Problem set 7, submit solutions by **12:00** on **02**.11.2012

The **Problem**s below will be discussed in the tutorial on **05**.11.2012.
(The **Exercise**s are additional and will be discussed only if time permits.)

Problem 2.8 (5 points each)**.** Let $d \in \mathbb{N}$, let μ be a Borel probability measure on $\mathbb{R}^d$ and let F_μ be the distribution function of μ. Prove the following statements:
(1) For any $(x_1, \ldots, x_d) \in \mathbb{R}^d$ and any $(h_1, \ldots, h_d) \in [0, \infty)^d$,

$$\begin{aligned} &\mu\big((x_1 - h_1, x_1] \times \cdots \times (x_d - h_d, x_d]\big) \\ &\quad = \sum_{(\alpha_1,\ldots,\alpha_d)\in\{0,1\}^d} (-1)^{\sum_{i=1}^d \alpha_i} F_\mu(x_1 - \alpha_1 h_1, \ldots, x_d - \alpha_d h_d) \geq 0, \end{aligned} \tag{2.17}$$

where $(a, a] := \emptyset$ for $a \in \mathbb{R}$. (Use the inclusion-exclusion formula (1.68).)
(2) For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$\lim_{\substack{(y_1,\ldots,y_d)\to x \\ y_i \geq x_i,\, i\in\{1,\ldots,d\}}} F_\mu(y_1, \ldots, y_d) = F_\mu(x). \tag{2.18}$$

(3) $\lim_{x\to\infty} F_\mu(x, \ldots, x) = 1$, and $\lim_{x_i\to-\infty} F_\mu(x_1, \ldots, x_i, \ldots, x_d) = 0$ for any $i \in \{1, \ldots, d\}$ and any $x_j \in \mathbb{R}$, $j \in \{1, \ldots, d\} \setminus \{i\}$
(4) μ is uniquely determined by its distribution function F_μ. (Show that $\mathcal{A} := \{\emptyset\} \cup \big\{(a_1, b_1] \times \cdots \times (a_d, b_d] \bigm| a_i, b_i \in \mathbb{R}, a_i < b_i, i \in \{1, \ldots, d\}\big\}$ is a π-system and that $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R}^d)$ by using Proposition 1.9, and then use (1) to apply Theorem 2.5.)

Exercise 2.9. Let $d \in \mathbb{N}$ and let μ be a Borel probability measure on $\mathbb{R}^d$. Define

$$C_{\mu,i} := \big\{a \in \mathbb{R} \bigm| \mu\big(H_i(a)\big) = 0\big\}, \quad \text{where } H_i(a) := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i = a\}, \tag{2.58}$$

for each $i \in \{1, \ldots, d\}$ and $C_\mu := C_{\mu,1} \times \cdots \times C_{\mu,d}$. Prove the following statements:
(1) $\mathbb{R} \setminus C_{\mu,i}$ is a countable set for any $i \in \{1, \ldots, d\}$. (Use Problem 1.14.)
(2) The distribution function $F_\mu : \mathbb{R}^d \to [0, 1]$ of μ is continuous at x for any $x \in C_\mu$.

Problem 2.10. Let $(X, \mathcal{M})$ be a measurable space. Let $n \in \mathbb{N}$, and for each $i \in \{1, \ldots, n\}$, let $(S_i, \mathcal{B}_i)$ be a measurable space and let $f_i : X \to S_i$. Prove that the map $f = (f_1, \ldots, f_d) : X \to S_1 \times \cdots \times S_n$ is $\mathcal{M}/\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$-measurable if and only if f_i is $\mathcal{M}/\mathcal{B}_i$-measurable for any $i \in \{1, \ldots, n\}$. (For "if" part, use Problem 1.17-(1) with $S = S_1 \times \cdots \times S_n$ and $\mathcal{A} = \mathcal{B}_1 \times \cdots \times \mathcal{B}_n$.)

Problem 2.11. Let $n \in \mathbb{N}$. For each $i \in \{1, \ldots, n\}$, let $(X_i, \mathcal{M}_i, \mu_i)$ be a σ-finite measure space and let $f_i : X_i \to [-\infty, \infty]$ be $\mathcal{M}_i$-measurable. For each $i \in \{1, \ldots, n\}$ define $F_i : X_1 \times \cdots \times X_n \to [-\infty, \infty]$ by $F_i(x_1, \ldots, x_n) := f_i(x_i)$, and define $F : X_1 \times \cdots \times X_n \to [-\infty, \infty]$ by $F(x_1, \ldots, x_n) := f_1(x_1) \cdots f_n(x_n)$. Prove the following statements:
(1) F_i is $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$-measurable for any $i \in \{1, \ldots, n\}$.

