Contraction properties and differentiability of p-energy forms with applications to nonlinear potential theory on self-similar sets

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Abstract

We introduce a new contraction property, which we call the *generalized p-contraction property*, for p-energy forms as generalizations of many well-known inequalities, such as p-Clarkson's inequality, the strong subadditivity and the Markov property in the theory of nonlinear Dirichlet forms, and show that any p-energy form satisfying p-Clarkson's inequality is Fréchet differentiable. We also verify the generalized p-contraction property for p-energy forms on fractals constructed by Kigami [*Mem. Eur. Math. Soc.* **5** (2023)] and by Cao–Gu–Qiu [*Adv. Math.* **405** (2022), no. 108517]. As a general framework of p-energy forms taking the generalized p-contraction property into consideration, we introduce the notion of *p-resistance form* and investigate fundamental properties of p-harmonic functions with respect to p-resistance forms. In particular, some new estimates on scaling factors of self-similar p-energy forms on self-similar sets are obtained by establishing Hölder regularity estimates for p-harmonic functions, and the p-walk dimensions of any generalized Sierpiński carpet and the D-dimensional level-l Sierpiński gasket are shown to be strictly greater than p.

Keywords: generalized p-contraction property, p-Clarkson's inequality, p-energy measure, p-resistance form, p-harmonic function, self-similar p-energy form, p-walk dimension

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Contents

1 Introduction

In the late 1980s, Goldstein [Gol87] and Kusuoka [Kus87] independently constructed a Brownian motion (a canonical diffusion process) on the Sierpiński gasket (the left of Figure 1.2) as a scaling limit of the simple random walks on pre-gaskets (approximating graphs), and Barlow–Perkins [BP88] established detailed estimates called the sub-Gaussian heat kernel estimates for its transition density. Subsequently, Kigami [Kig89] directly constructed the Laplacian on the Sierpiński gasket as a scaling limit of the discrete Laplacians on pre-gaskets, and Fukushima–Shima [FS92] indicated that the theory of Dirichlet forms was well-applicable to the field of analysis on fractals; more precisely, Fukushima and Shima gave a direct description of the regular symmetric Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$ corresponding to the Friedrichs extension of Kigami's Laplacian, which is an analogue of the pair of the Dirichlet 2-energy $\int |\nabla u|^2 \, dx =: \mathcal{E}_2(u)$ and the associated $(1,2)$-Sobolev space $W^{1,2} =: \mathcal{F}_2$ on smooth spaces, and used it to investigate the eigenvalue problems for Kigami's Laplacian[1]. Later, Kigami [Kig93] extended the method in [FS92] to *post-critically finite self-similar sets* (Definition 5.3), and Kusuoka–Zhou [KZ92] constructed regular symmetric Dirichlet forms $(\mathcal{E}_2, \mathcal{F}_2)$ on a large class of self-similar sets including the Sierpiński carpet (the right of Figure 1.2) through a subsequential scaling limit of discrete Dirichlet forms. (The first construction of a Brownian motion on the Sierpiński carpet was done by Barlow–Bass [BB89] by establishing a subsequential convergence of scaled Brownian motions on pre-carpets.) See, e.g., [Bar13, Kig01] for further background on the field of analysis on fractals. As another advantage of the theory of Dirichlet forms, once we obtain a regular symmetric Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$, we can capture the associated energy measure $\Gamma_2\langle u\rangle$ playing the role of $|\nabla u|^2 \, dx$ although the density "$|\nabla u|$" usually does not make sense on fractals due to the singularity of $\Gamma_2\langle u\rangle$ with respect to the canonical volume measure (see [Hin05, KM20] for details of this singularity of $\Gamma_2\langle u\rangle$).

The main purpose of this article is to develop a general theory of L^p-analogues of $(\mathcal{E}_2, \mathcal{F}_2, \Gamma_2\langle\,\cdot\,\rangle)$, where $p \in (1,\infty)$, on the basis of the new contraction property which we call the *generalized p-contraction property*. For a large class of triples (K, m, p) of a self-similar set K, a natural self-similar measure m on K and $p \in (1,\infty)$, an L^p-analogue of $(\mathcal{E}_2, \mathcal{F}_2)$ on (K, m), namely a p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ playing the role of $\int |\nabla u|^p \, dx$ and the associated $(1,p)$-Sobolev space $W^{1,p}$, where $\mathcal{F}_p$ is a linear subspace of $L^p(K, m)$ and $\mathcal{E}_p \colon \mathcal{F}_p \to [0,\infty)$ is such that $\mathcal{E}_p^{1/p}$ is a seminorm on $\mathcal{F}_p$, has been constructed in several works [CGQ22, HPS04, Kig23, KO+, MS25+, Shi24][2], most of which are very recent. Furthermore, the associated p-energy measure $\Gamma_p\langle u\rangle$, which is a finite Borel measure on K and an analogue of $|\nabla u|^p \, dx$, has been introduced in [MS25+, Shi24] with the help of

[1]The results in [FS92, Kig89] were proved for the D-dimensional level-2 Sierpiński gasket (Framework 9.9), where $D \in \mathbb{N}$ with $D \geq 2$.

[2]The main difference among these works is the classes of (K, m, p) on which $(\mathcal{E}_p, \mathcal{F}_p)$ is constructed. Let us briefly summarize what classes of (K, m, p) is treated in these works (see [KS23+, Introduction] for details). In [CGQ22, HPS04], K is assumed to be a post-critically finite self-similar set (Definition 5.3) so that the Sierpiński gasket is included while the Sierpiński carpet is excluded. The case where K is the Sierpiński carpet is allowed in [Kig23, KO+, MS25+, Shi24], but we need to assume that p is strictly greater than the Ahlfors regular conformal dimension of K (Definition 8.5-(4)) in [Kig23, KO+, Shi24].

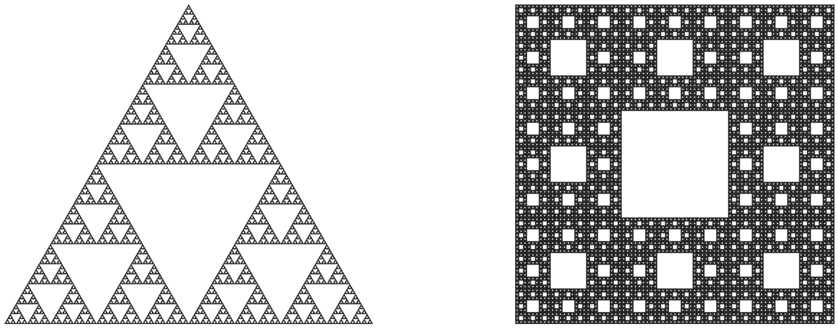

Figure 1.1: The Sierpiński gasket (left) and the Sierpiński carpet (right)

the self-similarity of $(\mathcal{E}_p, \mathcal{F}_p)$. See Section 5 for details on the self-similarity of a p-energy form, and Example 4.2 for examples of p-energy measures which do not rely on the self-similarity. Compared with the case of $p = 2$, where the theory of symmetric Dirichlet forms is applicable, very little has been established for $p \in (1, \infty) \setminus \{2\}$ in the direction of dealing with $(\mathcal{E}_p, \mathcal{F}_p, \Gamma_p\langle \cdot \rangle)$ in a general framework. In particular, there are two missing pieces in known results for $(\mathcal{E}_p, \mathcal{F}_p, \Gamma_p\langle \cdot \rangle)$: first, useful contraction properties of it, and secondly, the (Fréchet) differentiabilities of $\mathcal{E}_p$ and of Γ_p. In the first half of this paper (Sections 2–5), we aim at establishing general results filling these missing pieces. We shall explain more details of the main results of these sections below.

The first missing piece is contraction properties of $(\mathcal{E}_p, \mathcal{F}_p, \Gamma_p\langle \cdot \rangle)$. Every p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ constructed in the previous studies is known to satisfy the following *unit contractivity:*

$$u^+ \wedge 1 \in \mathcal{F}_p \quad \text{and} \quad \mathcal{E}_p(u^+ \wedge 1) \leq \mathcal{E}_p(u) \quad \text{for any } u \in \mathcal{F}_p. \tag{1.1}$$

In the case of $p = 2$, by using some helpful expressions of $\mathcal{E}_2$, e.g., [FOT, Lemma 1.3.4 and (3.2.12)], (1.1) can be improved to the following *normal contractivity* (see [MR, Theorem I.4.12] for example): if $n \in \mathbb{N}$ and $T\colon \mathbb{R}^n \to \mathbb{R}$ satisfy $|T(x)| \leq \sum_{k=1}^{n} |x_k|$ and $|T(x) - T(y)| \leq \sum_{k=1}^{n} |x_k - y_k|$ for any $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, then for any $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathcal{F}_2^n$ we have

$$T(\boldsymbol{u}) \in \mathcal{F}_2 \quad \text{and} \quad \mathcal{E}_2(T(\boldsymbol{u}))^{\frac{1}{2}} \leq \sum_{k=1}^{n} \mathcal{E}_2(u_k)^{\frac{1}{2}}. \tag{1.2}$$

It is natural to expect that $(\mathcal{E}_p, \mathcal{F}_p)$ for $p \in (1, \infty) \setminus \{2\}$ also has a similar property to (1.2) since $\mathcal{E}_p(u)$ is an analogue of $\int |\nabla u|^p \, dx$; nevertheless, it is not clear whether (1.1) can be improved in such a way without going back to the constructions of $(\mathcal{E}_p, \mathcal{F}_p)$ in the previous studies. Not only (1.2) but also other useful inequalities like the following *strong subadditivity* and *p-Clarkson's inequality*, were not mentioned in [CGQ22, HPS04, Kig23, MS25+, Shi24]:

(Strong subadditivity) For any $u, v \in \mathcal{F}_p$, we have $u \vee v, u \wedge v \in \mathcal{F}_p$ and

$$\mathcal{E}_p(u \vee v) + \mathcal{E}_p(u \wedge v) \leq \mathcal{E}_p(u) + \mathcal{E}_p(v). \tag{1.3}$$

