Uniform Amenability at Infinity

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May 2026

$\Gamma \curvearrowright \Gamma$ by left translation $\rightsquigarrow (sf)(x) = f(s^{-1}x)$ for a function f on Γ.
$\operatorname{Prob}\Gamma \subset \ell_1\Gamma$, equipped with the norm topology (= pointwise conv topo).
$\lambda\colon \Gamma \curvearrowright \ell_2\Gamma$ left reg repn, $\lambda(s)f = sf \rightsquigarrow \lambda\colon \mathbb{C}\Gamma \to \mathbb{B}(\ell_2\Gamma)$, $\lambda(f)g = f * g$.
$\mathrm{C}^*_\lambda\Gamma :=$ norm-closure $\lambda(\mathbb{C}\Gamma) \subset \mathbb{B}(\ell_2\Gamma)$, the reduced group C*-algebra.

Theorem (von Neumann, Reiter, Hulanicki, Følner, Kesten, . . .)

Γ is *amenable* if the following equivalent conditions hold.

- $\exists$ Γ-invariant finitely additive prob measure on Γ (*invariant mean*); i.e., $\exists\ m\colon \ell_\infty\Gamma \to \mathbb{C}$ positive unital linear functional that is Γ-invariant.
- $\exists\ \mu_i \in \operatorname{Prob}\Gamma$ such that $\forall s\ \lim_i \|\mu_i - s\mu_i\|_1 = 0$ (*approx inv mean*); i.e., $\forall F \Subset \Gamma\, \forall\varepsilon > 0\, \exists\mu \in \operatorname{Prob}\Gamma.\ \sum_{s\in F} \|\mu - s\mu\|_1 < \varepsilon$.
- $\|\lambda(\mu)\| = \lim_n \sqrt[2n]{\mu^{*2n}(1)} = 1$ for $\forall\mu \in \operatorname{Prob}\Gamma$, symmetric.

The class of amenable groups is stable under:
subgroups, quotients, extensions, directed unions.
$\rightsquigarrow$ finite grps, (v.) solvable grps, $\mathrm{Sym}(\infty) := \bigcup_n \mathrm{Sym}(n)$, $\mathrm{Alt}(5) \wr \mathbb{Z}$.
Non-cyclic free groups $\mathbf{F}$ are not amenable (Banach–Tarski Paradox).

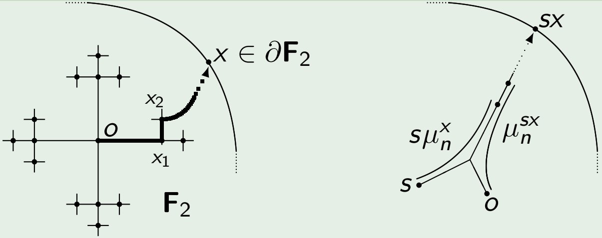
Many interesting groups are not amenable!

Definition (Anantharaman-Delaroche, AD–Renault (after Zimmer))

An action $\Gamma \curvearrowright X$ on a cpt topo space X is *topologically amenable* if $\exists \mu_i \colon X \to \operatorname{Prob}\Gamma$ **continuous** such that $\forall s$ $\lim_i \sup_{x\in X} \|\mu_i^{sx} - s\mu_i^x\|_1 = 0$.

- $\mu \colon X \ni x \mapsto \mu^x \in \operatorname{Prob}\Gamma$ is continuous iff $\forall t$ $x \mapsto \mu^x(t)$ is continuous. Moreover, w.m.a. by perturbation $\exists E \Subset \Gamma$ s.t. $\operatorname{supp}\mu^x \subset E$ for $\forall x$.
- If Γ is amenable, then every $\Gamma \curvearrowright X$ is topo amenable.
- If $\Gamma \curvearrowright X$ topo amen and $\operatorname{Prob}^\Gamma X \neq \emptyset$ (e.g., $X = \{\mathrm{pt}\}$), then Γ amen.

Prototype (Topo amenability of $\mathbf{F}_r \curvearrowright \partial\mathbf{F}_r \subset \{g_1, g_1^{-1} \ldots, g_r, g_r^{-1}\}^{\mathbb{N}}$)



For $x \in \partial\mathbf{F}_r$, put $\mu_n^x :=$ unif prob on the first n-segment of $[o, x]$. Then

$$\|\mu_n^{sx} - s\mu_n^x\|_1 \leq \frac{2|s|}{n}.$$

□

$\Gamma \curvearrowright X$ t.a. $\Leftrightarrow_{\mathrm{def}} \exists \mu_i \colon X \to \operatorname{Prob}\Gamma$ cts s.t. $\forall s\ \lim_i \sup_x \|\mu_i^{sx} - s\mu_i^x\|_1 = 0$.

