Problem set 13, submit solutions by **11**.12.2012

The **Problem**s and **Exercise**s below will be discussed in the tutorial on **11**.12.2012.

Remark. As usual, submission of solutions to the **Exercise**s is not required, but the **Exercise**s below are not difficult and will be treated in detail in the tutorial. Please **DO NOT SKIP** the **Exercise**s below when writing solutions to this problem set.

In the problems and the exercises below, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and all random variables are assumed to be defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Exercise 4.5. Let X be a real random variable and let $t \in \mathbb{R}$. Prove the following assertions:
(1) If X has the binomial distribution $B(n, p)$, $n \in \mathbb{N}$, $p \in [0, 1]$, then

$$\varphi_X(t) = \bigl(1 + p(e^{it} - 1)\bigr)^n. \tag{4.39}$$

(2) If X has the Poisson distribution $\mathrm{Po}(\lambda)$, $\lambda \in (0, \infty)$, then

$$\varphi_X(t) = \exp\bigl(\lambda(e^{it} - 1)\bigr). \tag{4.40}$$

(3) If X has the geometric distribution $\mathrm{Geom}(\alpha)$, $\alpha \in [0, 1)$, then

$$\varphi_X(t) = \frac{1-\alpha}{1-\alpha e^{it}}. \tag{4.41}$$

(4) If X has the uniform distribution $\mathrm{Unif}(-a, a)$ on $(-a, a)$, $a \in (0, \infty)$, then

$$\varphi_X(t) = \frac{\sin at}{at}. \tag{4.42}$$

Problem 4.6. Let $\mu \in \mathcal{P}(\mathbb{R})$ be the *Laplace distribution*, that is, the law on $\mathbb{R}$ given by

$$\mu(dx) := \frac{1}{2}e^{-|x|}dx. \tag{4.55}$$

(μ is also called the *double exponential distribution*.) Prove that for any $t \in \mathbb{R}$,

$$\varphi_\mu(t) = \frac{1}{1+t^2}. \tag{4.56}$$

(The result for $\mathrm{Exp}(\alpha)$ in Example 4.21 can be used with $\alpha = 1$.)

For Problem 4.7 and Exercises 4.8, 4.9 and 4.10 below, recall Proposition 4.18 and Examples 4.20, 4.22, 4.23 and 4.24. Note also the following immediate corollary of Theorem 4.25:

Corollary. *Let $d \in \mathbb{N}$, $\mu \in \mathcal{P}(\mathbb{R}^d)$ and let X be a d-dimensional random variable. If $\varphi_X = \varphi_\mu$ then $X \sim \mu$.*

Problem 4.7. (1) (Problem 3.13) Let X, Y be independent real random variables with $X \sim N(m_1, v_1)$ and $Y \sim N(m_2, v_2)$. Prove that $X + Y \sim N(m_1 + m_2, v_1 + v_2)$. (Use Proposition 4.18 and (4.44) of Example 4.22 to show that $\varphi_{X+Y} = \varphi_{N(m_1+m_2,v_1+v_2)}$.)
(2) (Exercise 3.14) Let $n \in \mathbb{N}$, and let $\{X_k\}_{k=1}^n$ be independent real random variables with $X_k \sim N(m_k, v_k)$ for any $k \in \{1, \ldots, n\}$. Set $X := \sum_{k=1}^n X_k$, $m := \sum_{k=1}^n m_k$ and $v := \sum_{k=1}^n v_k$. Prove that $X \sim N(m, v)$. (Similarly to (1), verify $\varphi_X = \varphi_{N(m,v)}$.)

Recall that Problem 4.7 already appeared as Problem 3.13 and Exercise 3.14, where some tedious calculations on density functions were necessary. Here the same assertions can be verified rather easily by virtue of Proposition 4.18 and Theorem 4.25. The same argument applies to Poisson, gamma and Cauchy random variables, as follows.

Exercise 4.8 (Exercise 3.18)**.** Let $n \in \mathbb{N}$, and let $\{X_k\}_{k=1}^n$ be independent real random variables with $X_k \sim \mathrm{Po}(\lambda_k)$ for any $k \in \{1, \ldots, n\}$. Set $X := \sum_{k=1}^n X_k$ and $\lambda := \sum_{k=1}^n \lambda_k$. Prove that $X \sim \mathrm{Po}(\lambda)$.

Exercise 4.9. Let $n \in \mathbb{N}$, $\beta \in (0, \infty)$ and let $\{X_k\}_{k=1}^n$ be independent real random variables with $X_k \sim \mathrm{Gamma}(\alpha_k, \beta)$ for any $k \in \{1, \ldots, n\}$. Set $X := \sum_{k=1}^n X_k$ and $\alpha := \sum_{k=1}^n \alpha_k$. Prove that $X \sim \mathrm{Gamma}(\alpha, \beta)$.

Exercise 4.10. Let $n \in \mathbb{N}$, and let $\{X_k\}_{k=1}^n$ be independent real random variables with $X_k \sim \mathrm{Cauchy}(m_k, \alpha_k)$ for any $k \in \{1, \ldots, n\}$. Set $X := \sum_{k=1}^n X_k$, $m := \sum_{k=1}^n m_k$ and $\alpha := \sum_{k=1}^n \alpha_k$. Prove that $X \sim \mathrm{Cauchy}(m, \alpha)$.

Problem 4.11. Let X be a real random variable with $X \sim N(0, 1)$. Calculate $\mathbb{E}[X^n]$ for any $n \in \mathbb{N}$. (Use the Taylor series expansion of $\varphi_X(t) = e^{-t^2/2}$ to apply (4.33) of Theorem 4.15.)

Problem 4.12. Let $m \in \mathbb{R}$, $v \in [0, \infty)$ and let X be a real random variable with $X \sim N(m, v)$. Prove that $\mathbb{E}\big[e^{sX}\big] = \exp\big(sm + s^2 v/2\big)$ for any $s \in \mathbb{R}$.

Remark. Formally, replacing s by it in Problem 4.12 yields the characteristic function (4.44) of $N(m, v)$ in Example 4.22, but some task is required to justify this reasoning.

Problem 4.13. Let $d \in \mathbb{N}$ and let X be a d-dimensional random variable.
(1) Prove that $\varphi_{-X}(t) = \overline{\varphi_X(t)}$ for any $t \in \mathbb{R}^d$.
(2) Prove that φ_X is real-valued (i.e. $\varphi_X(t) \in \mathbb{R}$ for any $t \in \mathbb{R}^d$) if and only if $\mathcal{L}(-X) = \mathcal{L}(X)$. (Use (1) and Theorem 4.25. Recall that for $z \in \mathbb{C}$, $z \in \mathbb{R}$ if and only if $\overline{z} = z$.)