(p-Clarkson's inequality) For any $u, v \in \mathcal{F}_p$,

$$\begin{cases} \mathcal{E}_p(u+v) + \mathcal{E}_p(u-v) \geq 2\bigl(\mathcal{E}_p(u)^{\frac{1}{p-1}} + \mathcal{E}_p(v)^{\frac{1}{p-1}}\bigr)^{p-1} & \text{if } p \in (1,2], \\ \mathcal{E}_p(u+v) + \mathcal{E}_p(u-v) \leq 2\bigl(\mathcal{E}_p(u)^{\frac{1}{p-1}} + \mathcal{E}_p(v)^{\frac{1}{p-1}}\bigr)^{p-1} & \text{if } p \in (2,\infty). \end{cases} \tag{Cla$_p$}$$

These inequalities play significant roles in the *nonlinear potential theory* with respect to $(\mathcal{E}_p, \mathcal{F}_p)$. For example, (1.3) will be important to consider the p-capacity associated with $(\mathcal{E}_p, \mathcal{F}_p)$; see [BV05, (H3)]. Also, we will frequently use (Cla)$_p$ in this paper; see Theorem 1.3 below for one of the most important consequences of (Cla)$_p$. Since it is not known, unlike the case of $p = 2$, whether such desirable inequalities as (1.2), (1.3) and (Cla)$_p$ are implied by the unit contractivity (1.1), one needs to go back to the constructions of $(\mathcal{E}_p, \mathcal{F}_p)$ in the preceding works if one wishes to show them. The situation is similar for p-energy measures. While it is natural to expect that contraction properties of $(\mathcal{E}_p, \mathcal{F}_p)$ are inherited by the associated p-energy measures, in order to show them for p-energy measures, we need to recall how p-energy measures are constructed, partially because no canonical way to define p-energy measures for a given p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ is known (see [MS25+, Problem 10.4]).

To overcome this situation, in this paper we develop a general theory of p-energy forms on the basis of the *generalized p-contraction property*, which is arguably the strongest possible form of contraction properties of p-energy forms and defined as follows. Throughout the rest of this section, we fix $p \in (1,\infty)$, a measure space $(X, \mathcal{B}, m)$, and the pair $(\mathcal{E}_p, \mathcal{F}_p)$ of a linear subspace $\mathcal{F}_p$ of $L^0(X,m)$[3] and a functional $\mathcal{E}_p \colon \mathcal{F}_p \to [0,\infty)$ which is *p-homogeneous*, i.e., satisfies $\mathcal{E}_p(au) = |a|^p\,\mathcal{E}_p(u)$ for any $u \in \mathcal{F}_p$ and any $a \in \mathbb{R}$. The pair $(\mathcal{E}_p, \mathcal{F}_p)$ is said to be a *p-energy form* on (X,m) if and only if $\mathcal{E}_p^{1/p}$ is a seminorm on $\mathcal{F}_p$.

Definition 1.1 (Generalized p-contraction property; Definition 2.2)**.** We say that $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies the *generalized p-contraction property*, (GC)$_p$ for short, if and only if the following holds: if $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy $T(0) = 0$ and $\|T(x) - T(y)\|_{\ell^{q_2}} \leq \|x - y\|_{\ell^{q_1}}$ for any $x, y \in \mathbb{R}^{n_1}$, then for any $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathcal{F}_p^{n_1}$ we have

$$T(\boldsymbol{u}) \in \mathcal{F}_p^{n_2} \quad \text{and} \quad \Bigl\| \bigl(\mathcal{E}_p(T_l(\boldsymbol{u}))^{\frac{1}{p}}\bigr)_{l=1}^{n_2} \Bigr\|_{\ell^{q_2}} \leq \Bigl\| \bigl(\mathcal{E}_p(u_k)^{\frac{1}{p}}\bigr)_{k=1}^{n_1} \Bigr\|_{\ell^{q_1}}. \tag{GC$_p$}$$

Note that the particular case of (GC)$_p$ for $(p, n_1, n_2, q_1, q_2) = (2, n, 1, 1, p)$ is nothing but the normal contractivity (1.2). As recorded in the following proposition, (GC)$_p$ is actually a generalization of many useful inequalities like (1.2), (1.3) and (Cla)$_p$.

Proposition 1.2 (Proposition 2.3)**.** *Let $\varphi \in C(\mathbb{R})$ satisfy $\varphi(0) = 0$ and $|\varphi(t) - \varphi(s)| \leq |t - s|$ for any $s, t \in \mathbb{R}$. Assume that $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies* (GC)$_p$. *Then the following hold.*

(a) (Triangle inequality and strict convexity) *$\mathcal{E}_p^{1/p}$ is a seminorm on $\mathcal{F}_p$, and for any $\lambda \in (0,1)$ and any $f, g \in \mathcal{F}_p$ with $\mathcal{E}_p(f) \wedge \mathcal{E}_p(g) \wedge \mathcal{E}_p(f - g) > 0$,*

$$\mathcal{E}_p(\lambda f + (1-\lambda) g) < \lambda \mathcal{E}_p(f) + (1-\lambda)\mathcal{E}_p(g).$$

[3] We set $L^0(X,m) \coloneqq \{\text{the } m\text{-equivalence class of } f \mid f \colon X \to \mathbb{R},\ f \text{ is } \mathcal{B}\text{-measurable}\}$; see (2.1).

(b) (Lipschitz contractivity) $\varphi(u) \in \mathcal{F}_p$ *and* $\mathcal{E}_p(\varphi(u)) \leq \mathcal{E}_p(u)$ *for any* $u \in \mathcal{F}_p$.

(c) (Strong subadditivity) *Assume that* φ *is non-decreasing. Then for any* $f, g \in \mathcal{F}_p$,

$$\mathcal{E}_p\bigl(f - \varphi(f-g)\bigr) + \mathcal{E}_p\bigl(g + \varphi(f-g)\bigr) \leq \mathcal{E}_p(f) + \mathcal{E}_p(g).$$

In particular, (1.3) *holds.*

(d) (Leibniz rule) *For any* $f, g \in \mathcal{F}_p \cap L^\infty(X,m)$, *we have*

$$f \cdot g \in \mathcal{F}_p \quad \textit{and} \quad \mathcal{E}_p(f \cdot g)^{\frac{1}{p}} \leq \|g\|_{L^\infty(K,m)} \mathcal{E}_p(f)^{\frac{1}{p}} + \|f\|_{L^\infty(K,m)} \mathcal{E}_p(g)^{\frac{1}{p}}.$$

(e) (p-Clarkson's inequality) *Let* $f, g \in \mathcal{F}_p$. *If* $p \in (1,2]$, *then*

$$2\bigl(\mathcal{E}_p(f) + \mathcal{E}_p(g)\bigr) \geq \mathcal{E}_p(f+g) + \mathcal{E}_p(f-g) \geq 2\bigl(\mathcal{E}_p(f)^{\frac{1}{p-1}} + \mathcal{E}_p(g)^{\frac{1}{p-1}}\bigr)^{p-1}.$$

If $p \in [2,\infty)$, *then*

$$2\bigl(\mathcal{E}_p(f) + \mathcal{E}_p(g)\bigr) \leq \mathcal{E}_p(f+g) + \mathcal{E}_p(f-g) \leq 2\bigl(\mathcal{E}_p(f)^{\frac{1}{p-1}} + \mathcal{E}_p(g)^{\frac{1}{p-1}}\bigr)^{p-1}.$$

In particular, $(\mathrm{Cla})_p$ *holds.*

Since the generalized p-contraction property is introduced as arguably the strongest possible formulation of the contraction property of $(\mathcal{E}_p, \mathcal{F}_p)$, it is highly non-trivial whether p-energy forms constructed in the previous studies satisfy it. In Section 8, we see that the existing constructions of p-energy forms in the previous studies do yield ones satisfying $(\mathrm{GC})_p$. (See also [KS24+] for another approach, which is based on Korevaar–Schoen p-energy forms, to obtain p-energy forms satisfying $(\mathrm{GC})_p$.)

In the rest of this section, we assume that $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-energy form on (X, m). The other missing piece in the previous studies on p-energy forms is their differentiability, which should be useful to study *p-harmonic functions* with respect to $\mathcal{E}_p$. (See [KM23, Problem 7.7] and [MS25+, Conjecture 10.8] for some motivations to investigate p-harmonic functions on fractals.) In [CGQ22, HPS04, Shi24], p-harmonic functions are defined as functions minimizing $\mathcal{E}_p$ under prescribed boundary values. However, it is still unclear how to give an equivalent definition of p-harmonic function in a weak sense due to the lack of a "two-variable version" $\mathcal{E}_p(u;\varphi)$ [Kig23, Problem 2 in Section 6.3]. We shall recall the Euclidean case to explain the importance of this object. Let $D \in \mathbb{N}$ and let U be an open subset of $\mathbb{R}^D$. A function $u \in W^{1,p}(\mathbb{R}^D)$ is said to be p-harmonic on U in the weak sense if and only if

$$\int_{\mathbb{R}^D} |\nabla u(x)|^{p-2} \langle \nabla u(x), \nabla \varphi(x) \rangle_{\mathbb{R}^D} \, dx = 0 \quad \text{for every } \varphi \in C_c^\infty(U), \tag{1.4}$$

where $\langle \cdot\,, \cdot \rangle_{\mathbb{R}^D}$ denotes the inner product of $\mathbb{R}^D$. It is well known that (1.4) is equivalent to the variational equality

$$\int_{\mathbb{R}^D} |\nabla u(x)|^p \, dx = \inf\biggl\{ \int_{\mathbb{R}^D} |\nabla v(x)|^p \, dx \biggm| v \in W^{1,p}(\mathbb{R}^D),\, v - u \in W_0^{1,p}(U) \biggr\}. \tag{1.5}$$