- If $\Gamma \curvearrowright X$ is topo amen and $Y \twoheadrightarrow_\Gamma X$ (i.e. $C(X) \hookrightarrow C(Y)$), then $\Gamma \curvearrowright Y$.
- If $\exists\, \Gamma \curvearrowright X$ topo amen, then $\Gamma \curvearrowright \beta\Gamma$ topo amen ($\Leftrightarrow \Gamma \curvearrowright \ell_\infty\Gamma$ amen).

Theorem (Anantharaman-Delaroche, Kirchberg, Pisier, Wassermann, O., Guentner–Kaminker, Dykema, Adams, Guentner–Higson–Weinberger,. . .

Γ is *exact*, a.k.a. *amenable at infinity*, if the following equiv cond hold.

- $\exists\, \Gamma \curvearrowright X$ topo amen, or equiv $\Gamma \curvearrowright \ell_\infty\Gamma$ is amenable.
- $\forall F \Subset \Gamma\, \forall \varepsilon > 0\, \exists E \Subset \Gamma\, \exists \mu \colon \Gamma \to \operatorname{Prob}\Gamma$ such that $\operatorname{supp} \mu^x \subset E$ and
$$\sum_{s \in F} \sup_{x \in \Gamma} \|\mu^{sx} - s\mu^x\|_1 < \varepsilon.$$
- $\mathrm{C}^*_\lambda\Gamma$ is *exact*, that is, $(\,\cdot\,) \otimes \mathrm{C}^*_\lambda\Gamma$ is exact.

The class of exact groups contains hyperbolic groups, linear groups, etc.

The class of exact groups is stable under:
subgroups, ~~quotients~~, extensions, directed unions,
amalgamated free products, HNN extensions.

Exactness is an important notion in Analytic Group Theory that has applications to the Baum–Connes conjecture (Yu, Tu, Higson, Kasparov, Roe,. . .), classification theory for group vN algebras, etc.

How can we estimate $\|\lambda(f)\|$?

Spectral radius formula

$$\|\lambda(f)\| = \lim_n \sqrt[2n]{(f^* * f)^{*2n}(1)}$$

requires global info and is not "computable" in the algorithmic sense.

For $E \Subset \Gamma$, let $\Phi_E \colon \mathbb{B}(\ell_2\Gamma) \to \mathbb{B}(\ell_2 E)$ denote the compression, given by

$$\Phi_E(T) := P_{\ell_2 E} T|_{\ell_2 E}.$$

A group Γ is exact iff $\forall F \Subset \Gamma \, \forall \varepsilon > 0 \, \exists E \Subset \Gamma \, \exists \mu \colon \Gamma \to \mathrm{Prob}\, E$ such that

$$\sum_{s \in F} \sup_{x \in \Gamma} \|\mu^{sx} - s\mu^x\|_1 < \varepsilon.$$

$$\rightsquigarrow \quad (1-\varepsilon)\|\lambda(f)\| \le \|\Phi_E(\lambda(f))\|_{\mathbb{B}(\ell_2 E)} \le \|\lambda(f)\|$$

for every $f \in \mathrm{C}^*_\lambda\Gamma$ with $\mathrm{supp}\, f \subset F$ (and $f \in A \otimes \mathrm{C}^*_\lambda\Gamma$ with A arbitrary).

$\because$ There is $\Psi_\mu \colon \mathbb{B}(\ell_2 E) \to \mathbb{B}(\ell_2\Gamma)$ s.t. $\sum_{s \in F} \|\Psi_\mu(\Phi_E(\lambda(s))) - \lambda(s)\| < \varepsilon$, which is implemented by an isometry $V \colon \ell_2\Gamma \to \ell_2 E \otimes \ell_2\Gamma$;

$$V\delta_x := \sum_t \mu^x(xt^{-1})^{1/2}\delta_{xt^{-1}} \otimes \delta_t \quad \text{and} \quad \Psi_\mu(T) := V^*(T \otimes 1)V. \qquad \square$$

$\rightsquigarrow$ One can recover $\lambda(F)$ on $\ell_2\Gamma$ from the partially def'd action $F \curvearrowright E$ and $\|\lambda(f)\|$ is "computable" modulo μ.

