Proximality and selflessness for group C*-algebras

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The (*reduced*) *group C*-algebra* of a countable discrete group Γ is

$\mathrm{C}^*_{\mathrm{r}}\Gamma := \overline{\mathbb{C}\Gamma} \subset \mathbb{B}(\ell_2\Gamma)$, where $\Gamma \curvearrowright \ell_2\Gamma$ by left translation (convolution).

$(\mathrm{C}^*_{\mathrm{r}}\mathbb{Z} \subset \mathbb{B}(\ell_2\mathbb{Z})) \cong (C(\mathbb{T}) \subset \mathbb{B}(L^2(\mathbb{T})))$ via Fourier transform $\ell_2\mathbb{Z} \cong L^2(\mathbb{T})$.
Hulanicki 66: $\mathrm{C}^*_{\mathrm{r}}\Gamma$ is not simple if $\exists$ an amenable normal subgroup $\neq \mathbf{1}$.
Powers 75: $\mathrm{C}^*_{\mathrm{r}}\mathbf{F}_d$, $d \in \{2, 3, \ldots, \infty\}$, is simple and uniquely tracial.

C*-simple, i.e., $\mathrm{C}^*_{\mathrm{r}}\Gamma$ simple $\underset{\text{Kalantar–Kennedy 17}}{\Longleftrightarrow}$ $\Gamma \curvearrowright \partial_{\mathrm{F}}\Gamma$ topo. free

$\Downarrow$ $\not\Uparrow$ (Le Boudec 17)

trivial amenable radical $\underset{\text{Breuillard–K–K–O 17}}{\Longleftrightarrow}$ $\mathrm{C}^*_{\mathrm{r}}\Gamma$ is uniquely tracial

C*-simple grps: ($R_{\mathrm{amen}} = \mathbf{1}$) linear grps, acylindrically hyperbolic grps, ...

Consider a unital simple C*-alg A with a unique tracial state τ. E.g., $\mathrm{C}^*_{\mathrm{r}}\Gamma$.

Key regularity properties that we expect A to have

- Strict Comparison: $\forall a, b \in \mathbb{M}_n(A)_+$, $\tau(\mathrm{s}(a)) < \tau(\mathrm{s}(b)) \Rightarrow a \precsim_{\mathrm{Cu}} b$.
- Stable Rank One: Invertible elements are dense in A.

Dykema–Haagerup–Rørdam 97 & D–R 00: Infinite free products. E.g., $\mathbf{F}_\infty$.
Rørdam 04: $\mathcal{Z}$-stable C*-algebras. E.g., $A \otimes \mathcal{Z}$ and $(\mathbf{F}_2)^\infty$.
(Matui–Sato 12: SC + **amenable** (nuclear) $\Rightarrow$ $\mathcal{Z}$-stable. — "Classifiable")

¿😐? $\mathbf{F}_2$, $\mathbf{F}_\infty \times \mathbf{F}_\infty$, $\mathrm{PSL}(d,\mathbb{Z})$, ... ¿😐?

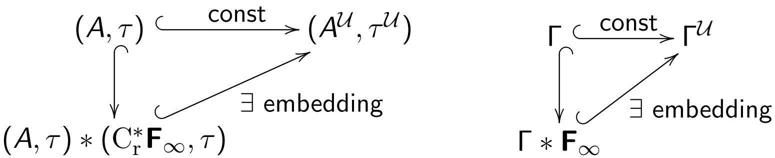
Def/Thm (Robert 23): Model theory of C*-algebras

A tracial C*-algebra (A,τ) is *selfless* if $(A,\tau) \equiv$ an infinite free product.
$\rightsquigarrow$ (A,τ) is simple, uniquely tracial, strict comparison, stable rank one, etc.

$\mathcal{U}$ a free ultrafilter on $\mathbb{N}$, i.e., $\lim_{\mathcal{U}} \colon \ell_\infty(\mathbb{N}) \to \mathbb{C}$ $*$-hom extending $\lim_{n\to\infty}$.
$A^{\mathcal{U}} := \ell_\infty(\mathbb{N},A)/\{(a_n)_n : \lim_{\mathcal{U}} \|a_n\| = 0\}$, $\tau^{\mathcal{U}}([a_n]_n) := \lim_{\mathcal{U}} \tau(a_n)$.

⚠ $\tau^{\mathcal{U}}$ is not faithful on $A^{\mathcal{U}}$. ⚠ In particular $\mathrm{C}^*_{\mathrm{r}}(\Gamma^{\mathcal{U}}) \not\hookrightarrow (\mathrm{C}^*_{\mathrm{r}}\Gamma)^{\mathcal{U}}$.

Def: (A,τ) is *selfless* if $\exists a \in A^{\mathcal{U}}$ "**strongly free**" from A in $(A^{\mathcal{U}},\tau^{\mathcal{U}})$.



Def: Γ is *MIF* (mixed identity free) if $\exists z \in \Gamma^{\mathcal{U}} \setminus \mathbf{1}$ free from Γ in $\Gamma^{\mathcal{U}}$.