The issue in considering an analogue of (1.4) for $\mathcal{E}_p$ is that we do not have a satisfactory counterpart, $\mathcal{E}_p(u;\varphi)$, of $\int |\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle\,dx$ associated with $\mathcal{E}_p$. As mentioned in [SW04, (2.1)], the ideal definition of $\mathcal{E}_p(u;\varphi)$[4] is

$$\mathcal{E}_p(u;\varphi) := \frac{1}{p}\,\frac{d}{dt}\mathcal{E}_p(u+t\varphi)\bigg|_{t=0}, \tag{1.6}$$

but the existence of this derivative is unclear[5] because the constructions of $\mathcal{E}_p$ in the previous studies include many steps such as the operation of taking a subsequential scaling limit of discrete p-energy forms. Similarly, in respect of p-energy measures, no suitable way is known to define a "two-variable version" $\Gamma_p\langle u;\varphi\rangle$ which plays the role of $|\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle\,dx$. The ideal definition of $\Gamma_p\langle u;\varphi\rangle$ is similar to (1.6), i.e., for any Borel subset A of K,

$$\Gamma_p\langle u;\varphi\rangle(A) := \frac{1}{p}\,\frac{d}{dt}\Gamma_p\langle u+t\varphi\rangle(A)\bigg|_{t=0}. \tag{1.7}$$

Such a signed measure was discussed in [BV05, Section 5], but the existence of the derivative in (1.7) (in some uniform manner) was an assumption in [BV05]; see [BV05, (H4) and the beginning of Section 5] for details. Similarly, in [Cap07], the (scale-invariant) elliptic Harnack inequality for p-harmonic functions on *metric fractals* ([Cap07, Definition 2.3]) was proved under some assumptions including the existence of $\Gamma_p\langle u;\varphi\rangle$, which was called the *measure-valued p-Lagrangian* and denoted by $\mathcal{L}^{(p)}(u,\varphi)$ in [Cap07]. However, for situations where no explicit expression of the p-energy measure $\Gamma_p\langle u\rangle$ is available unlike the case of the Euclidean spaces, there is no proof of the existence of the derivative in (1.7) in the literature. (The p-energy form on the Sierpiński gasket constructed in [HPS04] is discussed in [Cap07, Section 5] as a concrete examples and it is stated in [Cap07, p. 1315] that "we can define the corresponding Lagrangian $\mathcal{L}^{(p)}(u,v)$", but we have been unable to find in the literature a rigorous proof of the existence of the derivatives in [Cap07, p. 1315] defining $\mathcal{E}_g(u,v)$ and in [Cap07, p. 1303, (L5)] defining $\mathcal{L}^{(p)}(u,v)$ for the p-energy form on the Sierpiński gasket obtained in [HPS04].)

As another main contribution of this paper, we make a key observation that p-Clarkson's inequality $(\mathrm{Cla})_p$ implies the desired differentiability of $\mathcal{E}_p$. In addition to this result, we record basic properties of $\mathcal{E}_p(u;\varphi)$ given by (1.6) in the following theorem.

Theorem 1.3 (Proposition 3.6 and Theorem 3.7)**.** *Assume that $(\mathcal{E}_p,\mathcal{F}_p)$ satisfies $(\mathrm{Cla})_p$. Then the function $\mathbb{R}\ni t\mapsto \mathcal{E}_p(f+tg)\in[0,\infty)$ is differentiable for any $f,g\in\mathcal{F}_p$, and for any $c\in(0,\infty)$,*

$$\lim_{\delta\downarrow 0}\ \sup_{f,g\in\mathcal{F}_p;\,\mathcal{E}_p(f)\le c/(p-2)^+,\,\mathcal{E}_p(g)\le 1}\left|\frac{\mathcal{E}_p(f+\delta g)-\mathcal{E}_p(f)}{\delta}-\frac{d}{dt}\mathcal{E}_p(f+tg)\bigg|_{t=0}\right| = 0,$$

where $c/0 := \infty$. Moreover, define $\mathcal{E}_p(\,\cdot\,;\,\cdot\,)\colon \mathcal{F}_p\times\mathcal{F}_p\to\mathbb{R}$ by $\mathcal{E}_p(f;g) := \frac{1}{p}\frac{d}{dt}\mathcal{E}_p(f+tg)\big|_{t=0}$, and let $a\in\mathbb{R}$, $f,f_1,f_2,g\in\mathcal{F}_p$ and $h\in\mathcal{E}_p^{-1}(0)$. Then the following hold.

[4] Strichartz and Wong [SW04] proposed an approach based on the *subderivative* instead of (1.6), i.e., they defined $\mathcal{E}_p(u;\varphi)$ as the interval $\left[\mathcal{E}_p^-(u;\varphi),\mathcal{E}_p^+(u;\varphi)\right]$, where $\frac{d^\pm}{dt}\mathcal{E}_p(u+t\varphi)\big|_{t=0} =: \mathcal{E}_p^\pm(u;\varphi)$.

[5] The case of $p=2$ is special because of the parallelogram law. Indeed, $\mathcal{E}_2$ is known to be a quadratic form and hence $\mathcal{E}_2(u,v) := \frac{1}{4}(\mathcal{E}_2(u+v)-\mathcal{E}_2(u-v))$ is a symmetric form satisfying (1.6).

(a) $\mathcal{E}_p(f;f) = \mathcal{E}_p(f)$ *and* $\mathcal{E}_p(af;g) = \mathrm{sgn}(a)\,|a|^{p-1}\,\mathcal{E}_p(f;g)$.
(b) *The map* $\mathcal{E}_p(f;\,\cdot\,)\colon \mathcal{F}_p \to \mathbb{R}$ *is linear.*
(c) $\mathcal{E}_p(f;h) = 0$ *and* $\mathcal{E}_p(f+h;g) = \mathcal{E}_p(f;g)$.
(d) $\mathbb{R} \ni t \mapsto \mathcal{E}_p(f+tg;g) \in \mathbb{R}$ *is strictly increasing if and only if* $\mathcal{E}_p(g) > 0$.
(e) $|\mathcal{E}_p(f;g)| \le \mathcal{E}_p(f)^{\frac{p-1}{p}}\mathcal{E}_p(g)^{\frac{1}{p}}$.
(f) $|\mathcal{E}_p(f_1;g) - \mathcal{E}_p(f_2;g)| \le C_p\big(\mathcal{E}_p(f_1) \vee \mathcal{E}_p(f_2)\big)^{\frac{p-1-\alpha_p}{p}}\mathcal{E}_p(f_1-f_2)^{\frac{\alpha_p}{p}}\mathcal{E}_p(g)^{\frac{1}{p}}$, *where* $\alpha_p := \frac{1}{p} \wedge \frac{p-1}{p}$ *and* $C_p \in (0,\infty)$ *is a constant determined solely and explicitly by* p.

We also establish a similar result for p-energy measures as follows, which is the first rigorous result on the existence of the derivative in (1.7) for p-energy measures on fractals. (Recall that the existence of p-energy measures in a general setting not assuming the self-similarity of the space and the p-energy form is unknown; see [MS25+, Problem 10.4].)

Theorem 1.4 (Propositions 4.3, 4.8 and Theorem 4.5)**.** *Let* $\mathcal{B}_0$ *be a* σ*-algebra in* X*, and assume that* $\{\Gamma_p\langle u\rangle\}_{u\in\mathcal{F}_p}$ *is a family of measures on* $(X,\mathcal{B}_0)$ *such that* $\Gamma_p\langle f\rangle(X) \le \mathcal{E}_p(f)$ *for any* $f \in \mathcal{F}_p$ *and such that* $(\Gamma_p\langle\,\cdot\,\rangle(A), \mathcal{F}_p)$ *is a* p*-energy form on* (X,m) *satisfying* (Cla)$_p$ *for any* $A \in \mathcal{B}_0$*. Then* $\mathbb{R} \ni t \mapsto \Gamma_p\langle f+tg\rangle(A) \in [0,\infty)$ *is differentiable for any* $f,g \in \mathcal{F}_p$ *and any* $A \in \mathcal{B}_0$*, and for any* $c \in (0,\infty)$*,*

$$\lim_{\delta\downarrow 0}\sup_{A\in\mathcal{B}_0,\,f,g\in\mathcal{F}_p;\,\mathcal{E}_p(f)\le c/(p-2)^+,\,\mathcal{E}_p(g)\le 1}\left|\frac{\Gamma_p\langle f+\delta g\rangle(A) - \Gamma_p\langle f\rangle(A)}{\delta} - \frac{d}{dt}\Gamma_p\langle f+tg\rangle(A)\Big|_{t=0}\right| = 0.$$

Moreover, the set function $\Gamma_p\langle f;g\rangle\colon \mathcal{B}_0 \to \mathbb{R}$ *defined by* $\Gamma_p\langle f;g\rangle(A) := \frac{1}{p}\frac{d}{dt}\Gamma_p\langle f + tg\rangle(A)\big|_{t=0}$ *is a signed measure on* $(X,\mathcal{B}_0)$ *for any* $f,g \in \mathcal{F}_p$*, and the following hold for any* $A \in \mathcal{B}_0$*, any* $a \in \mathbb{R}$ *and any* $f,f_1,f_2,g,h \in \mathcal{F}_p$ *with* $\Gamma_p\langle h\rangle(A) = 0$*:*

(a) $\Gamma_p\langle f;f\rangle(A) = \Gamma_p\langle f\rangle$ *and* $\Gamma_p\langle af;g\rangle(A) = \mathrm{sgn}(a)\,|a|^{p-1}\,\Gamma_p\langle f;g\rangle(A)$.
(b) *The map* $\Gamma_p\langle f;\,\cdot\,\rangle(A)\colon \mathcal{F}_p \to \mathbb{R}$ *is linear.*
(c) $\Gamma_p\langle f;h\rangle(A) = 0$ *and* $\Gamma_p\langle f+h;g\rangle(A) = \Gamma_p\langle f;g\rangle(A)$.
(d) $\mathbb{R} \ni t \mapsto \Gamma_p\langle f+tg;g\rangle(A) \in \mathbb{R}$ *is strictly increasing if and only if* $\Gamma_p\langle g\rangle(A) > 0$.
(e) *For any* $\mathcal{B}_0$*-measurable functions* $\varphi,\psi\colon X \to [0,\infty]$*,*

$$\int_X \varphi\psi\,d\,|\Gamma_p\langle f;g\rangle| \le \left(\int_X \varphi^{\frac{p}{p-1}}\,d\Gamma_p\langle f\rangle\right)^{\frac{p-1}{p}}\left(\int_X \psi^p\,d\Gamma_p\langle g\rangle\right)^{\frac{1}{p}}.$$

