\ Y_n \text{ the standard nbhd RW on } \mathbb{Z}.\ \text{Then } \mathbb{P}(|Y_n| \leq c\sqrt{n} \text{ for all } n) > 0. \end{array}\right.$

Recall that G is amenable iff $\frac{\sum_g \mu(g)\|\mu^{*n/2} - g\cdot\mu^{*n/2}\|^2}{2\|\mu^{*n/2}\|^2} = \frac{\mu^{*n}(e) - \mu^{*n+1}(e)}{\mu^{*n}(e)} \to 0$.

Corollary (Erschler–Oz.)

Let G be a f.g. amenable grp without virtual surjection onto $\mathbb{Z}$.
(E.g. Grigorchuk's grps, Matui–Juschenko–Monod, ...) Assume $\exists \mu^{*1/2}$.

Then,
$$\lim_{m\to\infty} \lim_{n\to\infty} \sum_g \mu^{*m}(g) \left| \frac{\mu^{*n}(g) - \mu^{*n+m}(e)}{\mu^{*n}(e) - \mu^{*n+m}(e)} \right| = 0.$$