- C*-selfless $\Rightarrow$ C*-simple $\Rightarrow$ $R_{\mathrm{amen}} = \mathbf{1}$. **Converse?**
- MIF $\not\Rightarrow$ C*-selfless. $\because$ $\exists$ MIF amenable groups.
- $\Gamma = \Gamma_1 \times \Gamma_2$ not MIF. $\because$ Fix $g_i \in \Gamma_i \setminus \mathbf{1}$. Then $\forall x \in \Gamma$ $[[x,g_1],g_2] = 1$.

Theorem (O. 25):

- Unital simple uniquely tracial **amenable** $\mathcal{Z}$-stable C*-alg's are selfless.
- Groups with **property** $\mathbf{P_{PHP}}$ are C*-selfless.
  - $\rightsquigarrow$ Acylindrically hyperbolic grps with $R_{\text{amen}} = \mathbf{1}$. (Generalizing Amrutam–Gao–Kunnawalkam Elayavalli–Patchell 24.)
  - All non-elementary amalgamated free products with $R_{\text{amen}} = \mathbf{1}$.
  - Zariski dense subgroups of $\mathrm{PSL}(d, \mathbb{R})$. (Gen'd by Vigdorovich 26 for all linear groups with $R_{\text{amen}} = \mathbf{1}$.)
  - Closed under direct products.

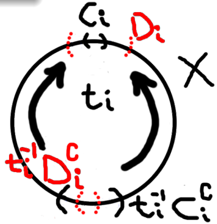
OPEN: ¿☺? C*-simple ⇒ C*-selfless ¿☺?

Def: Γ has property $\mathrm{P_{PHP}}$ if $\exists \Gamma \curvearrowright X$ (or equiv. $X = \Gamma$) $\forall F \Subset \Gamma\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ \exists t_i \in \Gamma\ \exists C_i \subset D_i \subset X$, $i \in [n]$, such that the members of

$$\{aC_i : a \in F,\ i \in [n]\} \cup \{bt_j^{-1} D_j^{\complement} : b \in F,\ j \in [n]\}$$

are mutually disjoint and that

$$\sup_{x \in \Gamma} |\{i : x \in D_i\} \cup \{j : x \in t_j^{-1} C_j^{\complement}\}| \leq \varepsilon n^{1/2}.$$



E.g., imagine $t = \begin{bmatrix} 2 & & \\ & 1 & \\ & & 1/2 \end{bmatrix}^{\infty}$ on $\mathbb{RP}^2$ maps $\begin{bmatrix} 0 \\ * \\ * \end{bmatrix}^{\complement} \subset \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}^{\complement}$ to $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \subset \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$.

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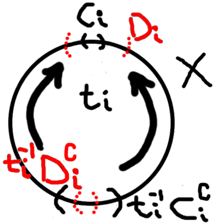

Claim. $\frac{1}{\sqrt{2n}} \sum_{i=1}^n (t_i + t_i^*) \in \mathrm{C}^*_{\mathrm{r}}\Gamma =: A$ is strongly asymp. free from A.

$P_i \leq Q_i$ (resp. $P'_j \leq Q'_j$) proj's on $\ell_2\Gamma$ for $C_i \subset D_i$ (resp. $t_j^{-1}D_j^{\complement} \subset t_j^{-1}C_j^{\complement}$).

$\rightsquigarrow$ P_i and P'_j "very small" (mutually $\perp$) and Q_i and Q'_j "quite small".

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$$T := \frac{1}{\sqrt{2n}} \sum_i (P_i t_i + P'_i t_i^{-1}) = \frac{1}{\sqrt{2n}} \sum_i (P_i t_i + (Q_i^{\perp} t_i)^*) \in \mathbb{B}(\ell_2\Gamma).$$

satisfies $\|T^*T - 1\| \approx 0$. Moreover, $T^*aT = 0 = \tau(a)$ for $a \in F \setminus \{1\}$.

$\rightsquigarrow$ a suit. limit $\tilde{T} \in \mathbb{B}(\ell_2\Gamma)^{\mathcal{U}}$ satisfies $\tilde{T}^* a \tilde{T} = \tau(a)$ for $\forall a \in A \subset \mathbb{B}(\ell_2\Gamma)^{\mathcal{U}}$.

$\rightsquigarrow$ $\mathrm{C}^*(A, \tilde{T}) \subset \mathbb{B}(\ell_2\Gamma)^{\mathcal{U}}$ is isom to the **Toeplitz alg** $(A, \tau) * (\mathcal{T}, \omega_0)$ over A.

⚠ **But $\tilde{T} \in \mathbb{B}(\ell_2\Gamma)^{\mathcal{U}}$ is an isometry and $\tilde{T} \notin A^{\mathcal{U}}$.** ☹

💡 $\tilde{T} + \tilde{T}^* = [\frac{1}{\sqrt{2n}} \sum_i (t_i + t_i^{-1})]_{F,\varepsilon} \in A^{\mathcal{U}}$. !☺! □

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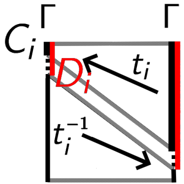

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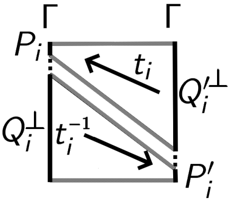

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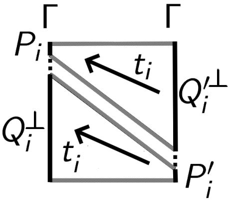

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