Problem set 9, submit solutions by 14.11.2012

The **Problem**s below will be discussed in the tutorial on 16.11.2012.
(The **Exercise** is additional and will be discussed only if time permits.)

In the problems and the exercise below, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and all random variables are assumed to be defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Problem 3.1. Let $d \in \mathbb{N}$ and let $x \in \mathbb{R}^d$. Prove that the unit mass δ_x at x defined by $\delta_x(A) := \mathbf{1}_A(x)$, $A \in \mathcal{B}(\mathbb{R}^d)$ (recall Example 1.5-(2)), does not have a density.

Problem 3.2. Calculate $\mathbb{E}[X]$ and $\mathrm{var}(X)$ for a real random variable X with
(1) the binomial distribution $B(n, p)$, $n \in \mathbb{N}$, $p \in [0, 1]$.
(2) the Poisson distribution $\mathrm{Po}(\lambda)$, $\lambda \in (0, \infty)$.
(3) the geometric distribution $\mathrm{Geom}(\alpha)$, $\alpha \in [0, 1)$.

Problem 3.3. Calculate $\mathbb{E}[X]$ and $\mathrm{var}(X)$ for a real random variable X with
(1) the uniform distribution $\mathrm{Unif}(a, b)$, $a, b \in \mathbb{R}$, $a < b$.
(2) the exponential distribution $\mathrm{Exp}(\alpha)$, $\alpha \in (0, \infty)$.
(3) the gamma distribution $\mathrm{Gamma}(\alpha, \beta)$, $\alpha, \beta \in (0, \infty)$.

Problem 3.4. Let X be an exponential random variable. Prove that

$$\mathbb{P}[X > s + t \mid X > s] = \mathbb{P}[X > t] \qquad \text{for any } s, t \in [0, \infty) \tag{3.79}$$

(recall (1.66) for the definition of conditional probabilities).

(3.79) is known as the "memoryless property" of exponential random variables. Due to this property, exponential random variables are often used as *"random alarm clocks with no memory"*.

Exercise 3.5. Let X be a real random variable such that $\mathbb{P}[X > 0] > 0$, and suppose $\mathbb{P}[X > s + t \mid X > s] = \mathbb{P}[X > t]$ for any $s, t \in (0, \infty)$ with $\mathbb{P}[X > s] > 0$. Define $h : \mathbb{R} \to [0, 1]$ by $h(t) := \mathbb{P}[X > t]$. Prove the following statements:
(1) h is right-continuous and $h(s + t) = h(s)h(t)$ for any $s, t \in [0, \infty)$.
(2) There exists $\alpha \in (0, \infty)$ such that $h(t) = e^{-\alpha t}$ for any $t \in [0, \infty)$.
(3) X is an exponential random variable of parameter α.

Problem 3.6. Let X be a normal random variable with mean m and variance $v \in (0, \infty)$. Prove that the real random variable $Y := e^X$ has a density ρ_Y given by

$$\rho_Y(x) = \frac{1}{x\sqrt{2\pi v}} \exp\left(-\frac{(\log x - m)^2}{2v}\right)\mathbf{1}_{(0,\infty)}(y). \tag{3.80}$$

The law of Y is called the *lognormal distribution with parameters m, v*.

Problem 3.7. Let X be a normal random variable with mean 0 and variance 1. Prove that the real random variable $Z := X^2$ has a density ρ_Z given by

$$\rho_Z(x) = \frac{1}{\sqrt{2\pi x}} e^{-x/2} \mathbf{1}_{(0,\infty)}(x). \tag{3.81}$$

The law of Z is called the *chi square distribution with one degree of freedom* and denoted as χ_1^2. (In fact, (3.81) and (3.21) easily imply that $\chi_1^2 = \text{Gamma}(1/2, 1/2)$.)

Problem 3.8. Let $m \in \mathbb{R}$, $\alpha \in (0, \infty)$ and let X be a Cauchy random variable with parameters m, α. Prove that X does not admit the mean, i.e. $\mathbb{E}[X^+] = \mathbb{E}[X^-] = \infty$.