Problem set 4, submit solutions by 10.10.2012

The **Problem**s below will be discussed in the tutorial on 12.10.2012.
(The **Exercise** is additional and will be discussed only if time permits.)

Throughout this problem set, $(X, \mathcal{M}, \mu)$ denotes a given measure space.

Problem 1.20. Let $\varphi : [0, \infty] \to [0, \infty]$ be non-decreasing and let $f : X \to [0, \infty]$ be $\mathcal{M}$-measurable. Prove the following assertions.
(1) $\varphi \circ f$ is $\mathcal{M}$-measurable. (For $a \in [0, \infty)$, set $M := \inf \varphi^{-1}\big((a, \infty]\big)$ ($\inf \emptyset := \infty$) and show that $\varphi^{-1}\big((a, \infty]\big)$ is either $(M, \infty]$ or $[M, \infty]$ (either $\emptyset$ or $\{\infty\}$ if $M = \infty$).)
(2) (Chebyshev's inequality) For any $a \in [0, \infty]$ with $\varphi(a) \in (0, \infty)$,

$$\mu\big(\{x \in X \mid f(x) \geq a\}\big) \leq \frac{1}{\varphi(a)} \int_X (\varphi \circ f) d\mu. \tag{1.77}$$

Problem 1.21. Let $f_n : X \to [-\infty, \infty]$ be $\mathcal{M}$-measurable for each $n \in \mathbb{N}$ and suppose that $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$. Prove that $\lim_{n\to\infty} f_n(x) = 0$ for μ-a.e. $x \in X$. (Use Proposition 1.26 and then Proposition 1.30-(2).)

Problem 1.22. Find the limits as $N \to \infty$ of the following series:

$$(1) \sum_{n=1}^{\infty} 2^{-n} \left(1 + \frac{\sin(2^N n)}{N}\right)^{-1} \quad (2) \sum_{n=1}^{\infty} \frac{1}{n(n+N)} \quad (3) \sum_{n=1}^{\infty} \left(1 + \frac{n}{N}\right)^{-N}$$

((1): Consider the case $N \geq 2$ only. (3): Recall that $f(x) := (1 + 1/x)^{-x}$, $x \in (0, \infty)$, is strictly decreasing and $\lim_{x\to\infty} f(x) = e^{-1}$. Again consider the case $N \geq 2$ only.)

Problem 1.23. Let m_1 be the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ introduced in Example 1.8.
(1) Prove that $\mathrm{m}_1(\{a\}) = 0$ for any $a \in \mathbb{R}$.
(2) Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous. For each $n \in \mathbb{N}$, define $f_n : [a, b] \to \mathbb{R}$ by

$$f_n := \sum_{k=1}^{n} f\left(a + \frac{k}{n}(b-a)\right) \mathbf{1}_{\left(a + \frac{k-1}{n}(b-a),\, a + \frac{k}{n}(b-a)\right]} + f(a) \mathbf{1}_{\{a\}}. \tag{1.78}$$

(i) Prove that $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in [a, b]$. (Use the continuity of f.)
(ii) By considering $\lim_{n\to\infty} \int_{[a,b]} f_n d\mathrm{m}_1$, prove that

$$\int_{[a,b]} f d\mathrm{m}_1 = \int_a^b f(x) dx, \tag{1.79}$$

where the integral in the right-hand side denotes the Riemann integral on $[a, b]$. (Noting that $\int_{[a,b]} f_n d\mathrm{m}_1$ gives a Riemann sum for f, use Theorem 1.33 to get $\int_{[a,b]} f d\mathrm{m}_1$ and use the definition of the Riemann integral to get $\int_a^b f(x) dx$.)

(3) Let $a \in \mathbb{R}$ and let $f : [a, \infty) \to \mathbb{R}$ be continuous. Prove that f is m_1-integrable on $[a, \infty)$ if and only if $\lim_{b\to\infty} \int_a^b |f(x)|dx < \infty$,[1] and in that case

$$\int_{[a,\infty)} f d\mathrm{m}_1 = \lim_{b\to\infty} \int_a^b f(x)dx. \tag{1.80}$$

(Suffices to show when $f \geq 0$. Use (1.79) and then use Theorem 1.24 to let $b \to \infty$.)

By Problem 1.23-(2), for a continuous function on a bounded closed interval, its integral with respect to the Lebesgue measure m_1 coincides with its Riemann integral. In fact, this fact can be generalized to any Riemann integrable function f on a bounded closed interval of any dimension. See Section 2.6 below for details.

On the other hand, Problem 1.23-(3) says that the same is true also for a continuous function on an unbounded interval *provided the improper Riemann integral* $\lim_{b\to\infty} \int_a^b f(x)dx$ *is absolutely convergent*. Here the assumption of the absolute convergence is necessary; see Problem 2.14 in this connection.

Problem 1.24. Find the limits as $n \to \infty$ of the following integrals:

$$(1) \int_0^\infty \frac{1}{1+x^n} dx \quad (2) \int_0^\infty \frac{\sin e^x}{1+nx^2} dx \quad (3) \int_0^1 \frac{n \cos x}{1+n^2 x^{3/2}} dx$$

((1): Consider $\int_0^1$ and $\int_1^\infty$ separately. (3): Imitate Example 1.34.)

Exercise 1.25 ([1, Section 4.3, Problem 1])**.** Let $f \in \mathcal{L}^1(\mu)$ and $\{f_n\}_{n=1}^\infty \subset \mathcal{L}^1(\mu)$. Suppose that $f_n \geq 0$ on X for any $n \in \mathbb{N}$, that $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in X$, and that $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$. Prove that $\lim_{n\to\infty} \int_X |f - f_n| d\mu = 0$. (It suffices to show $\lim_{n\to\infty} \int_X (f - f_n)^+ d\mu = 0$. Note that $0 \leq (f - f_n)^+ \leq f$.)

[1]Note that the limit $\lim_{b\to\infty} \int_a^b |f(x)|dx$ always exists in $[0, \infty]$, since $\int_a^b |f(x)|dx$ is non-decreasing in $b \in (a, \infty)$.