Problem set 10, submit solutions by 21.11.2012

The **Problem**s below will be discussed in the tutorial on 23.11.2012.
(The **Exercise**s are additional and will be discussed only if time permits.)

In the problems and the exercise below, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and all random variables are assumed to be defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Problem 3.9. Let X, Y be independent geometric random variables of parameter $1/2$. Let $k \in \mathbb{N} \cup \{0\}$. Calculate the following probabilities:

$$\text{(i)} \quad \mathbb{P}[\min\{X, Y\} \leq k] \quad \text{(ii)} \quad \mathbb{P}[X < Y] \quad \text{(iii)} \quad \mathbb{P}[X = Y]$$

Problem 3.10. Let X be a real random variable with $X \sim \mathrm{Unif}(0, \pi/2)$ and set $Y := \sin X$. Find the following quantities:

$$\text{(i)} \quad \text{a density of } Y \quad \text{(ii)} \quad \mathbb{E}[Y] \quad \text{(iii)} \quad \mathrm{var}(Y)$$

Exercise 3.11 (Not difficult – **DO NOT SKIP!**)**.** Define $\rho : \mathbb{R}^2 \to [0, \infty)$ by

$$\rho(x, y) := \frac{1}{2}(x + y)e^{-x-y}\mathbf{1}_{(0,\infty)^2}(x, y). \tag{3.81}$$

(1) Prove that $\int_{\mathbb{R}^2} \rho(z)dz = 1$, so that $\mu := \rho \cdot \mathrm{m}_2$ is a probability law on $\mathbb{R}^2$.
(2) Let X, Y be real random variables with $(X, Y) \sim \mu$. Find the following quantities:

$$\begin{array}{llll} \text{(i)} \quad \text{a density of } X & \text{(ii)} \quad \mathbb{E}[X] & \text{(iii)} \quad \mathrm{var}(X) \\ \text{(iv)} \quad \mathrm{cov}(X, Y) & \text{(v)} \quad \text{a density of } X + Y & \end{array}$$

((iv): $\mathbb{E}[XY] = \int_{\mathbb{R}^2} xy\rho(x, y)d\mathrm{m}_2(x, y)$ by Theorem 3.14. (v): find a density of $(X + Y, X - Y)$ in the same way as Example 3.30 and then use Proposition 3.17.)

In Exercise 3.11-(2), you will see that $\mathrm{cov}(X, Y) \neq 0$, which together with (3.31) in Proposition 3.32 implies that $\{X, Y\}$ is not independent.

Problem 3.12. Let X be a real random variable with $X \sim N(m, v)$. Let $\alpha \in \mathbb{R}$. Prove that $\alpha X \sim N(\alpha m, \alpha^2 v)$. (Note that a special treatment is required if $v = 0$ or $\alpha = 0$.)

Problem 3.13. Let X, Y be independent real random variables with $X \sim N(m_1, v_1)$ and $Y \sim N(m_2, v_2)$. Prove that $X + Y \sim N(m_1 + m_2, v_1 + v_2)$. (Use Propositions 3.36 and 3.38. Note again that a special treatment is required if $v_1 = 0$ or $v_2 = 0$.)

Exercise 3.14. Let $n \in \mathbb{N}$, and let $\{X_i\}_{i=1}^n$ be independent real random variables with $X_i \sim N(m_i, v_i)$ for any $i \in \{1, \ldots, n\}$. Set $X := \sum_{i=1}^n X_i$, $m := \sum_{i=1}^n m_i$ and $v := \sum_{i=1}^n v_i$. Prove that $X \sim N(m, v)$. (Induction in n. Use Proposition 3.31 and Problem 3.13.)

Problem 3.15. Let $\{X_n\}_{n=1}^{\infty}$ be real random variables. Prove the following statements:
(1) $\{\lim_{n\to\infty} X_n$ exists in $\mathbb{R}\}$ is a tail event for $\{X_n\}_{n=1}^{\infty}$. (The results of Example 3.48 can be used.)
(2) If $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ satisfies $\lim_{n\to\infty} a_n = 0$, then $\limsup_{n\to\infty} a_n \sum_{i=1}^{n} X_i$ and $\liminf_{n\to\infty} a_n \sum_{i=1}^{n} X_i$ are $\sigma_\infty(\{X_n\}_{n=1}^{\infty})$-measurable. (Imitate Example 3.48.)

Exercise 3.16. Let $d \in \mathbb{N}$, and let $\{X_n\}_{n=1}^{\infty}$ be d-dimensional random variables. Prove that $\left\{\lim_{n\to\infty} X_n \text{ exists in } \mathbb{R}^d\right\}$ is a tail event for $\{X_n\}_{n=1}^{\infty}$. (Problem 3.15-(1) can be used.)