(2) F is $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$-measurable. ($F = F_1 \cdots F_n$. Proposition 1.15-(2) applies.)
(3) If f_i is μ_i-integrable for any $i \in \{1, \ldots, n\}$, then F is $\mu_1 \times \cdots \times \mu_n$-integrable and

$$\int_{X_1 \times \cdots \times X_n} F d(\mu_1 \times \cdots \times \mu_n) = \int_{X_1} f_1 d\mu_1 \cdots \int_{X_n} f_n d\mu_n. \tag{2.59}$$

(Induction in n. Use Proposition 2.23 and Corollary 2.27 to apply Theorem 2.30-(2).)

Problem 2.12. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, let $f : X \to [0, \infty]$ be $\mathcal{M}$-measurable and set $S_f := \{(x, t) \in X \times \mathbb{R} \mid 0 \le t < f(x)\}$.
(1) Prove that $S_f \in \mathcal{M} \otimes \mathcal{B}(\mathbb{R})$ and that $[0, \infty) \ni t \mapsto \mu\bigl(\{x \in X \mid f(x) > t\}\bigr) \in [0, \infty]$ is Borel measurable. (To show $S_f \in \mathcal{M} \otimes \mathcal{B}(\mathbb{R})$, apply Problem 2.11-(1) to $X \times \mathbb{R} \ni (x, t) \mapsto f(x)$ and $X \times \mathbb{R} \ni (x, t) \mapsto t$ and then use Problem 1.15-(1).)
(2) Prove that $\int_X f d\mu = \mu \times \mathrm{m}_1(S_f)$ and that for any $p \in (0, \infty)$,

$$\int_X f^p d\mu = p \int_0^\infty t^{p-1} \mu\bigl(\{x \in X \mid f(x) > t\}\bigr) dt. \tag{2.60}$$

(3) Prove that $\mathrm{m}_2\bigl(\{x \in \mathbb{R}^2 \mid |x| < r\}\bigr) = \pi r^2$ for any $r \in (0, \infty)$.

Exercise 2.13 ([7, Counterexamples 8.9])**.** (1) Let # denote the counting measure on $[0, 1]$ and set $\Delta_{[0,1]} := \{(x, y) \in [0, 1]^2 \mid x = y\}$, which is closed in $\mathbb{R}^2$. Prove that

$$\int_0^1 \left(\int_{[0,1]} \mathbf{1}_{\Delta_{[0,1]}}(x, y) d\#(y) \right) dx = 1 \neq 0 = \int_{[0,1]} \left(\int_0^1 \mathbf{1}_{\Delta_{[0,1]}}(x, y) dx \right) d\#(y). \tag{2.61}$$

(2) Let $\{\delta_n\}_{n=0}^\infty \subset [0, 1)$ be such that $\delta_0 = 0$, $\delta_{n-1} < \delta_n$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} \delta_n = 1$. Also for each $n \in \mathbb{N}$, let $g_n : [0, 1) \to \mathbb{R}$ be a continuous function such that $g_n|_{[0,1)\setminus(\delta_{n-1},\delta_n)} = 0$ and $\int_0^1 g_n(x) dx = 1$. Define $f : [0, 1)^2 \to \mathbb{R}$ by

$$f(x, y) := \sum_{n=1}^\infty \bigl(g_n(x) - g_{n+1}(x)\bigr) g_n(y). \tag{2.62}$$

Prove the following statements:
(i) f is continuous and $\int_0^1 \bigl(\int_0^1 |f(x, y)| dx\bigr) dy = \infty$.
(ii) For any $x, y \in [0, 1)$, $f(x, \cdot), f(\cdot, y) \in \mathcal{L}^1\bigl([0, 1), \mathrm{m}_1\bigr)$, $\int_0^1 f(x, z) dz = g_1(x)$ and $\int_0^1 f(z, y) dz = 0$. In particular,

$$\int_0^1 \left(\int_0^1 f(x, y) dy \right) dx = 1 \neq 0 = \int_0^1 \left(\int_0^1 f(x, y) dx \right) dy. \tag{2.63}$$

Problem 2.14. (1) Prove that

$$\int_0^\infty \left| \frac{\sin x}{x} \right| dx = \infty. \tag{2.64}$$

(Estimate $\int_{n-1}^n \left| \frac{\sin x}{x} \right| dx$ from below for each $n \in \mathbb{N}$.)
(2) Use $x^{-1} = \int_0^\infty e^{-xt} dt$, $x \in (0, \infty)$, to prove that

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x} dx = \int_0^\infty \frac{1 - \cos x}{x^2} dx = \int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}. \tag{2.65}$$