(f) *Let* $\alpha_p = \frac{1}{p} \wedge \frac{p-1}{p}$ *and* C_p *be the same constants as in Theorem 1.3-(f). Then*

$$\begin{aligned}&|\Gamma_p\langle f_1;g\rangle(A) - \Gamma_p\langle f_2;g\rangle(A)|\\ &\le C_p\big(\Gamma_p\langle f_1\rangle(A) \vee \Gamma_p\langle f_2\rangle(A)\big)^{\frac{p-1-\alpha_p}{p}}\Gamma_p\langle f_1-f_2\rangle(A)^{\frac{\alpha_p}{p}}\Gamma_p\langle g\rangle(A)^{\frac{1}{p}}.\end{aligned}$$

In the second part of this paper (Sections 6 and 7), we aim at developing a general theory of p-energy forms taking (GC)$_p$ into account and focusing on a "low-dimensional" setting. Namely, we introduce the notion of *p-resistance form* as defined in Definition 1.5

below and establish fundamental properties of this class of p-energy forms, as a natural extension of the theory of resistance forms for $p = 2$ introduced in [Kig95] and developed further in [Kig01, Kig12] by Kigami. In the rest of this section, we consider the situation where $(\mathcal{B}, m)$ is the pair of $2^X = \{A \mid A \subseteq X\}$ and the counting measure on X, so that $L^0(X, m) = \mathbb{R}^X$; see also Remark 2.1.

Definition 1.5 (p-Resistance form; Definition 6.1)**.** We say that $(\mathcal{E}_p, \mathcal{F}_p)$ is a *p-resistance form* on X if and only if the following conditions hold:

$(\mathrm{RF1})_p$ $\mathcal{F}_p$ is a linear subspace of $\mathbb{R}^X$ containing $\mathbb{1}_X$ and $\mathcal{E}_p(\,\cdot\,)^{1/p}$ is a seminorm on $\mathcal{F}_p$ satisfying $\{u \in \mathcal{F}_p \mid \mathcal{E}_p(u) = 0\} = \mathbb{R}\mathbb{1}_X$.

$(\mathrm{RF2})_p$ The quotient normed space $(\mathcal{F}_p/\mathbb{R}\mathbb{1}_X, \mathcal{E}_p(\,\cdot\,)^{1/p})$ is a Banach space.

$(\mathrm{RF3})_p$ If $x \neq y \in X$, then there exists $u \in \mathcal{F}_p$ such that $u(x) \neq u(y)$.

$(\mathrm{RF4})_p$ For any $x, y \in X$,

$$R_{\mathcal{E}_p}(x, y) \coloneqq \sup\left\{ \frac{|u(x) - u(y)|^p}{\mathcal{E}_p(u)} \;\middle|\; u \in \mathcal{F}_p \setminus \mathbb{R}\mathbb{1}_X \right\} < \infty.$$

$(\mathrm{RF5})_p$ $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies the generalized p-contraction property $(\mathrm{GC})_p$.

We verify that the p-energy forms on *p-conductively homogeneous* compact metric spaces (K, d) (Definition 8.11) constructed by Kigami in [Kig23, Theorem 3.21], where p is assumed to be strictly greater than the Ahlfors regular conformal dimension of (K, d) (Definition 8.5-(4)), are p-resistance forms. In addition, we prove that the p-energy forms on post-critically finite self-similar sets constructed by Cao–Gu–Qiu in [CGQ22, Proposition 5.3] are also p-resistance forms for any $p \in (1, \infty)$ under the condition $(\mathbf{R})$ in [CGQ22, p. 18]. See Section 8 for details of the frameworks treated in [CGQ22, Kig23]. Similar to the case of $p = 2$, developing a general theory of p-resistance forms allows us to investigate p-energy forms provided by these broad frameworks in a synthetic manner.

It is immediate that if $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on X, then $R_{\mathcal{E}_p}(\,\cdot\,,\,\cdot\,)^{1/p}$ is a metric on X and any function in $\mathcal{F}_p$ is a Lipschitz function on K with respect to this metric. In the theory of resistance forms ($p = 2$), it is well known that $R_{\mathcal{E}_2}(\,\cdot\,,\,\cdot\,)$ is a metric, which is called the *resistance metric* of the resistance form $(\mathcal{E}_2, \mathcal{F}_2)$; see [Kig01, Theorem 2.3.4] for a proof. In view of this fact for $p = 2$, it is natural to seek the largest exponent q such that $R_{\mathcal{E}_p}(\,\cdot\,,\,\cdot\,)^q$ is a metric. The following theorem gives the answer.

Theorem 1.6 (Corollary 6.32)**.** *If $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on X, then $R_{\mathcal{E}_p}(\,\cdot\,,\,\cdot\,)^{\frac{1}{p-1}}$ is a metric on X.*

The power $1/(p-1)$ in Theorem 1.6 is sharp; see Example 6.34. Let us call $R_{\mathcal{E}_p}(\,\cdot\,,\,\cdot\,)^{\frac{1}{p-1}}$ the *p-resistance metric* of $(\mathcal{E}_p, \mathcal{F}_p)$. Theorem 1.6 was proved in [ACFP19, Her10] for the canonical p-energy forms (i.e., those given by (6.2)) on finite weighted graphs (V, L) and in [Shi21] for this class of forms on infinite graphs. Theorem 1.6 establishes the same result for the first time for p-energy forms which are not of the form (6.2) and for ones on continuous spaces.

We also investigate p-harmonic functions with respect to p-resistance forms, which should be considered as part of *nonlinear potential theory* under the condition that each point has a positive p-capacity. Let us explain some basic results in this introduction. The following definition is a natural analogue of (1.4) (or of (1.5)).

Definition 1.7 ($\mathcal{E}_p$-Harmonic function; see Definition 6.12)**.** Let $(\mathcal{E}_p, \mathcal{F}_p)$ be a p-resistance form on X and let B be a non-empty subset of X. A function $h \in \mathcal{F}_p$ is said to be $\mathcal{E}_p$-*harmonic* on $X \setminus B$ if and only if

$$\mathcal{E}_p(h; \varphi) = 0 \quad \text{for any } \varphi \in \mathcal{F}_p \text{ with } \varphi|_B = 0,$$

or equivalently (see Proposition 6.11 for this equivalence),

$$\mathcal{E}_p(h) = \inf\{\mathcal{E}_p(u) \mid u \in \mathcal{F}_p,\, u|_B = h|_B\}.$$

A standard argument in variational analysis ensures the existence and uniqueness of $\mathcal{E}_p$-harmonic functions with given boundary values.

Proposition 1.8 (Part of Theorem 6.13)**.** *Let $(\mathcal{E}_p, \mathcal{F}_p)$ be a p-resistance form on X and let B be a non-empty subset of X. Define $\mathcal{F}_p|_B \coloneqq \{u|_B \mid u \in \mathcal{F}_p\}$. Then for any $u \in \mathcal{F}_p|_B$, there exists a unique function $h_B^{\mathcal{E}_p}[u] \in \mathcal{F}_p$ satisfying $h_B^{\mathcal{E}_p}[u]\big|_B = u$ and $\mathcal{E}_p(h_B^{\mathcal{E}_p}[u]) = \inf\{\mathcal{E}_p(v) \mid v \in \mathcal{F}_p,\, v|_B = u\}$.*

Using the (nonlinear) operator $h_B^{\mathcal{E}_p}[\,\cdot\,]\colon \mathcal{F}_p|_B \to \mathcal{F}_p$ given in Proposition 1.8, we can introduce a new p-resistance form on the boundary set B, which is called the *trace* of $(\mathcal{E}_p, \mathcal{F}_p)$ to B. This notion is at the core of our theory of p-resistance forms, and turns out to be a powerful tool especially when we work on post-critically finite self-similar sets; see Subsection 8.3 for example. Here we just record fundamental results on traces in the following theorem.

Theorem 1.9 (Trace of p-resistance form; part of Theorem 6.13)**.** *Let $(\mathcal{E}_p, \mathcal{F}_p)$ be a p-resistance form on X and let B be a non-empty subset of X. Define $\mathcal{E}_p|_B\colon \mathcal{F}_p|_B \to [0, \infty)$ by $\mathcal{E}_p|_B(u) \coloneqq \mathcal{E}_p(h_B^{\mathcal{E}_p}[u])$ for $u \in \mathcal{F}_p|_B$. Then $(\mathcal{E}_p|_B, \mathcal{F}_p|_B)$ is a p-resistance form on B. Furthermore, $R_{\mathcal{E}_p|_B} = R_{\mathcal{E}_p}|_{B \times B}$ and*

$$\mathcal{E}_p|_B(u; v) = \mathcal{E}_p\bigl(h_B^{\mathcal{E}_p}[u]; h_B^{\mathcal{E}_p}[v]\bigr) \quad \textit{for any } u, v \in \mathcal{F}_p|_B.$$

Now let us state results on behavior of $\mathcal{E}_p$-harmonic functions. We start with *comparison principles* for $\mathcal{E}_p$-harmonic functions, namely monotonicity properties of $h_B^{\mathcal{E}_p}[u]$ with respect to the boundary value u. Because of the *nonlinearity* of the operator $h_B^{\mathcal{E}_p}$, a *maximum principle* does not imply a comparison principle unlike the case of $p = 2$. Fortunately, by virtue of Proposition 1.8 and the strong subadditivity (1.3), we can prove the following *weak comparison principle* for $\mathcal{E}_p$-harmonic functions (Proposition 6.26):

$$\text{If } \emptyset \neq B \subseteq X \text{ and } u, v \in \mathcal{F}_p|_B \text{ satisfy } u \le v \text{ on } B, \text{ then } h_B^{\mathcal{E}_p}[u] \le h_B^{\mathcal{E}_p}[v] \text{ on } X. \tag{1.8}$$

We also show a localized version of (1.8) under suitable assumptions (Proposition 6.30). Furthermore, by employing the approach in [Cap07], we show the following *(scale-invariant) elliptic Harnack inequality* for non-negative $\mathcal{E}_p$-harmonic functions under some extra assumptions including the existence of nice p-energy measures (see Theorem 6.36 for the precise statement): there exists a constant $C \in (0,\infty)$ such that for any $(x,s) \in X \times (0,\infty)$ and any non-negative $h \in \mathcal{F}_p$ that is $\mathcal{E}_p$-harmonic on $B_{\widehat{R}_p}(x,2s)$, where $\widehat{R}_p \coloneqq R_{\mathcal{E}_p}^{1/(p-1)}$,

$$\sup_{B_{\widehat{R}_p}(x,s)} h \leq C \inf_{B_{\widehat{R}_p}(x,s)} h, \tag{1.9}$$

which is well known to imply a local Hölder continuity of h. Regarding continuity estimates for $\mathcal{E}_p$-harmonic functions, we also obtain the following sharp Hölder regularity estimate, which in fact implies Theorem 1.6 as an easy corollary.

