Problem set 6, submit solutions by 24.10.2012

The **Problem**s below will be discussed in the tutorial on 26.10.2012.
(The **Exercise**s are additional and will be discussed only if time permits.)

In Exercise 1.35, $(X, \mathcal{M}, \mu)$ denotes a given measure space. The next problem requires the following definition.

Exercise 1.35 (30 points)**.** Assume that $(X, \mathcal{M}, \mu)$ is σ-finite (see Definition 2.25). Let $p \in (1, \infty)$, $q := p/(p-1)$, and let $f : X \to [0, \infty]$ be $\mathcal{M}$-measurable. Prove that

$$\|f\|_{L^p} = \sup\left\{ \int_X fg d\mu \;\middle|\; g : X \to [0,\infty],\, g \text{ is } \mathcal{M}\text{-measurable and } \|g\|_{L^q} \le 1 \right\}. \tag{1.87}$$

("$\ge$" is immediate from Hölder's inequality. For "$\le$", let $g := (h/\|h\|_{L^p})^{p-1}$ for a suitable h with $\|h\|_{L^p} \in (0, \infty)$. Treat the case of $\|f\|_{L^p} < \infty$ and that of $\|f\|_{L^p} = \infty$ separately.)

Problem 2.1. Let X be a set and let $\mathcal{D} \subset 2^X$. Prove that $\mathcal{D}$ is a Dynkin system in X if and only if $\mathcal{D}$ satisfies the conditions (D1) and (D2) of Definition 2.1-(2) and the following condition (D3)$'$:

(D3)$'$ If $\{A_n\}_{n=1}^\infty \subset \mathcal{D}$ and $A_i \cap A_j = \emptyset$ for any $i, j \in \mathbb{N}$ with $i \neq j$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{D}$.

Problem 2.2. Let $\Omega := \{0,1\}^{\mathbb{N}} = \{(\omega_n)_{n=1}^\infty \mid \omega_n \in \{0,1\}\}$, let $\mathcal{F}$ be the σ-algebra in Ω defined by (1.11) and let $p \in [0, 1]$. Prove the uniqueness of the Bernoulli measure $\mathbb{P}_p$ on $(\Omega, \mathcal{F})$ of probability p stated in Example 1.12. (Show that the π-system $\mathcal{A} := \{\{\omega\} \times \{0,1\}^{\mathbb{N}\setminus\{1,\ldots,n\}} \mid n \in \mathbb{N}, \omega \in \{0,1\}^n\} \cup \{\emptyset\}$ satisfies $\sigma(\mathcal{A}) = \mathcal{F}$ and apply Theorem 2.5.)

The next exercise requires the following definition.

Definition. Let X be a set and let $\mathcal{A}, \mathcal{M} \subset 2^X$.
(1) $\mathcal{A}$ is called an *algebra in* X if and only if it possesses the following properties:

(A1) $\emptyset \in \mathcal{A}$.

(A2) If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$, where $A^c := X \setminus A$.

(A3) If $n \in \mathbb{N}$ and $\{A_i\}_{i=1}^n \subset \mathcal{A}$ then $\bigcup_{i=1}^n A_i \in \mathcal{A}$.

(2) $\mathcal{M}$ is called a *monotone class in* X if and only if it satisfies the following conditions:

(M1) If $\{A_n\}_{n=1}^\infty \subset \mathcal{M}$ and $A_n \subset A_{n+1}$ for any $n \in \mathbb{N}$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{M}$.

(M2) If $\{A_n\}_{n=1}^\infty \subset \mathcal{M}$ and $A_n \supset A_{n+1}$ for any $n \in \mathbb{N}$, then $\bigcap_{n=1}^\infty A_n \in \mathcal{M}$.

Exercise 2.3. Let X be a set and let $\mathcal{A} \subset 2^X$.
(1) Prove that

$$\mathcal{M}(\mathcal{A}) := \mathcal{M}_X(\mathcal{A}) := \bigcap_{\mathcal{M}\text{: monotone class in } X,\, \mathcal{A} \subset \mathcal{M}} \mathcal{M} \tag{1}$$

is the smallest monotone class in X that includes $\mathcal{A}$, and that $\mathcal{M}(\mathcal{A}) \subset \sigma(\mathcal{A})$.
(2) (Monotone class theorem) Suppose $\mathcal{A}$ is an algebra in X. Prove that

$$\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A}). \tag{2}$$

Problem 2.4. Let X be a set, let $\mathcal{A} \subset 2^X$ and let $\nu : \mathcal{A} \to [0, \infty]$. Suppose $\emptyset \in \mathcal{A}$ and $\nu(\emptyset) = 0$. Prove that the set function $\nu_* : 2^X \to [0, \infty]$ defined by

$$\nu_*(A) := \inf\left\{ \sum_{n=1}^{\infty} \nu(A_n) \,\middle|\, \{A_n\}_{n=1}^{\infty} \subset \mathcal{A},\ A \subset \bigcup_{n=1}^{\infty} A_n \right\} \quad (\inf \emptyset := \infty) \tag{2.11}$$

is an outer measure on X.

Problem 2.5 ([4, Corollary 7.1]). Let μ be a Borel probability measure on $\mathbb{R}$ and let F be its distribution function. Recalling that F is non-decreasing, we define $F(x-) := \lim_{y \uparrow x} F(y)$ for each $x \in \mathbb{R}$. Let $a, b \in \mathbb{R}$, $a < b$. Prove the following equalities:
(1) $\mu\big([a, b]\big) = F(b) - F(a-)$.
(2) $\mu\big([a, b)\big) = F(b-) - F(a-)$.
(3) $\mu\big((a, b)\big) = F(b-) - F(a)$.
(4) $\mu(\{a\}) = F(a) - F(a-)$. (Thus $\mu(\{a\}) = 0$ if and only if F is continuous at a.)

Problem 2.6. Let F be the distribution function of a Borel probability measure on $\mathbb{R}$. Prove that the set $\{x \in \mathbb{R} \mid F(x) \neq F(x-)\}$ is countable, where $F(x-)$ is as in Problem 2.5. (Noting Problem 2.5-(4), use Problem 1.14.)

Problem 2.7 ([4, Exercise 7.18]). Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F := \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbf{1}_{[n^{-1}, \infty)}. \tag{3}$$

(1) Prove that F is the distribution function of a Borel probability measure μ on $\mathbb{R}$.
(2) (5 points each) Let μ be as in (1). Calculate the following values (i)–(vi):

(i)	$\mu\big([1, \infty)\big)$	(ii)	$\mu\big([1/10, \infty)\big)$	(iii)	$\mu(\{0\})$
(iv)	$\mu\big([0, 1/2)\big)$	(v)	$\mu\big((-\infty, 0)\big)$	(vi)	$\mu\big((0, \infty)\big)$