Problem set 12, submit solutions by 05.12.2012

The **Problem**s below will be discussed in the tutorial on **10**.12.2012.
(The **Exercise**s are additional and will be discussed only if time permits.)

In the problems and the exercises below, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and all random variables are assumed to be defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Exercise 3.25. Let $X, \{X_n\}_{n=1}^\infty$ be real random variables with $X_n \xrightarrow{\text{P}} X$ and suppose $X \neq 0$ a.s. Prove that $X_n^{-1}\mathbf{1}_{\{X_n \neq 0\}} \xrightarrow{\text{P}} X^{-1}$. (Use Theorem 3.52, similarly to the proof of Corollary 3.53-(2).)

Problem 3.26. Let $(S, \mathcal{B})$ be a measurable space and let $\{X_n\}_{n=1}^\infty$ be i.i.d. $(S, \mathcal{B})$-valued random variables. Let $(E, \mathcal{E})$ be a measurable space and let $f : S \to E$ be $\mathcal{B}/\mathcal{E}$-measurable. Prove that $\{f(X_n)\}_{n=1}^\infty$ is i.i.d. $(E, \mathcal{E})$-valued random variables.

Problem 3.27. Let $\{X_n\}_{n=1}^\infty \subset \mathcal{L}^1(\mathbb{P})$ be i.i.d. and set $Y_n := e^{X_n}$ for each $n \in \mathbb{N}$. Prove that

$$(Y_1 \cdots Y_n)^{1/n} \xrightarrow{\text{a.s.}} \exp\bigl(\mathbb{E}[X_1]\bigr). \tag{3.85}$$

($(Y_1 \cdots Y_n)^{1/n} = \exp\bigl(\frac{1}{n}\sum_{k=1}^n X_k\bigr)$, to which Theorem 3.61 applies.)

Problem 3.28. Let $N \in \mathbb{N}$ and let $\{X_n\}_{n=1}^\infty \subset \mathcal{L}^N(\mathbb{P})$ be i.i.d. Prove that

$$\frac{1}{n}\sum_{k=1}^n X_k^N \xrightarrow{\text{a.s.}} \mathbb{E}[X_1^N]. \tag{3.86}$$

(Apply Theorem 3.61 to $\{X_n^N\}_{n=1}^\infty$, which is i.i.d. by Problem 3.26.)

Problem 3.29. Let $m \in \mathbb{R}$, $v \in (0, \infty)$ and let $\{X_n\}_{n=1}^\infty$ be i.i.d. with $X_1 \sim N(m, v)$. Prove that

$$\frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n X_k^2} \xrightarrow{\text{a.s.}} \frac{m}{m^2 + v}. \tag{3.87}$$

(Divide both the numerator and the denominator by n and apply Theorem 3.61.)

Problem 3.30. Let $\{X_n\}_{n=1}^\infty \subset \mathcal{L}^2(\mathbb{P})$ be i.i.d. Prove that

$$\frac{1}{n}\sum_{k=1}^n \bigl(X_k - \mathbb{E}[X_1]\bigr)^2 \xrightarrow{\text{a.s.}} \mathrm{var}(X_1). \tag{3.88}$$

Problem 4.1. Let $\{X_n\}_{n=1}^\infty$ be i.i.d. real random variables with $X_1 \sim \mathrm{Po}(1)$, and set $S_n := \sum_{k=1}^n X_k$ for each $n \in \mathbb{N}$. Prove the following statements:

(1) $\mathcal{L}\left(\frac{S_n - n}{\sqrt{n}}\right) \xrightarrow{\mathcal{L}} N(0, 1)$. (Simply apply Theorem 4.4-(1).)

(2) $\mathbb{P}[S_n \leq n] = e^{-n}\sum_{k=0}^n \frac{n^k}{k!}$ for any $n \in \mathbb{N}$. (Use Exercise 3.18.)

(3) $\lim_{n\to\infty} e^{-n}\sum_{k=0}^n \frac{n^k}{k!} = \frac{1}{2}$. (Theorem 4.4-(2) applies by (2) above.)

