Finite-Propagation Operators versus Quasi-Local Operators

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N. Ozawa; Embeddings of matrix algebras into uniform Roe algebras and quasi-local algebras. Preprint. arxiv:2310.03677

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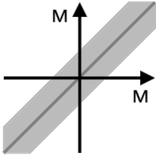
M complete Riemannian mfld & D differential op, self-adjoint on L^2M

M	$\mathrm{Spec}(D)$	$\ker D$	
compact	discrete	finite dim	operator theory
non-cpt	continuous	infinite dim	op algebra theory



$k\colon M \times M \to \mathbb{C}$ (continuous) has **finite-propagation** if

$$\exists R > 0 \ \ \mathrm{supp}\, k \subset \{(x,y) \in M \times M : \mathrm{d}(x,y) \leq R\}.$$

$(T_k f)(x) := \int_M k(x,y) f(y)\, dy$ f.p. integral op on L^2M

$\mathrm{C}^*_\rho M := \text{norm-cl}\,\{\, T_k : \|T_k\| < \infty\} \subset \mathbb{B}(L^2M)$ **Roe algebra**; $e^{itD} \in \mathrm{C}^*_\rho M$

J. Roe: Analytic indices of D take values in $K_\bullet(\mathrm{C}^*_\rho M)$.

- The isom. class of $\mathrm{C}^*_\rho M$ depends only on the "coarse structure" of the metric space (M, d).

In general, we set $\mathrm{C}^*_\rho M := \text{norm-cl}\,\{\, T : \text{loc. cpt \& f.p.}\} \subset \mathbb{B}(\mathcal{H})$, where

(M, d) proper metric space (of bounded geometry)

$\mathcal{H}$ Hilbert M-module: $C_0M \subset \mathbb{B}(\mathcal{H})$ non-deg & $C_0M \cap \mathbb{K}(\mathcal{H}) = 0$

$T \in \mathbb{B}(\mathcal{H})$ is

locally compact $\overset{\text{def}}{\iff} fT, Tf \in \mathbb{K}(\mathcal{H})$ for $f \in C_0M$

finite-propagation $\overset{\text{def}}{\iff} \exists R > 0\ \mathrm{d}(\mathrm{supp}\, f, \mathrm{supp}\, g) > R \Rightarrow fTg = 0$

(M, d) proper metric space (of bounded geometry)
$\mathcal{H}$ s.t. $C_0M \subset \mathbb{B}(\mathcal{H})$ non-deg & $C_0M \cap \mathbb{K}(\mathcal{H}) = 0$
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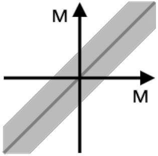


locally compact $\overset{\text{def}}{\iff}$ $fT, Tf \in \mathbb{K}(\mathcal{H})$ for $f \in C_0M$
finite-propagation $\overset{\text{def}}{\iff}$ $\exists R > 0\ \mathrm{d}(\operatorname{supp} f, \operatorname{supp} g) > R \Rightarrow fTg = 0$
quasi-local $\overset{\text{def}}{\iff}$ $\forall \varepsilon > 0\ \exists R > 0\ \mathrm{d}(\operatorname{supp} f, \operatorname{supp} g) > R \Rightarrow \|fTg\| < \varepsilon \|f\| \|g\|$

$\mathrm{C}^*_\rho M :=$ norm-cl $\{\, T : \text{loc. cpt \& f.p.} \,\}$ **Roe algebra**
$\mathrm{C}^*_\varrho M := \{\, T : \text{loc. cpt \& q.l.} \,\}$ **quasi-local algebra**

The C^*-algebras $\mathrm{C}^*_\rho M \subset \mathrm{C}^*_\varrho M$ were introduced by John Roe in 1980s.

⸮😐? $\mathrm{C}^*_\rho M \overset{?}{=} \mathrm{C}^*_\varrho M$ ⸮😐?

Is every quasi-local operator approx. finite-propagation?

- YES if M is sufficiently nice (Roe ?,...,Špakula–Zhang 2018)
- NO in general (O. 2023)

J. Roe (1980s)

If $M \cong_{\mathrm{c}} N$ (**coarsely isomorphic**), then $\mathrm{C}^*_\rho M \cong \mathrm{C}^*_\rho N$ and $\mathrm{C}^*_\varrho M \cong \mathrm{C}^*_\varrho N$.