Theorem 1.10 (Theorem 6.31)**.** *Let $(\mathcal{E}_p,\mathcal{F}_p)$ be a p-resistance form on X and let B be a non-empty subset of X. Define $B^{\mathcal{F}_p} \coloneqq \bigcap_{u \in \mathcal{F}_p; u|_B = 0} u^{-1}(0)$ and, for $x \in X \setminus B^{\mathcal{F}_p}$,*

$$\widehat{R}_p(x,B) \coloneqq \left(\sup\left\{ \frac{|u(x)|^p}{\mathcal{E}_p(u)} \;\middle|\; u \in \mathcal{F}_p,\, u|_B = 0,\, u(x) \neq 0 \right\}\right)^{\frac{1}{p-1}}.$$

Let $x \in X \setminus B^{\mathcal{F}_p}$ and $y \in X$. Then

$$h_{B\cup\{x\}}^{\mathcal{E}_p}\bigl[\mathbb{1}_B^{B\cup\{x\}}\bigr](y) \leq \frac{\widehat{R}_p(x,y)}{\widehat{R}_p(x,B)}.$$

Moreover, for any $h \in \mathcal{F}_p$ that is $\mathcal{E}_p$-harmonic on $X \setminus B$ and satisfies $\sup_B |h| < \infty$,

$$|h(x) - h(y)| \leq \frac{\widehat{R}_p(x,y)}{\widehat{R}_p(x,B)} \sup_{x',y' \in B} |h(x') - h(y')|.$$

Next let us move to applications of our general theory of p-resistance forms. In their forthcoming papers [KS+a, KS+b], the authors will heavily use this theory to make some essential progress in the setting of post-critically finite self-similar structures; see [KS23+] for a survey of these results described in the setting of the Sierpiński gasket. In Section 9 of this paper, we shall give another application to strict inequalities for the p-walk dimensions of two classes of self-similar fractals, the generalized Sierpiński carpets and the D-dimensional level-l Sierpiński gasket (see Figure 1.2). Let K be a generalized Sierpiński carpet or the D-dimensional level-l Sierpiński gasket, equip K with the Euclidean metric d, let $p \in (1,\infty)$, and assume in the former case that p is strictly greater than the Ahlfors regular conformal dimension of (K,d). Then by Theorem 8.30 in the former case and by Theorem 8.51 in the latter case, we can construct a canonical p-resistance form $(\mathcal{E}_p,\mathcal{F}_p)$ on K. To be more precise, let $\{F_i\}_{i \in S}$, with S a suitable non-empty finite set, be the family of contractive similitudes defining K, i.e., such that $K = \bigcup_{i \in S} F_i(K)$. Then there exists a p-resistance form $(\mathcal{E}_p,\mathcal{F}_p)$ on K which satisfies $\mathcal{F}_p \subseteq C(K)$ and the following *self-similarity* for some $\sigma_p \in (1,\infty)$(, which we call the *weight* of $(\mathcal{E}_p,\mathcal{F}_p)$):

$$\mathcal{E}_p(u) = \sigma_p \sum_{i \in S} \mathcal{E}_p(u \circ F_i), \quad u \in \mathcal{F}_p. \tag{1.10}$$

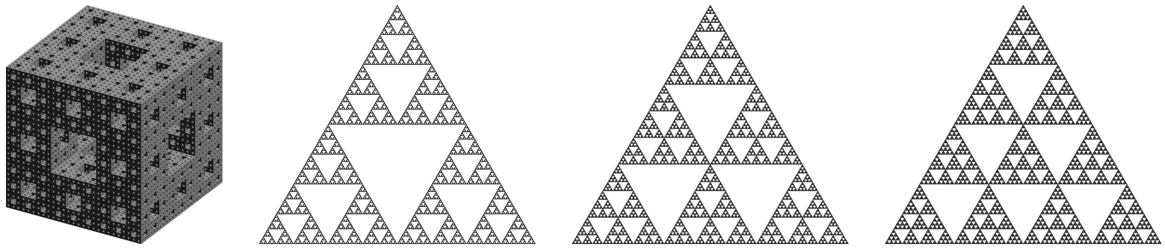

Figure 1.2: From the left, a non-planar generalized Sierpiński carpet (Menger Sponge) and the 2-dimensional level-l Sierpiński gaskets ($l = 2, 3, 4$)

Letting $r_* \in (0, 1)$ denote the common contraction ratio of the similitudes $\{F_i\}_{i\in S}$, we define the *p-walk dimension* $d_{\mathrm{w},p}$ of K by

$$d_{\mathrm{w},p} \coloneqq \frac{\log\bigl((\#S)\sigma_p\bigr)}{\log(r_*^{-1})},$$

which coincides with the walk dimension of K if $p = 2$. As shown in [MS25+, Theorem 7.1], the value $d_{\mathrm{w},p}$ shows up as a space-scaling exponent in the following manner:

$$\mathcal{E}_p(u) \asymp \limsup_{r\downarrow 0} \int_K \fint_{|x-y|<r} \frac{|u(x) - u(y)|^p}{r^{d_{\mathrm{w},p}}}\,\mu(dy)\,\mu(dx), \quad u \in \mathcal{F}_p,$$

where μ denotes the $\log(\#S)/\log(r_*^{-1})$-dimensional Hausdorff measure on (K, d). In the case of $p = 2$, the strict inequality $d_{\mathrm{w},2} > 2$ has been verified for various self-similar fractals, and has been shown to imply a number of anomalous features of the diffusion associated with $(\mathcal{E}_2, \mathcal{F}_2)$; see, e.g., [Kaj23] and the references therein for further details. Compared with the case of $p = 2$, the class of self-similar fractals for which $d_{\mathrm{w},p} > p$ has been proved in [Shi24, Theorem 2.27] is limited to the *planar* generalized Sierpiński carpets due to the lack of counterparts of many useful tools available in the case of $p = 2$. As an application of the differentiability in (1.6), in Section 9, we show $d_{\mathrm{w},p} > p$ for *any* generalized Sierpiński carpet and for the D-dimensional level-l Sierpiński gasket with *any* $D, l \in \mathbb{N} \setminus \{1\}$. The proof for the former follows closely the argument in [Kaj23], whereas for the latter we need a different argument from that in [Kaj23].

We would also like to mention a geometric role of σ_p appearing in (1.10). As done in [Kig20, Kig23], the constant σ_p is obtained by seeking the behavior of *conductance constants* ([Kig23, Definition 2.17]) on approximating graphs of K; see Theorem 8.12 for details. A remarkable fact is that the behavior of σ_p as a function of p is deeply related to the *Ahlfors regular conformal dimension* $\dim_{\mathrm{ARC}}(K, d)$ of (K, d) (see Definition 8.5-(4) for its definition); indeed, $\sigma_p > 1$ if and only if $p > \dim_{\mathrm{ARC}}(K, d)$ (see, e.g., [Kig20, Theorem 4.7.6]). Therefore, knowing properties of the function $p \mapsto \sigma_p$ is very important to understand the Ahlfors regular conformal dimension and related geometric information. Nevertheless, we do not know anything other than the following:

(Continuity; [Kig20, Proposition 4.7.5]) σ_p is continuous in p.

(Simple monotonicity; [Kig20, Proposition 4.7.5]) σ_p is non-decreasing in p.

(Hölder-type monotonicity; [Kig20, Lemma 4.7.4]) $d_{\mathrm{w},p}/p$ is non-increasing in p.
(Relation with $\dim_{\mathrm{ARC}}$; [Kig20, Theorem 4.7.6]) $\sigma_p > 1$ if and only if $p > \dim_{\mathrm{ARC}}(K,d)$.

As yet another application of our theory of p-resistance forms, we prove in Theorems 8.32 and 7.9 the following new monotonicity behavior of σ_p (in suitably general settings including any generalized Sierpiński carpet and the D-dimensional level-l Sierpiński gasket with any $D, l \in \mathbb{N} \setminus \{1\}$[6]):

$$(\dim_{\mathrm{ARC}}(K,d), \infty) \ni p \mapsto \sigma_p^{1/(p-1)} \in (0,\infty) \text{ is non-decreasing,} \tag{1.11}$$

which is good evidence that properties of $p \mapsto \sigma_p^{1/(p-1)}$ are also important to deepen our understanding of $(\mathcal{E}_p, \mathcal{F}_p)$ and, possibly, of $\dim_{\mathrm{ARC}}(K,d)$.

