Problem set 2, submit solutions by 26.09.2012

The **Problem**s below will be discussed in the tutorial on 28.09.2012.
(The **Exercise** is additional and will be discussed only if time permits.)

Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $B \in \mathcal{F}$ satisfy $\mathbb{P}[B] > 0$. For each $A \in \mathcal{F}$, We define the *conditional probability* $\mathbb{P}[A \mid B]$ *of* A *given* B by

$$\mathbb{P}[A \mid B] := \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}. \tag{1.60}$$

Problem 1.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $B \in \mathcal{F}$ satisfy $\mathbb{P}[B] > 0$.
(1) Let $A \in \mathcal{F}$. Prove that $\{A, B\}$ is independent if and only if $\mathbb{P}[A \mid B] = \mathbb{P}[A]$.
(2) Prove that the set function $\mathcal{F} \ni A \mapsto \mathbb{P}[A \mid B]$ is a probability measure on $(\Omega, \mathcal{F})$. This probability measure is called the *conditional probability measure given* B.

Problem 1.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\Omega_n\}_{n=1}^N \subset \mathcal{F}$, where $N \in \mathbb{N} \cup \{\infty\}$, satisfy $\mathbb{P}[\Omega_n] > 0$ for any n, $\Omega_i \cap \Omega_j = \emptyset$ for any i, j with $i \neq j$ and $\bigcup_{n=1}^N \Omega_n = \Omega$. Also let $A \in \mathcal{F}$. Prove the following statements:
(1) $\mathbb{P}[A] = \sum_{n=1}^N \mathbb{P}[A \mid \Omega_n]\mathbb{P}[\Omega_n]$.
(2) (Bayes' theorem) If $\mathbb{P}[A] > 0$, then for each n,

$$\mathbb{P}[\Omega_n \mid A] = \frac{\mathbb{P}[A \mid \Omega_n]\mathbb{P}[\Omega_n]}{\sum_{k=1}^N \mathbb{P}[A \mid \Omega_k]\mathbb{P}[\Omega_k]}. \tag{1.61}$$

Exercise 1.8. Suppose people have a certain disease with probability 0.001. Doctors use a test to detect the disease, and suppose that the test gives a positive result on a patient with the disease with probability 0.99 and on a patient without it with probability 0.004. Evaluate the probability that one has this disease under the condition that
(1) the result of the test was positive.
(2) the result of the test was negative.

In the rest of this problem set, $(X, \mathcal{M}, \mu)$ denotes a given measure space.

Problem 1.9. Let $n \in \mathbb{N}$ and let $\{A_i\}_{i=1}^n \subset \mathcal{M}$ satisfy $\mu\left(\bigcup_{i=1}^n A_i\right) < \infty$. Prove the following *inclusion-exclusion formula*:

$$\mu\left(\bigcup_{i=1}^n A_i\right) = \sum_{k=1}^n \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{k-1} \mu\left(\bigcap_{\ell=1}^k A_{i_\ell}\right). \tag{1.62}$$

(Conduct an induction in n.)

Problem 1.10. Prove the following *countable subadditivity* of μ: for $\{A_n\}_{n=1}^\infty \subset \mathcal{M}$,

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n). \tag{1.63}$$

(Set $B_1 := A_1$ and $B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i$, $n \geq 2$, and show that $\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty B_n$.)

Problem 1.11. Let $\{A_n\}_{n=1}^{\infty} \subset 2^X$ and define $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ by

$$\limsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \qquad \liminf_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k, \tag{1.64}$$

so that they belong to $\mathcal{M}$ if $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$. Prove the following assertions.
(1) $\left(\limsup_{n\to\infty} A_n\right)^c = \liminf_{n\to\infty} A_n^c$ and

$$\begin{aligned} \limsup_{n\to\infty} A_n &= \{x \in X \mid x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}, \\ \liminf_{n\to\infty} A_n &= \{x \in X \mid x \in A_n \text{ for sufficiently large } n \in \mathbb{N}\}. \end{aligned} \tag{1.65}$$

(2) (First Borel-Cantelli lemma) If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ and $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then

$$\mu\Bigl(\limsup_{n\to\infty} A_n\Bigr) = \mu\Bigl(\Bigl(\liminf_{n\to\infty} A_n^c\Bigr)^c\Bigr) = 0. \tag{1.66}$$

(Noting $\limsup_{n\to\infty} A_n \subset \bigcup_{n=k}^{\infty} A_n$, use the countable subadditivity (1.63) of μ.)

Problem 1.12. Let # be the counting measure on $\mathbb{N}$ (recall Example 1.5-(1)). Provide an example of $\{A_n\}_{n=1}^{\infty} \subset 2^{\mathbb{N}}$ such that $A_n \supset A_{n+1}$ for any $n \in \mathbb{N}$ but $\lim_{n\to\infty} \#A_n \neq \#\bigl(\bigcap_{n=1}^{\infty} A_n\bigr)$.

Problem 1.12 shows that the conclusion of Proposition 1.4-(4) is not necessarily valid if the assumption "$\mu(A_1) < \infty$" is dropped.