$\mathcal{U}\subset\mathfrak{P}(\mathbb{N})$ a free ultrafilter on $\mathbb{N}$ (all arguments indep of the choice of $\mathcal{U}$)

- $\mathbb{N}\in\mathcal{U}$ • $A\in\mathcal{U}$, $A\subset B \Rightarrow B\in\mathcal{U}$ • $\forall A.\ A\in\mathcal{U}$ or $A^{\mathrm{c}}\in\mathcal{U}$
- $\mathcal{U}$ is *free* if $\nexists n$ such that $(A\in\mathcal{U}\Leftrightarrow n\in A)$.

$\rightsquigarrow$ a hom $\mathrm{Lim}_{\mathcal{U}}\colon \ell_\infty\mathbb{N}\to\mathbb{C}$; $a=\mathrm{Lim}_{\mathcal{U}}\, a_n$ iff $\{n : |a_n-a|<\varepsilon\}\in\mathcal{U}$.

$X^{\mathcal{U}} := \prod_{\mathbb{N}} X/\sim$, where $(x_n)_n\sim(y_n)_n$ if $\{n : x_n=y_n\}\in\mathcal{U}$.
Write $[x_n]_n\in X^{\mathcal{U}}$ for the equiv class for $(x_n)_n$.

- $\exists E_n\Subset X$ w. $|E_n|=k$ and $E=\{[x_n]_n : x_n\in E_n\}\Leftrightarrow E\Subset X^{\mathcal{U}}$ w. $|E|=k$.

Theorem (Keller, Bożejko, Wysoczański, . . .)

Γ is *uniformly amenable* if the following equivalent conditions hold.

- $\Gamma^{\mathcal{U}}$ is amenable.
- $\exists k\colon\mathbb{N}\times\mathbb{R}_+\to\mathbb{N}$ (*modulus of u.a.*) $\forall F\Subset\Gamma\,\forall\varepsilon>0\,\exists\mu\in\mathrm{Prob}\,\Gamma$
  $|\operatorname{supp}\mu|\le k(|F|,\varepsilon)$ and $\sum_{s\in F}\|\mu-s\mu\|_1<\varepsilon$.
- $\lim_n \sqrt[2n]{\nu^{*2n}(1)}=1$ *uniformly* for $\nu\in\mathrm{Prob}\,\Gamma$ symmetric, $|\operatorname{supp}\nu|$ bdd.

The class of uniformly amenable groups is stable under:
subgroups, quotients, extensions, ~~directed unions~~.
$\rightsquigarrow$ finite grps, (v.) solvable grps, ~~$\mathrm{Sym}(\infty):=\bigcup_n\mathrm{Sym}(n)$~~, $\mathrm{Alt}(5)\wr\mathbb{Z}$.

OPEN: ¿ Γ amen and $\Gamma^{\mathcal{U}}$ free-subgroup-free $\Rightarrow$ Γ unif amen ?

When $\Gamma^{\mathcal{U}}$ exact?

Γ hyperbolic (Kharlampovich–Myasnikov & Sela)
or linear, i.e., subgroups of $\mathrm{GL}_d(K)$ (Guentner–Higson–Weinberger).

This fact alone does **not** seem useful. We need a stronger property.

Theorem (following the traditional recipe)

Γ is *uniformly exact*, or *u.a. at infinity*, if the following equiv cond hold.

- $\Gamma^{\mathcal{U}} \curvearrowright (\ell_\infty\Gamma)^{\mathcal{U}}$ amenable (as opposed to $\Gamma^{\mathcal{U}} \curvearrowright \ell_\infty(\Gamma^{\mathcal{U}})$).
- $\exists k\colon \mathbb{N}\times\mathbb{R}_+ \to \mathbb{N}$ (*modulus of u.e.*) $\forall F \Subset \Gamma\, \forall\varepsilon>0\, \exists\mu\colon \Gamma\to \operatorname{Prob}\Gamma$ $|\bigcup_{x\in\Gamma}\operatorname{supp}\mu^x| < k(|F|,\varepsilon)$ and $\sum_{s\in F}\sup_{x\in\Gamma}\|\mu^{sx}-s\mu^x\|_1<\varepsilon$.
- The same as above, but $\bigcup_{x\in\Gamma}\operatorname{supp}\mu^x \subset (F\cup\{1\}\cup F^{-1})^{k(|F|,\varepsilon)}$.
- $A^{\mathcal{U}}\otimes \mathrm{C}^*_\lambda(\Gamma^{\mathcal{U}}) \hookrightarrow (A\otimes\mathrm{C}^*_\lambda\Gamma)^{\mathcal{U}}$ for every C*-algebra A.