Problem 4.2. Let $y \in \mathbb{R}$ and let $X, \{X_n\}_{n=1}^{\infty}, \{Y_n\}_{n=1}^{\infty}$ be real random variables such that

$$X_n \xrightarrow{\mathcal{L}} X \quad \text{and} \quad Y_n \xrightarrow{\mathrm{P}} y. \tag{4.78}$$

(1) Prove that $X_n + Y_n \xrightarrow{\mathcal{L}} X + y$ and that $X_n Y_n \xrightarrow{\mathcal{L}} yX$. (Since $(X_n, Y_n) \xrightarrow{\mathcal{L}} (X, y)$ by Proposition 4.11, Corollary 3.53-(3) applies to (X, y) and $\{(X_n, Y_n)\}_{n=1}^{\infty}$.)
(2) Suppose $y \neq 0$. Prove that

$$\frac{X_n}{Y_n}\mathbf{1}_{\{Y_n \neq 0\}} \xrightarrow{\mathcal{L}} \frac{X}{y}. \tag{4.79}$$

(Use Exercise 3.25 to apply the latter assertion of (1).)

Remark. Note that in the statements of Problem 4.2, the random variable X is involved *only in terms of its law* $\mathcal{L}(X)$ since the laws of $X + y, yX, X/y$ are determined solely by $\mathcal{L}(X)$ and y. In particular, the statements of Problem 4.2 are valid even if X is replaced by another real random variable X_0 with $\mathcal{L}(X_0) = \mathcal{L}(X)$ which is defined on a *different* probability space.

Exercise 4.3 ([2, Exercise 3.4.4])**.** Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. $[0, \infty)$-valued random variables with $\mathbb{E}[X_1] = 1$ and $v := \mathrm{var}(X_1) < \infty$. Set $S_n := \sum_{k=1}^{n} X_k$ for each $n \in \mathbb{N}$.
(1) Prove that for any $n \in \mathbb{N}$,

$$\sqrt{S_n} - \sqrt{n} = \frac{S_n - n}{\sqrt{n}} \frac{1}{1 + \sqrt{S_n/n}}. \tag{4.80}$$

(2) Prove that

$$\mathcal{L}\big(\sqrt{S_n} - \sqrt{n}\big) \xrightarrow{\mathcal{L}} N(0, v/4). \tag{4.81}$$

((4.81) can be rephrased as "$\sqrt{S_n} - \sqrt{n} \xrightarrow{\mathcal{L}} Z/2$" for a real random variable Z with $Z \sim N(0, v)$, and Theorem 4.4-(1) can be also rephrased in the same way. Apply this version of Theorem 4.4-(1) to $(S_n - n)/\sqrt{n}$ and then use (4.80) and the latter part of Problem 4.2-(1), noting that it is irrelevant on which probability space Z is defined.)

Problem 4.4 ([2, Exercise 3.4.5])**.** Let $\{X_n\}_{n=1}^{\infty} \subset \mathcal{L}^2(\mathbb{P})$ be i.i.d. with $\mathbb{E}[X_1] = 0$ and $v := \mathrm{var}(X_1) > 0$. Prove that

$$\mathcal{L}\left(\frac{\sum_{k=1}^{n} X_k}{\sqrt{\sum_{k=1}^{n} X_k^2}} \mathbf{1}_{\{\sum_{k=1}^{n} X_k^2 \neq 0\}} \right) \xrightarrow{\mathcal{L}} N(0, 1). \tag{4.82}$$

($\big(\sum_{k=1}^{n} X_k\big)/\sqrt{\sum_{k=1}^{n} X_k^2} = \big(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k\big)/\sqrt{\frac{1}{n} \sum_{k=1}^{n} X_k^2}$ on $\{\sum_{k=1}^{n} X_k^2 \neq 0\}$. Similarly to Exercise 4.3-(2), use Theorem 4.4-(1) and Problem 4.2-(2).)