Problem set 8, submission of solutions **NOT** required

The **Problem**s below will be discussed in the tutorial on **06**.11.2012.

Problem 8.1. Let $(X, \mathcal{M})$ be a measurable space and let $f, g : X \to [-\infty, \infty]$ be $\mathcal{M}$-measurable. Prove that the following sets belong to $\mathcal{M}$:

$$\{x \in X \mid f(x) < g(x)\}, \quad \{x \in X \mid f(x) = g(x)\}, \quad \{x \in X \mid f(x) > g(x)\}.$$

Problem 8.2. Find the limits as $N \to \infty$ of the following series:

$$(1) \sum_{n=1}^{\infty} 2^{-n} \left(1 + \frac{\sin(2^N n)}{N}\right)^{-1} \quad (2) \sum_{n=1}^{\infty} \frac{1}{n(n+N)} \quad (3) \sum_{n=1}^{\infty} \left(1 + \frac{n}{N}\right)^{-N}$$

Problem 8.3. Find the limits as $n \to \infty$ of the following integrals:

$$(1) \int_0^{\infty} \frac{1}{1+x^n} dx \quad (2) \int_0^{\infty} \frac{\sin e^x}{1+nx^2} dx \quad (3) \int_0^1 \frac{n \cos x}{1+n^2 x^{3/2}} dx$$

Problem 8.4. Let $(X, \mathcal{M}, \mu)$ be a measure space.
(1) Let $f_n : X \to [-\infty, \infty]$ be $\mathcal{M}$-measurable for each $n \in \mathbb{N}$ and suppose that $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$. Prove that $\lim_{n\to\infty} f_n(x) = 0$ for μ-a.e. $x \in X$.
(2) ([1, Section 4.3, Problem 1]) Let $f \in \mathcal{L}^1(\mu)$ and $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^1(\mu)$. Suppose that $f_n \geq 0$ on X for any $n \in \mathbb{N}$, that $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in X$, and that $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$. Prove that $\lim_{n\to\infty} \int_X |f - f_n| d\mu = 0$.

Problem 8.5. Let $(X, \mathcal{M})$ be a measurable space.
(1) Let S be a set, let $\mathcal{A} \subset 2^S$ and let $f : X \to S$. Prove that f is $\mathcal{M}/\sigma_S(\mathcal{A})$-measurable if and only if $f^{-1}(A) \in \mathcal{M}$ for any $A \in \mathcal{A}$.
(2) Let $n \in \mathbb{N}$, and for each $i \in \{1, \dots, n\}$ let $(S_i, \mathcal{B}_i)$ be a measurable space and let $f_i : X \to S_i$. Prove that the map $f := (f_1, \dots, f_n) : X \to S_1 \times \cdots \times S_n$ is $\mathcal{M}/\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$-measurable if and only if f_i is $\mathcal{M}/\mathcal{B}_i$-measurable for any $i \in \{1, \dots, n\}$.
(3) Let $d \in \mathbb{N}$ and let $f = (f_1, \dots, f_d) : X \to \mathbb{R}^d$, where $f_i : X \to \mathbb{R}$ for each $i \in \{1, \dots, d\}$. Prove that f is $\mathcal{M}/\mathcal{B}(\mathbb{R}^d)$-measurable if and only if f_i is $\mathcal{M}$-measurable for any $i \in \{1, \dots, d\}$. (Proposition 2.24-(2) may be used.)

Problem 8.6. Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, let $f : X \to \mathbb{R}$ be $\mathcal{M}$-measurable and let $g : Y \to \mathbb{R}$ be $\mathcal{N}$-measurable. Define $f \otimes g : X \times Y \to \mathbb{R}$ by $(f \otimes g)(x, y) := f(x)g(y)$. Prove the following statements:
(1) $f \otimes g$ is $\mathcal{M} \otimes \mathcal{N}$-measurable.
(2) If f is μ-integrable and g is ν-integrable, then $f \otimes g$ is $\mu \times \nu$-integrable and

$$\int_{X \times Y} f \otimes g d(\mu \times \nu) = \int_X f d\mu \int_Y g d\nu.$$