For a C*-algebra A, the ultrapower is

$$A^{\mathcal{U}} := \ell_\infty(\mathbb{N},A)/\{(a_n)_n : \mathrm{Lim}_{\mathcal{U}}\|a_n\|=0\}.$$

- $(\ell_\infty\Gamma)^{\mathcal{U}} \subset \ell_\infty(\Gamma^{\mathcal{U}})$, $[f_n]_n([x_n]_n) := \mathrm{Lim}_{\mathcal{U}} f_n(x_n)$, as “internal” functions, but $(\mathrm{C}^*_\lambda\Gamma)^{\mathcal{U}}$ on $(\ell_2\Gamma)^{\mathcal{U}}$ and $\mathrm{C}^*_\lambda(\Gamma^{\mathcal{U}})$ on $\ell_2(\Gamma^{\mathcal{U}})$ are *a priori* unrelated.

Conjecture: Hyperbolic groups and linear groups are uniformly exact.

Theorem (O. 2026)

Free groups **F** are uniformly exact.

- $\exists k\colon \mathbb{N}\times\mathbb{R}_+\to\mathbb{N}$ (*modulus of u.e.*) $\forall f\in \mathrm{C}^*_\lambda\Gamma$ with $\operatorname{supp} f\subset F\Subset\Gamma$

$$(1-\varepsilon)\|\lambda(f)\|\leq\|\Phi_E(\lambda(f))\|_{\mathbb{B}(\ell_2 E)}\leq\|\lambda(f)\|$$

  for $E=(F\cup\{1\}\cup F^{-1})^{k(|F|,\varepsilon)}$. Unif conv for the spectral radius formula

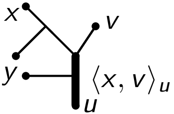
$$\|\lambda(f)\|=\lim_n\sqrt[2n]{(f^**f)^{*2n}(1)}.$$

- An amenable group is uniformly exact iff it is uniformly amenable.
  $\rightsquigarrow$ $\mathrm{Sym}(\infty)$ is not uniformly exact.
- The class of u.e. grps is stable under subgrps, extensions, free products.
  $\because\quad(\Gamma/\Delta)^{\mathcal{U}}\cong\Gamma^{\mathcal{U}}/\Delta^{\mathcal{U}};\quad \mathbf{1}\to\mathbf{F}\to\Gamma*\mathbb{Z}\to\Gamma\to\mathbf{1}.$
- The class of u.e. grps having the same m.o.u.e. is closed in the space of marked grps. Convergence of a seq of marked grps having the same m.o.u.e. implies *strong* convergence in the operator algebraic sense.
  $\rightsquigarrow$ Limit grps are the strong limit of free grps (Louder–Magee 2022).

OPEN: ¿ Amalgamated free products and HNN extensions ?

- A sketch of the proof of u.e. for a f.g. free group **F**.

F with the standard metric is a *tree* and $\mathbf{F}^{\mathcal{U}}$ is a $\mathbb{Z}^{\mathcal{U}}$-*tree*; $\mathrm{d}_{\mathcal{U}}(x,y) := [\mathrm{d}(x_n,y_n)]_n \in \mathbb{Z}^{\mathcal{U}}$ for $x=[x_n]_n$ and $y=[y_n]_n$.



$$\langle x,v\rangle_u := \tfrac{1}{2}(\mathrm{d}(x,u)+\mathrm{d}(v,u)-\mathrm{d}(x,v)). \qquad \langle x,v\rangle_u \ge \langle x,y\rangle_u \wedge \langle y,v\rangle_u$$

$\Lambda := \mathbb{Z}^{\mathcal{U}}$ is an ordered abelian group.
For $a,b\in\Lambda_+$, write $b\preceq a$ if $\exists N\ b<Na$; and $b\ll a$ if $b\preceq a$ and $a\not\preceq b$.
For $a\in\Lambda_+$, $\exists!\ \varepsilon^a\colon\{b : |b|\preceq a\}\to\mathbb{R}$, $a\mapsto 1$. One has $\varepsilon^a(b)=\mathrm{Lim}_{\mathcal{U}}\frac{b_n}{a_n}$.

- For amenability of $\mathbf{F}^{\mathcal{U}} \curvearrowright (\ell_\infty\mathbf{F})^{\mathcal{U}}$, we may pass to a f.g. $\Gamma=\langle F\rangle\le\mathbf{F}^{\mathcal{U}}$.

