Problem set 11, submit solutions by 28.11.2012

The **Problem**s below will be discussed in the tutorial on 30.11.2012.
(The **Exercise**s are additional and will be discussed only if time permits.)

In the problems and the exercises below, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and all random variables are assumed to be defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Problem 3.17. Let X, Y be independent real random variables with $X \sim \mathrm{Po}(\lambda_1)$ and $Y \sim \mathrm{Po}(\lambda_2)$. Prove that $X + Y \sim \mathrm{Po}(\lambda_1 + \lambda_2)$.

Exercise 3.18. Let $n \in \mathbb{N}$, and let $\{X_i\}_{i=1}^n$ be independent real random variables with $X_i \sim \mathrm{Po}(\lambda_i)$ for any $i \in \{1, \dots, n\}$. Set $X := \sum_{i=1}^n X_i$ and $\lambda := \sum_{i=1}^n \lambda_i$. Prove that $X \sim \mathrm{Po}(\lambda)$. (Induction in n. Similarly to Exercise 3.14, use Proposition 3.31 and Problem 3.17.)

Problem 3.19. Let $a, b \in [-\infty, \infty]$, $a < b$ and let μ be a law on $\mathbb{R}$. Prove that, if the distribution function F_μ of μ is C^1 on (a, b), $\lim_{x \uparrow b} F_\mu(x) = 1$ and $\lim_{x \downarrow a} F_\mu(x) = 0$, then $\mu(dx) = F'_\mu(x)\mathbf{1}_{(a,b)}(x)dx$. (Show $\int_{-\infty}^x F'_\mu(y)\mathbf{1}_{(a,b)}(y)dy = F_\mu(x)$, $x \in \mathbb{R}$.)

Problem 3.20. Let X, Y be independent real random variables with $X \sim \mathrm{Exp}(1)$ and $Y \sim \mathrm{Exp}(1)$. Find a density of the random variable $Z := X/Y$. (Calculate $F_Z(t) := \mathbb{P}[Z \leq t]$ for $t \in (0, \infty)$, differentiate F_Z and use Problem 3.19.)

Problem 3.21 (5 points each)**.** Let X, Y be independent real random variables with $X \sim \mathrm{Unif}(0, 1)$ and $Y \sim \mathrm{Unif}(0, 1)$. Find the following quantities:

(i)	a density of $X + Y$	(ii)	a density of XY	(iii)	a density of X^2
(iv)	$\mathbb{E}[\max\{X, Y\}]$	(v)	$\mathbb{E}[\min\{X, Y\}]$	(vi)	$\mathbb{E}[\max\{X, Y\} \cdot \min\{X, Y\}]$

((i): Use Propositions 3.36 and 3.38. (ii), (iii): Calculate $\mathbb{P}[XY \leq t], \mathbb{P}[X^2 \leq t]$ for $t \in (0, 1)$ and use Problem 3.19. (iv), (v), (vi): Apply Theorem 3.10 to the random variable (X, Y) and use the independence of X, Y.)

Problem 3.22. Let $X, Y, \{X_n\}_{n=1}^\infty, \{Y_n\}_{n=1}^\infty$ be real random variables such that

$$X_n \xrightarrow{\mathrm{P}} X \quad \text{and} \quad Y_n \xrightarrow{\mathrm{P}} Y. \tag{3.82}$$

(1) Prove that $(X_n, Y_n) \xrightarrow{\mathrm{P}} (X, Y)$. (Use $|(X_n, Y_n) - (X, Y)| \leq |X_n - X| + |Y_n - Y|$.)
(2) Prove that $X_n + Y_n \xrightarrow{\mathrm{P}} X + Y$ and that $X_n Y_n \xrightarrow{\mathrm{P}} XY$. (By (1), Corollary 3.53-(2) applies to (X, Y) and $\{(X_n, Y_n)\}_{n=1}^\infty$.)

Problem 3.23. Let $X, Y, \{X_n\}_{n=1}^\infty, \{Y_n\}_{n=1}^\infty$ be real random variables such that

$$\frac{1}{n}\sum_{k=1}^n X_k \xrightarrow{\mathrm{P}} X \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^n Y_k \xrightarrow{\mathrm{P}} Y. \tag{3.83}$$

Define $\{Z_n\}_{n=1}^{\infty}$ by $Z_{2n-1} := X_n$ and $Z_{2n} := Y_n$. Prove that

$$\frac{1}{n}\sum_{k=1}^{n} Z_k \xrightarrow{\mathrm{P}} \frac{X+Y}{2}. \tag{3.84}$$

(Use Problem 3.22-(2).)

Exercise 3.24. Let $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ and let $\{X_n\}_{n=1}^{\infty}$ be d-dimensional random variables with $X_n \xrightarrow{\mathcal{L}} x$. Prove that $X_n \xrightarrow{\mathrm{P}} x$. (For $\varepsilon \in (0,\infty)$, $\mathbb{P}[|X_n - x| \geq \varepsilon] = \mathbb{P}[\min\{2\varepsilon, |X_n - x|\} \geq \varepsilon]$. Apply Chebyshev's inequality (Problem 1.20-(2)) with $\varphi(x) = x$ and then use $X_n \xrightarrow{\mathcal{L}} x$, noting that $\mathbb{R}^d \ni y \mapsto \min\{2\varepsilon, |y - x|\}$ is a bounded continuous function on $\mathbb{R}^d$.)