Problem set 5, submit solutions by 17.10.2012

The **Problem**s below will be discussed in the tutorial on 19.10.2012.
(The **Exercise**s are additional and will be discussed only if time permits.)

Throughout this problem set, $(X, \mathcal{M}, \mu)$ denotes a given measure space.

Problem 1.26 ([7, Chapter 1, Exercise 9])**.** Let $\alpha \in (0, \infty)$, let $f : X \to [0, \infty]$ be $\mathcal{M}$-measurable and suppose $\int_X f d\mu \in (0, \infty)$. Find the limit (with $\log \infty := \infty^\alpha := \infty$)

$$\lim_{n\to\infty} \int_X n \log\bigl(1 + (f/n)^\alpha\bigr) d\mu.$$

(Note that $\log(1 + x) \geq 0$ for any $x \in [0, \infty]$. If $\alpha \geq 1$, then $\log(1 + x^\alpha) \leq \alpha x$ for any $x \in [0, \infty]$. If $\alpha < 1$, then Fatou's lemma (Theorem 1.27) applies.)

Exercise 1.27 (30 points)**.** Let $f : X \to [-\infty, \infty]$. Prove that the following three conditions are equivalent:
(1) f is $\overline{\mathcal{M}}^\mu$-measurable.
(2) There exist $\mathcal{M}$-measurable functions $f_1, f_2 : X \to [-\infty, \infty]$ such that $f_1 \leq f \leq f_2$ on X and $f_1 = f_2$ μ-a.e.
(3) There exists a $\mathcal{M}$-measurable function $f_0 : X \to [-\infty, \infty]$ such that $f_0 = f$ μ-a.e.
((1) $\Rightarrow$ (2): If f is $\overline{\mathcal{M}}^\mu$-simple, then (2) can be proved by using the definition of $\overline{\mathcal{M}}^\mu$. For a general $\overline{\mathcal{M}}^\mu$-measurable f, take non-decreasing sequences $\{s_n^\pm\}_{n=1}^\infty$ of $\overline{\mathcal{M}}^\mu$-simple functions converging to $f^\pm$, use (2) for $s_n^\pm$ and take $\limsup_{n\to\infty}$, $\liminf_{n\to\infty}$.)

Problem 1.28. Let $p \in (0, \infty)$ and let $f \in \mathcal{L}^p(\mu)$. Prove that

$$\lim_{n\to\infty} \int_X \bigl|f - f\mathbf{1}_{\{|f|\leq n\}}\bigr|^p d\mu = 0. \tag{1.81}$$

Problem 1.29. Let $p, q \in (0, \infty)$, $p < q$, and let $f : X \to [0, \infty]$ be $\mathcal{M}$-measurable. Prove that

$$\left(\int_X f^p d\mu\right)^{1/p} \leq \left(\int_X f^q d\mu\right)^{1/q} \mu(X)^{(q-p)/pq}. \tag{1.82}$$

By Problem 1.29, if $\mu(X) < \infty$, then $\mathcal{L}^q(X, \mu) \subset \mathcal{L}^p(X, \mu)$ for any $p, q \in (0, \infty)$ with $p < q$.

Problem 1.30 (Minkowski's inequality)**.** Let $p \in [1, \infty)$ and let $f, g : X \to [0, \infty]$ be $\mathcal{M}$-measurable. Prove that

$$\left(\int_X (f + g)^p d\mu\right)^{1/p} \leq \left(\int_X f^p d\mu\right)^{1/p} + \left(\int_X g^p d\mu\right)^{1/p}. \tag{1.83}$$

(Assuming $p > 1$, apply Hölder's inequality to $f(f + g)^{p-1}$ and $g(f + g)^{p-1}$.)

For the next problem, we need the following definition.

Definition. Let $f : X \to \mathbb{R}$ and $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, be $\mathcal{M}$-measurable. We say that $\{f_n\}_{n=1}^{\infty}$ *converges in μ-measure to f* if and only if for any $\varepsilon \in (0,\infty)$,

$$\lim_{n\to\infty} \mu\bigl(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}\bigr) = 0. \tag{1.84}$$

Problem 1.31. Let $f : X \to \mathbb{R}$ and $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, be $\mathcal{M}$-measurable.
(1) Let $p \in (0,\infty)$ and suppose $\lim_{n\to\infty} \|f_n - f\|_{L^p(\mu)} = 0$. Prove that $\{f_n\}_{n=1}^{\infty}$ converges in μ-measure to f. (Use Problem 1.20-(2) with $\varphi(x) = x^p$, $x \in [0,\infty]$.)
(2) Suppose that $\{f_n\}_{n=1}^{\infty}$ converges in μ-measure to f. Prove that there exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for μ-a.e. $x \in X$. (Choose $n_k \in \mathbb{N}$ so that $\mu\bigl(\{x \in X \mid |f_{n_k}(x) - f(x)| \geq 2^{-k}\}\bigr) \leq 2^{-k}$ and use Problem 1.11-(2) to show that $\mu\bigl(X \setminus \liminf_{k\to\infty}\{x \in X \mid |f_{n_k}(x) - f(x)| < 2^{-k}\}\bigr) = 0$.)

Problem 1.32. Let $A \in \mathcal{M}$, and define a measure $\mu|_A$ on $\mathcal{M}|_A = \{B \cap A \mid B \in \mathcal{M}\}$ by $\mu|_A := \mu|_{\mathcal{M}|_A}$ (note that $\mathcal{M}|_A \subset \mathcal{M}$). Let $f : X \to [-\infty,\infty]$ be $\mathcal{M}$-measurable. Prove that $\int_X f\mathbf{1}_A d\mu$ exists if and only if $\int_A f|_A d(\mu|_A)$ exists, and in this case

$$\left(\int_A f d\mu :=\right) \int_X f\mathbf{1}_A d\mu = \int_A f|_A \, d(\mu|_A). \tag{1.85}$$

(Modify the proof of Theorem 1.43. It suffices to prove (1.85) for $f^{\pm}$ and hence when $f \geq 0$. Show first for non-negative $\mathcal{M}$-simple functions by using Proposition 1.25 and then use Proposition 1.19 and Theorem 1.24 for general non-negative f.)

According to Problem 1.32, $\int_A f d\mu$ could alternatively be defined as the integral of $f|_A$ with respect to $\mu|_A = \mu|_{\mathcal{M}|_A}$, the *restriction of μ to A*.

Exercise 1.33. Let $\mathcal{N}$ be a σ-algebra in X such that $\mathcal{N} \subset \mathcal{M}$, and let $f : X \to [-\infty,\infty]$ be $\mathcal{N}$-measurable. Prove that $\int_X f d\mu$ exists if and only if $\int_X f d(\mu|_{\mathcal{N}})$ exists (note that $\mu|_{\mathcal{N}}$ is a measure on $(X,\mathcal{N})$), and in this case

$$\int_X f d\mu = \int_X f d(\mu|_{\mathcal{N}}). \tag{1.86}$$

Exercise 1.34. Let $f : X \to [0,\infty]$ be $\mathcal{M}$-measurable and μ-integrable. Prove that, for any $\varepsilon \in (0,\infty)$ there exists $\delta \in (0,\infty)$ such that $\int_A f d\mu < \varepsilon$ for any $A \in \mathcal{M}$ with $\mu(A) < \delta$. (Proof by contradiction. Problem 1.11-(2) can be used.)