Let us conclude this introduction by mentioning a significant difference between our theory and some recent results [BBR24, Kuw24] on p-energy forms based on strongly local regular symmetric Dirichlet forms. (Similar p-energy forms were considered earlier in [HRT13, Remark 6.1].) In the settings of [BBR24, Kuw24], the associated p-energy measure $\Gamma_p^{\mathrm{DF}}\langle u \rangle$ can be explicitly defined by using the "density" which plays the role of "$|\nabla u|$" and is independent of p (see Example 4.2-(3)), whereas it is almost impossible to find a priori such a density on fractals. Meanwhile, we can naturally define the *self-similar p-energy measure* $\Gamma_p\langle u \rangle$ of u by using (1.10); see Section 5 for details. (See also [KS24+] for p-energy measures associated with Korevaar–Schoen p-energy forms.) In [KS+b], the authors will show that $\Gamma_p\langle u_p \rangle$ and $\Gamma_q\langle u_q \rangle$ are mutually singular for any $p, q \in (1,\infty)$ with $p \neq q$ and any $(u_p, u_q) \in \mathcal{F}_p \times \mathcal{F}_q$ for a certain class of post-critically finite self-similar sets including the D-dimensional level-l Sierpiński gasket with any $D, l \in \mathbb{N} \setminus \{1\}$, by proving that $(1,\infty) \ni p \mapsto \sigma_p^{1/(p-1)}$ is *strictly* increasing. This phenomenon on the singularity of energy measures *never* happens if we consider the energy measures $\Gamma_p^{\mathrm{DF}}\langle \cdot \rangle, \Gamma_q^{\mathrm{DF}}\langle \cdot \rangle$ that naturally show up in the settings of [BBR24, Kuw24]. This point also motivates us to develop a general theory of p-energy forms in an abstract setting in order to deal with fractals.

This paper is organized as follows. In Section 2, we collect basic results on the generalized p-contraction property $(\mathrm{GC})_p$. In Section 3, we prove the differentiability of p-energy forms satisfying p-Clarkson's inequality (Theorem 1.3). Moreover, we see that the (Fréchet) derivative in (1.6) gives a homeomorphism between $\mathcal{F}_p/\mathcal{E}_p^{-1}(0)$ and its dual. We also discuss regular and strong local properties of p-energy forms there. In Section 4, under the assumption of the existence of p-energy measures, we discuss their fundamental properties (Theorem 1.4 for example). We also formulate a chain rule for p-energy measures and observe some consequences of it. In Section 5, we recall standard notions on self-similar structures, discuss the self-similarity of p-energy forms and see that we can associate self-similar p-energy measures to a given self-similar p-energy form. Section 6 is devoted to the study of fundamental nonlinear potential theory for p-resistance forms,

[6] It is essentially known to experts that $\dim_{\mathrm{ARC}}(K,d) = 1$ for the D-dimensional level-l Sierpiński gasket K equipped with the Euclidean metric d. In Theorem B.8, we give a new proof of this fact, based on the existence of self-similar p-resistance forms proved in Theorem 8.50 as an extension of [CGQ22, Theorem 6.3], for a large class of post-critically finite self-similar sets with good geometric symmetry; see Subsection B.2 for details and relevant results in the literature.

most of which are mentioned in the introduction (see Theorems 1.6, 1.9, 1.10, Proposition 1.8, (1.8) and (1.9)). We further investigate the theory of p-resistance forms in the self-similar case in Section 7. In particular, we establish a Poincaré-type inequality in terms of self-similar p-energy measures under some geometric assumptions on the p-resistance metric. In Section 8, the generalized p-contraction property $(\mathrm{GC})_p$ is verified for the p-energy/p-resistance forms constructed in [CGQ22, Kig23]. More precisely, in Subsections 8.1 and 8.2, we recall the notion of p-conductively homogeneous compact metric space and the construction of p-energy forms due to [Kig23] and prove $(\mathrm{GC})_p$ for them. In Subsection 8.3, we focus on the case of post-critically finite self-similar structures and show that the *eigenforms* constructed in [CGQ22] are indeed p-resistance forms. In Subsection 8.4, we prove the existence of eigenforms for a large class of post-critically finite self-similar sets with good geometric symmetry (Theorem 8.50), extending [CGQ22, Theorem 6.3] by following the framework of [Kig01, Theorem 3.8.10]. In Section 9, we prove $d_{\mathrm{w},p} > p$ for the generalized Sierpiński carpets and the D-dimensional level-l Sierpiński gasket by using properties of p-harmonic functions established in Section 6. In Appendix A, we show that $(\mathrm{GC})_2$ holds for any symmetric Dirichlet form, the (2-)energy measures associated with any regular symmetric Dirichlet form, and their densities. Lastly, in Appendix B we collect some miscellaneous results related to self-similar p-resistance forms on post-critically finite self-similar structures.

Notation. Throughout this paper, we use the following notation and conventions.

(1) For $[0,\infty]$-valued quantities A and B, we write $A \lesssim B$ to mean that there exists an implicit constant $C \in (0,\infty)$ depending on some unimportant parameters such that $A \leq CB$. We write $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$.

(2) For a set A, we let $\#A \in \mathbb{N} \cup \{0,\infty\}$ denote the cardinality of A.

(3) We set $\sup\emptyset := 0$, $\inf\emptyset := \infty$, $a/0 := \infty$ for $a \in (0,\infty]$ and $0^0 := 1$. We write $a \vee b := \max\{a,b\}$, $a \wedge b := \min\{a,b\}$ and $a^+ := a \vee 0$ for $a, b \in [-\infty,\infty]$, and we use the same notation also for $[-\infty,\infty]$-valued functions and equivalence classes of them. All numerical functions in this paper are assumed to be $[-\infty,\infty]$-valued.

(4) We define $\mathrm{sgn}\colon \mathbb{R} \to \mathbb{R}$ by $\mathrm{sgn}(a) := |a|^{-1}a$ for $a \in \mathbb{R} \setminus \{0\}$ and $\mathrm{sgn}(0) := 0$.

(5) Let $n \in \mathbb{N}$. For $x = (x_k)_{k=1}^n \in \mathbb{R}^n$, we set $\|x\|_{\ell_n^p} := \|x\|_{\ell^p} := (\sum_{k=1}^n |x_k|^p)^{1/p}$ for $p \in (0,\infty)$, $\|x\|_{\ell_n^\infty} := \|x\|_{\ell^\infty} := \max_{1\leq k\leq n}|x_k|$ and $\|x\| := \|x\|_{\ell^2}$. For $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ which is differentiable on $\mathbb{R}^n$ and for $k \in \{1,\dots,n\}$, its first-order partial derivative in the k-th coordinate is denoted by $\partial_k\Phi$ and its gradient is denoted by $\nabla\Phi := (\partial_k\Phi)_{k=1}^n$.

(6) Let X be a non-empty set. We define $\mathrm{id}_X\colon X \to X$ by $\mathrm{id}_X(x) := x$, $\mathbb{1}_A = \mathbb{1}_A^X \in \mathbb{R}^X$ for $A \subseteq X$ by $\mathbb{1}_A(x) := \mathbb{1}_A^X(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$ and set $\|u\|_{\sup} := \|u\|_{\sup,X} := \sup_{x\in X}|u(x)|$ for $u\colon X \to [-\infty,\infty]$. Also, set $\mathrm{osc}_X[u] := \sup_{x,y\in X}|u(x)-u(y)|$ for $u\colon X \to \mathbb{R}$ with $\|u\|_{\sup} < \infty$.

(7) Let X be a topological space. The Borel σ-algebra of X is denoted by $\mathcal{B}(X)$, the closure of $A \subseteq X$ in X by $\overline{A}^X$, and we say that $A \subseteq X$ is *relatively compact* in X if and only if $\overline{A}^X$ is compact. We set $C(X) := \{u \in \mathbb{R}^X \mid u \text{ is continuous}\}$,

$\mathrm{supp}_X[u] := \overline{X \setminus u^{-1}(0)}^X$ for $u \in C(X)$, $C_b(X) := \{u \in C(X) \mid \|u\|_{\sup} < \infty\}$, and $C_c(X) := \{u \in C(X) \mid \mathrm{supp}_X[u] \text{ is compact}\}$.

(8) Let X be a topological space having a countable open base. For a measure m on a σ-algebra $\mathcal{B}$ in X including $\mathcal{B}(X)$, we let $\mathrm{supp}_X[m]$ denote the support of m in X, i.e., the smallest closed subset F of X such that $m(X \setminus F) = 0$, and set $\mathrm{supp}_m[f] := \mathrm{supp}_X[|f|\,dm]$ for a $\mathcal{B}$-measurable function $f \colon X \to [-\infty, \infty]$ or an m-equivalence class f of such functions.

(9) Let (X, d) be a metric space. We set $B_d(x, r) := \{y \in X \mid d(x, y) < r\}$ for $(x, r) \in X \times (0, \infty)$, and $\mathrm{diam}(A, d) := \sup_{x,y \in A} d(x, y)$ and $\mathrm{dist}_d(A, B) := \inf\{d(x, y) \mid x \in A, y \in B\}$ for subsets A, B of X.

(10) Let $(X, \mathcal{B}, m)$ be a measure space. We set $⨍_A f\,dm := \frac{1}{m(A)} \int_A f\,dm$ for $f \in L^1(X, m)$ and $A \in \mathcal{B}$ with $m(A) \in (0, \infty)$, and set $m|_A := m|_{\mathcal{B}|_A}$ for $A \in \mathcal{B}$, where $\mathcal{B}|_A := \{B \cap A \mid B \in \mathcal{B}\}$. For a measure μ on $(X, \mathcal{B})$, we write $\mu \ll m$ to mean that μ is absolutely continuous with respect to m.

2 The generalized p-contraction property

In this section, we will introduce the generalized p-contraction property and establish basic results on this property. Throughout this section, we fix $p \in (1, \infty)$, a measure space $(X, \mathcal{B}, m)$, a linear subspace $\mathcal{F}$ of $L^0(X, m) := L^0(X, \mathcal{B}, m)$, where

$$L^0(X, \mathcal{B}, m) := \{\text{the } m\text{-equivalence class of } f \mid f \colon X \to \mathbb{R},\ f \text{ is } \mathcal{B}\text{-measurable}\}, \tag{2.1}$$

and a functional $\mathcal{E} \colon \mathcal{F} \to [0, \infty)$ which is *p-homogeneous*, i.e., satisfies $\mathcal{E}(au) = |a|^p\,\mathcal{E}(u)$ for any $(a, u) \in \mathbb{R} \times \mathcal{F}$.

