Problem set 1, submission of solutions **NOT** required

The **Problem**s below will be discussed in the tutorial on 21.09.2012.
(The **Exercise** is additional and will be discussed only if time permits.)

Problem 0.1. (1) Let $A \subset [-\infty, \infty]$ be non-empty. Prove that $\sup(-A) = -\inf A$, where $-A := \{-a \mid a \in A\}$.
(2) Let $\{a_n\}_{n=1}^{\infty} \subset [-\infty, \infty]$. Prove that $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.

Problem 0.2. Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset [-\infty, \infty]$.
(1) Suppose $a_n \leq b_n$ for any $n \in \mathbb{N}$. Prove that

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n \qquad \text{and} \qquad \liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

(2) Suppose that $\{\limsup_{n\to\infty} a_n, \limsup_{n\to\infty} b_n\} \neq \{\infty, -\infty\}$ and that $\{a_n, b_n\} \neq \{\infty, -\infty\}$ for any $n \in \mathbb{N}$. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{0.14}$$

and that the equality holds in (0.14) if $\lim_{n\to\infty} a_n$ exists in $[-\infty, \infty]$. Give an example of $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset [0, 1]$ for which the strict inequality holds in (0.14).

Problem 1.1. Let $X := \{1, 2, 3\}$. Provide all σ-algebras in X.

Problem 1.2. For a set X and $A \subset X$, prove that $\{\emptyset, A, A^c, X\}$ is a σ-algebra in X.

The notion of independence is very important in probability theory. The following definitions, problems and exercises provide some basics about independence of events.

Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
(1) A pair $\{A, B\}$ of events $A, B \in \mathcal{F}$ is called *independent* if and only if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$.
(2) A (possibly infinite) family $\{A_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{F}$ of events is called *independent* if and only if it holds that $\mathbb{P}\big[\bigcap_{\lambda\in\Lambda_0} A_\lambda\big] = \prod_{\lambda\in\Lambda_0} \mathbb{P}[A_\lambda]$ for any non-empty finite $\Lambda_0 \subset \Lambda$.

Problem 1.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
(1) Let $A, B \in \mathcal{F}$. Prove that if $\{A, B\}$ is independent then $\{A^c, B\}$, $\{A, B^c\}$ and $\{A^c, B^c\}$ are also independent.
(2) Let $\{A_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{F}$ be a (possibly infinite) family of events. Prove that $\{A_\lambda\}_{\lambda\in\Lambda}$ is independent if and only if $\mathbb{P}\big[\bigcap_{\lambda\in\Lambda_0} B_\lambda\big] = \prod_{\lambda\in\Lambda_0} \mathbb{P}[B_\lambda]$ for any non-empty finite $\Lambda_0 \subset \Lambda$ and any $B_\lambda \in \{\emptyset, A_\lambda, A_\lambda^c, \Omega\}$, $\lambda \in \Lambda_0$.

Problem 1.4. Give an example of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and events $A, B, C \in \mathcal{F}$ such that the pairs $\{A, B\}$, $\{B, C\}$ and $\{A, C\}$ are independent but $\mathbb{P}[A \cap B \cap C] \neq \mathbb{P}[A]\mathbb{P}[B]\mathbb{P}[C]$. (Consider $\Omega := \{1, 2, 3, 4\}$ and $\mathbb{P}[A] := \#A/4$, $A \subset \Omega$.)

Exercise 1.5. Give an example of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and events $A, B, C \in \mathcal{F}$ such that $\{A, B\}$ and $\{B, C\}$ are independent, $\mathbb{P}[A \cap B \cap C] = \mathbb{P}[A]\mathbb{P}[B]\mathbb{P}[C]$ but $\{A, C\}$ is not independent. (Consider $\Omega := \{1, \ldots, 16\}$ and $\mathbb{P}[A] := \#A/16$, $A \subset \Omega$.)