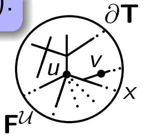
Fix $z=[z_n]_n\in\mathbf{F}^{\mathcal{U}}$ and set $c:=\sum_{s\in F}\mathrm{d}_{\mathcal{U}}(z,sz)=[\sum_{s\in F}\mathrm{d}(z_n,s_nz_n)]_n\in\Lambda$.

$$\mathbf{T} := \{x\in\mathbf{F}^{\mathcal{U}} : \mathrm{d}(x,z)\preceq c\}/{\sim_{\mathrm{d}}} \text{ with } \mathrm{d}(x,y):=\varepsilon^c(\mathrm{d}_{\mathcal{U}}(x,y))\in\mathbb{R}$$

is an $\mathbb{R}$-*tree* on which Γ acts isometrically.

$h_{u,v}(x) := \frac{1}{\mathrm{d}(u,v)}\langle x,v\rangle_u$ on **T** and $\mathcal{C}(\mathbf{T}) := \mathrm{C}^*(\{h_{u,v} : u,v\in\mathbf{T}\})$.

The Gelfand spectrum $\hat{\mathbf{T}}$ of $\mathcal{C}(\mathbf{T})$ is the *tree compactification*.
$\rightsquigarrow$ $\hat{\mathbf{T}}=\mathbf{T}\cup\partial\mathbf{T}$, for the visual boundary $\partial\mathbf{T}$, and with topo gen'd by $\{x\in\hat{T} : \langle x,v\rangle_u>0\}=\{x\in\hat{T} : u\notin[v,x]\}$.



$h_{u,v}(x)=\varepsilon^{\mathrm{d}(u,v)}(\langle x,v\rangle_u)$ is def'd on $\mathbf{F}^{\mathcal{U}}$ and $h_{u,v}=[h_{u_n,v_n}]_n\in(\ell_\infty\mathbf{F})^{\mathcal{U}}$.

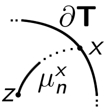
Key obs: $\Gamma = \langle F \rangle \curvearrowright \hat{\mathbf{T}}$ is topo amen modulo stabilizers at branch points.

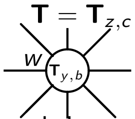
Theorem (Anantharaman-Delaroche, AD–Renault)

An action $\Gamma \curvearrowright X$ of a **c'ble** grp Γ on a **cpt** space X is topo amen iff $\exists \mu_i \colon X \to \operatorname{Prob}\Gamma$ **Borel**. $\forall s \forall m \in \operatorname{Prob} X$ $\lim_i \int \|\mu_i^{sx} - s\mu_i^x\|_1 \, dm(x) = 0$.

We may pass to a subtree and assume that $\mathbf{T}$ is a separable $\mathbb{R}$-tree.

- The countable Borel equiv rel $\mathcal{R}_{\Gamma \curvearrowright \mathbf{T}}$ is $\mathcal{M}$-amenable.
  $\because$ $\mathcal{R}_{\Gamma \curvearrowright \mathbf{T}}$ is a countable directed union of interval exchange trans o.e.'s.
- Except for branch points, stabilizer subgrps are amenable.
  $\because$ Arc stabilizers are abelian for limit $\mathbb{R}$-trees.
- $\{$branch points$\}$ is countable and $\{$amenable subgrps of $\Gamma\}$ is countable.
  (Kharlampovich–Myasnikov & Sela)
  $\rightsquigarrow \Gamma \curvearrowright \hat{T} \setminus \{$branch points$\}$ is Borel amenable.
- Amenability of $\Gamma \curvearrowright (\ell_\infty \mathbf{F})^{\mathcal{U}}$ reduces to that for





$\Gamma^w = \{t \in \Gamma : tw = w \text{ in } \mathbf{T}\} \curvearrowright \{y \in \mathbf{F}^{\mathcal{U}} : \mathrm{d}_{\mathcal{U}}(y, w) \ll c\}^\wedge$ for each b.p. w.

- We can make this process terminate as $\mathbf{F}$ has *tame geom at infinity*. ☐

For (G, ℓ) and $s, t \in G^{\mathcal{U}}$, set $s \approx t$ if $\exists N$ $\frac{1}{N}\ell^{\mathcal{U}}(s) < \ell^{\mathcal{U}}(t) < N\ell^{\mathcal{U}}(s)$ in $\mathbb{Z}^{\mathcal{U}}$.
G has t.g.a.i. if $\forall d$ $\exists r$ $\forall d$-generated subgrp $\Gamma \leq G^{\mathcal{U}}$. $|\Gamma/{\approx}| \leq r$.

¿Estimate for the modulus of u.e.? ¿ Hyperbolic groups ?