Remark 2.1. Note that the pair $(\mathcal{B}, m)$ is arbitrary. For example, $(\mathcal{B}, m)$ could be the pair of $2^X = \{A \mid A \subseteq X\}$ and the counting measure on X, in which case $L^0(X, \mathcal{B}, m) = \mathbb{R}^X$. We will make this choice of $(\mathcal{B}, m)$ later in Section 6.

Definition 2.2 (Generalized p-contraction property)**.** The pair $(\mathcal{E}, \mathcal{F})$ is said to satisfy the *generalized p-contraction property*, $(\mathrm{GC})_p$ for short, if and only if the following hold: if $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$ and $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy

$$T(0) = 0 \quad \text{and} \quad \|T(x) - T(y)\|_{\ell^{q_2}} \le \|x - y\|_{\ell^{q_1}} \quad \text{for any } x, y \in \mathbb{R}^{n_1}, \tag{2.2}$$

then for any $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}^{n_1}$ we have

$$T(\boldsymbol{u}) \in \mathcal{F}^{n_2} \quad \text{and} \quad \left\| \left(\mathcal{E}(T_l(\boldsymbol{u}))^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\mathcal{E}(u_k)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{GC$_p$}$$

The next proposition is a collection of useful inequalities included in $(\mathrm{GC})_p$.

Proposition 2.3. *Let $\varphi \in C(\mathbb{R})$ satisfy $\varphi(0) = 0$ and $|\varphi(t) - \varphi(s)| \le |t - s|$ for any $s, t \in \mathbb{R}$.*[7] *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{GC})_p$.*

[7] Note that any such φ satisfies $|\varphi \circ f| \le |f|$ on X for any $f \colon X \to \mathbb{R}$ and hence $\varphi \circ f \in L^p(X, m)$ for any $f \in L^p(X, m)$.

(a) *$T(x,y) \coloneqq x+y$, $x,y \in \mathbb{R}$, satisfies (2.2) with $(q_1,q_2,n_1,n_2) = (1,p,2,1)$. In particular, $\mathcal{E}^{1/p}$ is a seminorm on $\mathcal{F}$, and $\mathcal{E}$ is strictly convex on $\mathcal{F}/\mathcal{E}^{-1}(0)$, i.e., for any $\lambda \in (0,1)$ and any $f,g \in \mathcal{F}$, if $\mathcal{E}(f) \wedge \mathcal{E}(g) \wedge \mathcal{E}(f-g) > 0$, then*

$$\mathcal{E}(\lambda f + (1-\lambda)g) < \lambda \mathcal{E}(f) + (1-\lambda)\mathcal{E}(g). \tag{2.3}$$

(b) *$T \coloneqq \varphi$ satisfies (2.2) with $(q_1,q_2,n_1,n_2) = (1,p,1,1)$. In particular,*

$$\text{for any } \varphi \text{ as assumed above, } \varphi(u) \in \mathcal{F} \text{ and } \mathcal{E}(\varphi(u)) \le \mathcal{E}(u) \text{ for any } u \in \mathcal{F}. \tag{2.4}$$

(c) *Assume that φ is non-decreasing. Define $T = (T_1,T_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_1(x_1,x_2) = x_1 - \varphi(x_1-x_2) \quad \text{and} \quad T_2(x_1,x_2) = x_2 + \varphi(x_1-x_2), \quad (x_1,x_2) \in \mathbb{R}^2.$$

Then T satisfies (2.2) with $(q_1,q_2,n_1,n_2) = (p,p,2,2)$. In particular,

$$\mathcal{E}\bigl(f - \varphi(f-g)\bigr) + \mathcal{E}\bigl(g + \varphi(f-g)\bigr) \le \mathcal{E}(f) + \mathcal{E}(g) \quad \text{for any } f,g \in \mathcal{F}. \tag{2.5}$$

In particular, by considering the case of $\varphi(x) = x^+$, we have the following strong subadditivity*: for any $f,g \in \mathcal{F}$, $f \vee g, f \wedge g \in \mathcal{F}$ and*

$$\mathcal{E}(f \vee g) + \mathcal{E}(f \wedge g) \le \mathcal{E}(f) + \mathcal{E}(g). \tag{2.6}$$

(d) *For any $a_1,a_2 > 0$, define $T^{a_1,a_2}\colon \mathbb{R}^2 \to \mathbb{R}$ by*

$$T^{a_1,a_2}(x_1,x_2) \coloneqq \bigl(\bigl[(-a_1) \vee a_2^{-1}x_1\bigr] \wedge a_1\bigr) \cdot \bigl(\bigl[(-a_2) \vee a_1^{-1}x_2\bigr] \wedge a_2\bigr), \quad (x_1,x_2) \in \mathbb{R}^2.$$

Then T^{a_1,a_2} satisfies (2.2) with $(q_1,q_2,n_1,n_2) = (1,p,2,1)$. In particular, for any $f,g \in \mathcal{F} \cap L^\infty(X,m)$ we have

$$f \cdot g \in \mathcal{F} \quad \text{and} \quad \mathcal{E}(f \cdot g)^{1/p} \le \|g\|_{L^\infty(X,m)} \mathcal{E}(f)^{1/p} + \|f\|_{L^\infty(X,m)} \mathcal{E}(g)^{1/p}. \tag{2.7}$$

(e) *Assume that $p \in (1,2]$. Define $T = (T_1,T_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_1(x_1,x_2) = 2^{-(p-1)/p}(x_1+x_2) \quad \text{and} \quad T_2(x_1,x_2) = 2^{-(p-1)/p}(x_1-x_2), \quad (x_1,x_2) \in \mathbb{R}^2.$$

Then T satisfies (2.2) with $(q_1,q_2,n_1,n_2) = (p/(p-1),p,2,2)$. In particular, $(\mathcal{E},\mathcal{F})$ satisfies the following p-Clarkson's inequality:

$$\mathcal{E}(f+g) + \mathcal{E}(f-g) \ge 2\bigl(\mathcal{E}(f)^{1/(p-1)} + \mathcal{E}(g)^{1/(p-1)}\bigr)^{p-1} \quad \text{for any } f,g \in \mathcal{F}. \tag{2.8}$$

(f) *Assume that $p \in [2,\infty)$. Define $T = (T_1,T_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_1(x_1,x_2) = 2^{-1/p}(x_1+x_2) \quad \text{and} \quad T_2(x_1,x_2) = 2^{-1/p}(x_1-x_2), \quad (x_1,x_2) \in \mathbb{R}^2.$$

Then T satisfies (2.2) with $(q_1,q_2,n_1,n_2) = (p,p/(p-1),2,2)$. In particular, $(\mathcal{E},\mathcal{F})$ satisfies the following p-Clarkson's inequality:

$$\mathcal{E}(f+g) + \mathcal{E}(f-g) \le 2\bigl(\mathcal{E}(f)^{1/(p-1)} + \mathcal{E}(g)^{1/(p-1)}\bigr)^{p-1} \quad \text{for any } f,g \in \mathcal{F}. \tag{2.9}$$

Remark 2.4. (1) The property (2.5) is inspired by the *nonlinear Dirichlet form theory* due to Cipriani and Grillo [CG03]. See [Cla23, Theorem 4.7] and the reference therein for further background.

(2) There are two versions of p-Clarkson's inequality, one of which is stronger than the other. The inequalities (2.8) and (2.9) above are the stronger one for $p \in (1,2]$ and for $p \in [2,\infty)$, respectively; see Remark 3.3 below for the weaker one.

Proof of Proposition 2.3. (a): It is obvious that $T(x,y) \coloneqq x+y$ satisfies (2.2) with $(q_1,q_2,n_1,n_2) = (1,p,2,1)$ and hence the triangle inequality for $\mathcal{E}^{1/p}$ holds. Since $(0,\infty) \ni x \mapsto x^p$ is strictly convex, for any $\lambda \in (0,1)$ and any $f,g \in \mathcal{F}$ with $\mathcal{E}(f) \wedge \mathcal{E}(g) \wedge \mathcal{E}(f-g) > 0$,

$$\mathcal{E}(\lambda f + (1-\lambda)g) \leq \bigl(\lambda\mathcal{E}(f)^{1/p} + (1-\lambda)\mathcal{E}(g)^{1/p}\bigr)^p < \lambda\mathcal{E}(f) + (1-\lambda)\mathcal{E}(g),$$

where we used the triangle inequality for $\mathcal{E}^{1/p}$ in the first inequality.

(b): This is obvious.

(c): Let $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$. For ease of notation, set $z_i \coloneqq x_i - y_i$ and $A \coloneqq \varphi(x_1-x_2) - \varphi(y_1-y_2)$. Then $\|T(x)-T(y)\|_{\ell^p} \leq \|x-y\|_{\ell^p}$ ie equivalent to

$$|z_1 - A|^p + |z_2 + A|^p \leq |z_1|^p + |z_2|^p, \tag{2.10}$$

so we will show (2.10). Note that $|A| \leq |z_1 - z_2|$ since φ is 1-Lipschitz. The desired estimate (2.10) is evident when $z_1 = z_2$, so we consider the case of $z_1 \neq z_2$. Assume that $z_1 > z_2$ because the remaining case $z_1 < z_2$ is similar. Then $(x_1-x_2)-(y_1-y_2) = z_1 - z_2 > 0$ and thus $0 \leq A \leq z_1 - z_2$. Set $\psi_p(t) \coloneqq |t|^p$ $(t \in \mathbb{R})$ for brevity. If $0 \leq A < \frac{z_1-z_2}{2}$, then $z_2 \leq z_2 + A < z_1 - A \leq z_1$ and we see that

$$\begin{aligned}|z_1 - A|^p + |z_2 + A|^p - |z_1|^p - |z_2|^p &= \int_{z_2}^{z_2+A} \psi_p'(t)\,dt - \int_{z_1-A}^{z_1} \psi_p'(t)\,dt \\ &\leq A\bigl(\psi_p'(z_2+A) - \psi_p'(z_1-A)\bigr) \leq 0.\end{aligned}$$

If $A \geq \frac{z_1-z_2}{2}$, then $z_2 \leq z_1 - A \leq z_2 + A \leq z_1$ and thus

$$\begin{aligned}|z_1 - A|^p + |z_2 + A|^p - |z_1|^p - |z_2|^p &= \int_{z_2}^{z_1-A} \psi_p'(t)\,dt - \int_{z_2+A}^{z_1} \psi_p'(t)\,dt \\ &\leq (z_1 - z_2 - A)\bigl(\psi_p'(z_1-A) - \psi_p'(z_2+A)\bigr) \leq 0,\end{aligned}$$

which proves (2.10). The case of $\varphi(x) = x^+$ immediately implies (2.6).

