Problem set 14, submission of solutions **NOT** required

The **Problem**s below will be discussed in the tutorial on **14**.12.2012.

In the problems and the exercises below, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and all random variables are assumed to be defined on $(\Omega, \mathcal{F}, \mathbb{P})$ unless otherwise stated.

Problem 14.1. Calculate $\mathbb{E}[X]$ and $\mathrm{var}(X)$ for a real random variable X with
(1) the binomial distribution $B(n, p)$, $n \in \mathbb{N}$, $p \in [0, 1]$.
(2) the Poisson distribution $\mathrm{Po}(\lambda)$, $\lambda \in (0, \infty)$.
(3) the geometric distribution $\mathrm{Geom}(\alpha)$, $\alpha \in [0, 1)$.
(4) the uniform distribution $\mathrm{Unif}(a, b)$, $a, b \in \mathbb{R}$, $a < b$.
(5) the exponential distribution $\mathrm{Exp}(\alpha)$, $\alpha \in (0, \infty)$.
(6) the gamma distribution $\mathrm{Gamma}(\alpha, \beta)$, $\alpha, \beta \in (0, \infty)$.
(7) a density ρ_X given by $\rho_X(x) = (1 - |x|)^+$.

Problem 14.2. Let X, Y be independent real random variables with $X \sim \mathrm{Unif}(0, 1)$ and $Y \sim \mathrm{Unif}(0, 1)$. Find the following quantities:

(i) a density of $X + Y$ (ii) a density of XY (iii) a density of X^2
(iv) $\mathbb{E}[\max\{X, Y\}]$ (v) $\mathbb{E}[\min\{X, Y\}]$ (vi) $\mathbb{E}[\max\{X, Y\} \cdot \min\{X, Y\}]$

Problem 14.3. Let X, Y be independent real random variables with $X \sim \mathrm{Exp}(1)$ and $Y \sim \mathrm{Exp}(1)$. Find the following quantities:

(i) a density of $X + Y$ (ii) a density of X/Y (iii) a density of X^2
(iv) $\mathbb{E}[\max\{X, Y\}]$ (v) $\mathbb{E}[|X - Y|]$

Problem 14.4. Define $\rho : \mathbb{R}^2 \to [0, \infty)$ by

$$\rho(x, y) := \frac{1}{2}(x + y)e^{-x-y}\mathbf{1}_{(0,\infty)^2}(x, y). \tag{14.1}$$

(1) Prove that $\int_{\mathbb{R}^2} \rho(z)dz = 1$, so that $\mu(dz) := \rho(z)dz$ is a probability law on $\mathbb{R}^2$.
(2) Let X, Y be real random variables with $(X, Y) \sim \mu$. Find the following quantities:

(i) $\mathrm{cov}(X, Y)$ (ii) a density of $X + Y$

(3) Evaluate the characteristic function φ_μ of μ.

Problem 14.5. Let X be a real random variable and let $t \in \mathbb{R}$. Prove the following assertions:
(1) If X has the binomial distribution $B(n, p)$, $n \in \mathbb{N}$, $p \in [0, 1]$, then

$$\varphi_X(t) = \big(1 + p(e^{it} - 1)\big)^n. \tag{14.2}$$

(2) If X has the Poisson distribution $\mathrm{Po}(\lambda)$, $\lambda \in (0, \infty)$, then

$$\varphi_X(t) = \exp\big(\lambda(e^{it} - 1)\big). \tag{14.3}$$

(3) If X has the geometric distribution $\mathrm{Geom}(\alpha)$, $\alpha \in [0,1)$, then

$$\varphi_X(t) = \frac{1-\alpha}{1-\alpha e^{it}}. \tag{14.4}$$

(4) If X has the uniform distribution $\mathrm{Unif}(a,b)$, $a, b \in \mathbb{R}$, $a < b$, then

$$\varphi_X(t) = \frac{e^{itb} - e^{ita}}{it(b-a)}. \tag{14.5}$$

(5) If X has a density ρ_X given by $\rho_X(x) = (1-|x|)^+$, then

$$\varphi_X(t) = \frac{2}{t^2}(1-\cos t). \tag{14.6}$$

(6) If X has the Laplace distribution, that is, has a density ρ_X given by $\rho_X(x) = \frac{1}{2}e^{-|x|}$, then

$$\varphi_X(t) = \frac{1}{1+t^2}. \tag{14.7}$$

For the next problem, recall the following immediate corollary of Theorem 4.25:

Corollary. *Let $d \in \mathbb{N}$, $\mu \in \mathcal{P}(\mathbb{R}^d)$ and let X be a d-dimensional random variable. If $\varphi_X = \varphi_\mu$ then $X \sim \mu$.*

Problem 14.6. Let $n \in \mathbb{N}$, let $\{X_k\}_{k=1}^n$ be independent real random variables and set $X := \sum_{k=1}^n X_k$. Prove the following statements:
(1) If $X_k \sim N(m_k, v_k)$ for any $k \in \{1,\dots,n\}$, $m := \sum_{k=1}^n m_k$ and $v := \sum_{k=1}^n v_k$, then

$$X \sim N(m, v). \tag{14.8}$$

(2) If $X_k \sim \mathrm{Po}(\lambda_k)$ for any $k \in \{1,\dots,n\}$ and $\lambda := \sum_{k=1}^n \lambda_k$, then

$$X \sim \mathrm{Po}(\lambda). \tag{14.9}$$

(3) If $\beta \in (0,\infty)$, $X_k \sim \mathrm{Gamma}(\alpha_k, \beta)$ for any $k \in \{1,\dots,n\}$ and $\alpha := \sum_{k=1}^n \alpha_k$, then

$$X \sim \mathrm{Gamma}(\alpha, \beta). \tag{14.10}$$

(4) If $X_k \sim \mathrm{Cauchy}(m_k, \alpha_k)$ for any $k \in \{1,\dots,n\}$, $m := \sum_{k=1}^n m_k$ and $\alpha := \sum_{k=1}^n \alpha_k$, then

$$X \sim \mathrm{Cauchy}(m, \alpha). \tag{14.11}$$