Problem set 3, submit solutions by **02**.10.2012

The **Problem**s below will be discussed in the tutorial on 05.10.2012.
(The **Exercise** is additional and will be discussed only if time permits.)

Throughout this problem set, $(X, \mathcal{M}, \mu)$ denotes a given measure space.

Problem 1.13. Let Y be a set and define $\mathcal{N} := \{A \subset Y \mid \text{either } A \text{ or } A^c \text{ is countable}\}$ and $\mathcal{N}_0 := \{A \subset Y \mid \text{either } A \text{ or } A^c \text{ is finite}\}$. Prove that $\mathcal{N}$ is a σ-algebra in Y and that $\sigma(\mathcal{N}_0) = \mathcal{N}$.

Problem 1.14. Assume $\mu(X) < \infty$. Let Λ be a set and let $\{A_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{M}$ be such that $A_{\lambda_1} \cap A_{\lambda_2} = \emptyset$ for any $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$. Prove that $\{\lambda \in \Lambda \mid \mu(A_\lambda) > 0\}$ is a countable set. (Show that $\{\lambda \in \Lambda \mid \mu(A_\lambda) \geq 1/n\}$ is finite for any $n \in \mathbb{N}$.)

Problem 1.15. (1) Let $f, g : X \to [-\infty, \infty]$ be $\mathcal{M}$-measurable. Prove that the following sets belong to $\mathcal{M}$:

$$\{x \in X \mid f(x) < g(x)\}, \quad \{x \in X \mid f(x) = g(x)\}, \quad \{x \in X \mid f(x) > g(x)\}.$$

(2) Let $f_n : X \to [-\infty, \infty]$ be $\mathcal{M}$-measurable for each $n \in \mathbb{N}$ and let $h : X \to [-\infty, \infty]$ be $\mathcal{M}$-measurable. Define $f, g : X \to [-\infty, \infty]$ by

$$f(x) := \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if the limit } \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}, \\ h(x) & \text{otherwise,} \end{cases} \tag{1.73}$$

$$g(x) := \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if the limit } \lim_{n\to\infty} f_n(x) \text{ exists in } [-\infty, \infty], \\ h(x) & \text{otherwise.} \end{cases} \tag{1.74}$$

Prove that the functions f and g are $\mathcal{M}$-measurable.

We need the following definition for the following two problems.

Definition. Let $(S, \mathcal{B})$ be a measurable space. A map $\varphi : X \to S$ is called $\mathcal{M}/\mathcal{B}$-*measurable* if and only if $\varphi^{-1}(A) \in \mathcal{M}$ for any $A \in \mathcal{B}$.

Problem 1.16. Let $(S, \mathcal{B})$ be a measurable space, let $\varphi : X \to S$ be $\mathcal{M}/\mathcal{B}$-measurable and let $f : S \to [-\infty, \infty]$ be $\mathcal{B}$-measurable. Prove that $f \circ \varphi : X \to [-\infty, \infty]$ is $\mathcal{M}$-measurable.

Problem 1.17. (1) Let S be a set, let $\mathcal{A} \subset 2^S$ and let $f : X \to S$. Prove that f is $\mathcal{M}/\sigma_S(\mathcal{A})$-measurable if and only if $f^{-1}(A) \in \mathcal{M}$ for any $A \in \mathcal{A}$. (As in the proof of Proposition 1.14, show that $\mathcal{A} := \{A \subset S \mid f^{-1}(A) \in \mathcal{M}\}$ is a σ-algebra in S.)
(2) Let $d \in \mathbb{N}$ and let $f = (f_1, \ldots, f_d) : X \to \mathbb{R}^d$, where $f_i : X \to \mathbb{R}$ for each $i \in \{1, \ldots, d\}$. Prove that f is $\mathcal{M}/\mathcal{B}(\mathbb{R}^d)$-measurable if and only if f_i is $\mathcal{M}$-measurable for any $i \in \{1, \ldots, d\}$.

Exercise 1.18. Let $d \in \mathbb{N}$, let $S \subset \mathbb{R}^d$ and let $f : S \to [-\infty, \infty]$.
(1) Let $\varepsilon \in (0, \infty)$ and define $f^\varepsilon, f_\varepsilon : S \to [-\infty, \infty]$ by

$$f^\varepsilon(x) := \sup_{y \in B_S(x,\varepsilon)} f(y) \quad \text{and} \quad f_\varepsilon(x) := \inf_{y \in B_S(x,\varepsilon)} f(y). \tag{1.75}$$

Prove that f^ε and f_ε are Borel measurable. (Show that $(f^\varepsilon)^{-1}\big((a, \infty]\big)$ is open in S.)
(2) Prove that the functions $\overline{f}, \underline{f} : S \to [-\infty, \infty]$ defined by

$$\overline{f}(x) := \limsup_{S \ni y \to x} f(y) \quad \text{and} \quad \underline{f}(x) := \liminf_{S \ni y \to x} f(y) \tag{1.76}$$

are Borel measurable.
(3) Prove that $\{x \in S \mid \lim_{S \ni y \to x} f(y) = f(x)\}$ is a Borel set of S.

Problem 1.19. Let X be a countable set and let μ be a measure on $(X, 2^X)$.
(1) Prove that any function $f : X \to [-\infty, \infty]$ on X is 2^X-measurable.
(2) Let $f : X \to [0, \infty]$. Prove that $\int_X f d\mu = \sum_{x \in X} f(x)\mu(\{x\})$.