Recall $M \cong_{\mathrm{c}} N \overset{\text{def}}{\iff} \exists\theta\colon M \to N$ a coarsely surjective map such that

$$\forall x_n, y_n \in M \ \ \mathrm{d}_M(x_n, y_n) \to \infty \Leftrightarrow \mathrm{d}_N(\theta(x_n), \theta(y_n)) \to \infty$$

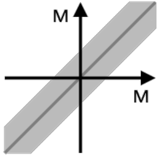
Examples

- M bounded $\Leftrightarrow M \cong_{\mathrm{c}} \{\mathrm{pt}\}$
- $\mathbb{R}^m \cong_{\mathrm{c}} \mathbb{Z}^m$, $\ \mathrm{asdim}\,\mathbb{Z}^m = m$
- M compact mfld $\Rightarrow \widetilde{M} \cong_{\mathrm{c}} \pi_1(M)$
- Γ discrete group with a word metric: $\mathrm{d}(x, y) := \ell(x^{-1}y)$

Henceforth, M is **uniformly locally finite**: $\forall R > 0\ \ \sup_x |\mathrm{Ball}(x; R)| < \infty$

and consider the **unif Roe algebra** $\mathrm{C}^*_{\mathrm{u}} M \subset \mathrm{C}^*_{\mathrm{q}} M$ instead of $\mathrm{C}^*_\rho M \subset \mathrm{C}^*_\varrho M$.

$$\mathrm{C}^*_{\mathrm{u}} M := \overline{\{T : \text{f.p.}\}}^{\|\,\|} \subset \mathrm{C}^*_{\mathrm{q}} M \subset \mathbb{B}(\ell_2 M)$$



$\rightsquigarrow$ $\mathrm{C}^*_{\mathrm{u}} M \cong \mathrm{C}^*_{\mathrm{r}} \mathcal{G}_M$ for the étale groupoid $\mathcal{G}_M$ assoc with M

e.g., $\ \mathrm{C}^*_{\mathrm{u}}\Gamma \cong \Gamma \ltimes \ell_\infty\Gamma \cong \mathrm{C}^*_{\mathrm{r}}\mathcal{G}_\Gamma \quad$ for $\mathcal{G}_\Gamma := \Gamma \ltimes \beta\Gamma$

M has **property A** $\overset{\text{def}}{\iff} \mathcal{G}_M$ amenable $\iff \cdots$

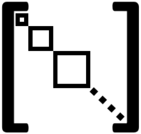
Špakula–Zhang (2018)

If M has property A, then $\mathrm{C}^*_{\mathrm{u}} M = \mathrm{C}^*_{\mathrm{q}} M$ and $\mathrm{C}^*_\rho M = \mathrm{C}^*_\varrho M$.

Baudier–Braga–Farah–Viganti–Willett (2022)

ʕ😐ʔ a von Neumann algebra $\overset{?}{\hookrightarrow} \mathrm{C}^*_{\mathrm{u}}M$ ʕ😐ʔ

$\prod \mathbb{M}_n =$

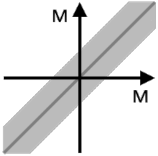


Theorem (BBFVW 2022). $L^\infty[0,1] \not\hookrightarrow \mathrm{C}^*_{\mathrm{q}}M$

Question. ¿ $\prod \mathbb{M}_n$?

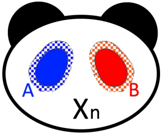
O. (2023)

- $\prod \mathbb{M}_n \not\hookrightarrow \mathrm{C}^*_{\mathrm{u}}M$ for any uniformly locally finite M
- $\prod \mathbb{M}_n \hookrightarrow \mathrm{C}^*_{\mathrm{q}}M$ if M contains expanders
- $\mathrm{C}^*_{\mathrm{u}}M \neq C^*_{\mathrm{q}}M$ and $\mathrm{C}^*_{\rho}M \neq C^*_{♀}M$ if $\exists$ expanders



Proof of the second assertion: By **expanders** we mean

X_n finite metric spaces with $|X_n| \to \infty$, $\exists R > 0\ \exists h > 1$
$|\mathcal{N}_{X_n}(E;R)| > h|E|$ for every $E \subset X_n$, $0 < |E| \leq |X_n|/2$



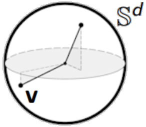
$\rightsquigarrow \forall A, B \subset X_n$ satisfy $\min\{\frac{|A|}{|X_n|}, \frac{|B|}{|X_n|}\} \leq h^{-\mathrm{d}(A,B)/2R}$.

Now, if $|X_n| \gg n$, a random rank n ortho projection P_n on $\ell_2 X_n$ satisfies

$$\max\{\|P_n|_{\ell_2 E}\| : E \subset X_n,\ \frac{|E|}{|X_n|} < \delta\} < 10\sqrt{\delta \log(1/\delta)}.$$

$$\rightsquigarrow \prod \mathbb{M}_n \cong \prod P_n \mathbb{B}(\ell_2 X_n) P_n \subset \mathrm{C}^*_{\mathrm{q}}M.$$

$\because \min\{\|P_n|_{\ell_2 A}\|, \|P_n|_{\ell_2 B}\|\}$ small whenever $\mathrm{d}(A,B)$ large. \\😐/



$\|P_E \mathbf{v}\| \approx \sqrt{|E|/d}$