(d): For any $a_1, a_2 > 0$ and $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$, we see that

$$\begin{aligned}&|T^{a_1,a_2}(x_1,x_2) - T^{a_1,a_2}(x_1,x_2)| \\ &\leq \bigl|(-a_1) \vee a_2^{-1}x_1 \wedge a_1\bigr| \bigl|\bigl((-a_2) \vee a_1^{-1}x_2 \wedge a_2\bigr) - \bigl((-a_2) \vee a_1^{-1}y_2 \wedge a_2\bigr)\bigr| \\ &\quad + \bigl|(-a_2) \vee a_1^{-1}y_2 \wedge a_2\bigr| \bigl|\bigl((-a_1) \vee a_2^{-1}x_1 \wedge a_1\bigr) - \bigl((-a_1) \vee a_2^{-1}y_1 \wedge a_1\bigr)\bigr| \\ &\leq a_1\left|a_1^{-1}x_2 - a_1^{-1}y_2\right| + a_2\left|a_2^{-1}x_1 - a_2^{-1}y_1\right| = |x_1 - y_1| + |x_2 - y_2|,\end{aligned}$$

whence T^{a_1,a_2} satisfies (2.2). We get (2.7) by applying $(\mathrm{GC})_p$ with $u_1 = \|g\|_{L^\infty(X,m)} f$, $u_2 = \|f\|_{L^\infty(X,m)} g$, $a_1 = \|f\|_{L^\infty(X,m)}$, $a_2 = \|g\|_{L^\infty(X,m)}$.

(e),(f): These statements follow from p-Clarkson's inequality for the ℓ^p-norm (see, e.g., [Cla36, Theorem 2]). □

The following corollary is easily implied by Proposition 2.3-(b),(d).

Corollary 2.5. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{GC})_p$.*

(a) *Let $u \in \mathcal{F} \cap L^\infty(X,m)$ and let $\Phi \in C^1(\mathbb{R})$ satisfy $\Phi(0) = 0$. Then*

$$\Phi(u) \in \mathcal{F} \quad \textit{and} \quad \mathcal{E}(\Phi(u)) \le \sup\bigl\{|\Phi'(t)|^p \bigm| t \in \mathbb{R}, |t| \le \|u\|_{L^\infty(X,m)}\bigr\}\mathcal{E}(u). \tag{2.11}$$

(b) *Let $\delta, M \in (0,\infty)$ and let $f, g \in \mathcal{F}$ satisfy $f \ge 0$, $g \ge 0$, $f \le M$ and $(f+g)|_{\{f\neq 0\}} \ge \delta$. Then there exists $C \in (0,\infty)$ depending only on p, δ, M such that*

$$\frac{f}{f+g} \in \mathcal{F} \quad \textit{and} \quad \mathcal{E}\biggl(\frac{f}{f+g}\biggr) \le C\bigl(\mathcal{E}(f) + \mathcal{E}(g)\bigr). \tag{2.12}$$

(c) *Let $n \in \mathbb{N}$, $q \in [1,p]$, $\boldsymbol{u} = (u_1,\dots,u_n) \in \mathcal{F}^n$ and $v \in L^0(X,m)$. If there exist m-versions of $\boldsymbol{u}, v$ such that $|v(x)| \le \|\boldsymbol{u}(x)\|_{\ell^q}$ and $|v(x) - v(y)| \le \|\boldsymbol{u}(x) - \boldsymbol{u}(y)\|_{\ell^q}$ for any $x, y \in X$, then $v \in \mathcal{F}$ and $\mathcal{E}(v) \le \bigl\|\bigl(\mathcal{E}(u_k)^{1/p}\bigr)_{k=1}^n\bigr\|_{\ell^q}$.*

Proof. (a): This is immediate from Proposition 2.3-(b).

(b): We follow [MS23+, Proposition 6.25(ii)][8]. Let $\varphi \in C(\mathbb{R})$ be a Lipschitz map such that $\varphi(x) = \frac{1}{x}$ for $x \ge \delta$ and $\sup_{x\neq y\in\mathbb{R}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|} \le C'$ for some constant C' depending only on δ. Since $f \cdot \varphi(f+g) = \frac{f}{f+g}$, we get (2.12) by using (2.4) and (2.7).

(c): The proof below is similar to [MR, Corollary I.4.13]. Fix m-versions of $\boldsymbol{u}, v$ satisfying $|v(x)| \le \|\boldsymbol{u}(x)\|_{\ell^q}$ and $|v(x) - v(y)| \le \|\boldsymbol{u}(x) - \boldsymbol{u}(y)\|_{\ell^q}$ for any $x, y \in X$. We define $T_0 \colon \boldsymbol{u}(X) \cup \{0\} \to \mathbb{R}$ by setting $T_0(0) \coloneqq 0$ and $T_0(\boldsymbol{z}) \coloneqq v(x)$ for each $\boldsymbol{z} \in \boldsymbol{u}(X)$, where $x \in X$ satisfies $\boldsymbol{z} = \boldsymbol{u}(x)$. This map T_0 is well-defined since $v(x) = 0$ for any $x \in X$ with $\boldsymbol{u}(x) = 0$ and $|v(x) - v(y)| \le \|\boldsymbol{u}(x) - \boldsymbol{u}(y)\|_{\ell^q} = 0$ for any $x, y \in X$ with $\boldsymbol{u}(x) = \boldsymbol{u}(y) \in \boldsymbol{u}(X)$. In addition, we easily see that $|T_0(\boldsymbol{z}_1) - T_0(\boldsymbol{z}_2)| \le \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{\ell^q}$ for any $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \boldsymbol{u}(X) \cup \{0\}$, i.e., $T_0 \colon (\boldsymbol{u}(X) \cup \{0\}, \|\cdot\|_{\ell^q}) \to \mathbb{R}$ is 1-Lipschitz. Noting that $(\mathbb{R}^n, \|\cdot\|_{\ell^q})$ is a metric space by $q \ge 1$, we obtain a 1-Lipschitz map $T \colon (\mathbb{R}^n, \|\cdot\|_{\ell^q}) \to \mathbb{R}$ satisfying $T(\boldsymbol{z}) = T_0(\boldsymbol{z})$ for any $\boldsymbol{z} \in \boldsymbol{u}(X) \cup \{0\}$ by applying the McShane–Whitney extension lemma (see, e.g., [HKST, p. 99]). Since T satisfies (2.2) with $(q_1, q_2, n_1, n_2) = (q, p, n, 1)$ and $T(\boldsymbol{u}) = v$, the assertions follow from $(\mathrm{GC})_p$. □

We also notice that $(\mathrm{GC})_p$ implies a new variant of p-Clarkson's inequality, which we call improved p-Clarkson's inequality. This result is not used in the paper, but we record it for potential future applications.

[8] The article [MS23+] is a detailed version of [MS25+]. Several statements and proofs have been removed from the latter, so we still refer to [MS23+] for those omitted results and arguments.

Proposition 2.6 (Improved p-Clarkson's inequality)**.** *Define* $\psi_p\colon (0,\infty) \to (0,\infty)$ *by*

$$\psi_p(s) \coloneqq (1+s)^{p-1} + \operatorname{sgn}(1-s)\,|1-s|^{p-1}, \quad s>0. \tag{2.13}$$

(a) *Assume that* $p \in (1,2]$. *For* $s \in (0,\infty)$, *define* $T^s = (T^s_1, T^s_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$T^s_1(x_1,x_2) \coloneqq 2^{-1}\psi_p(s)^{1/p}(x_1+x_2), \quad T^s_2(x_1,x_2) \coloneqq 2^{-1}\psi_p(s^{-1})^{1/p}(x_1-x_2).$$

Then T^s *satisfies* (2.2) *with* $(q_1,q_2,n_1,n_2) = (p,p,2,2)$ *for any* $s \in (0,\infty)$. *If* $(\mathcal{E},\mathcal{F})$ *satisfies* $(\mathrm{GC})_p$, *then*

$$\sup_{s>0}\bigl\{\psi_p(s)\mathcal{E}(f) + \psi_p(s^{-1})\mathcal{E}(g)\bigr\} \le \mathcal{E}(f+g) + \mathcal{E}(f-g) \quad \text{for any } f,g \in \mathcal{F}. \tag{2.14}$$

(b) *If* $\mathcal{E}$ *satisfies* (2.14), *then* (2.8) *holds.*

(c) *Assume that* $p \in [2,\infty)$. *For* $s \in (0,\infty)$, *define* $T^s = (T^s_1, T^s_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$T^s_1(x_1,x_2) \coloneqq \psi_p(s)^{-1/p}x_1 + \psi_p(s^{-1})^{-1/p}x_2, \quad T^s_2(x_1,x_2) \coloneqq \psi_p(s)^{-1/p}x_1 - \psi_p(s^{-1})^{-1/p}x_2.$$

Then T^s *satisfies* (2.2) *with* $(q_1,q_2,n_1,n_2) = (p,p,2,2)$ *for any* $s \in (0,\infty)$. *If* $p \in [2,\infty)$ *and* $(\mathcal{E},\mathcal{F})$ *satisfies* $(\mathrm{GC})_p$, *then*

$$\mathcal{E}(f+g) + \mathcal{E}(f-g) \le \inf_{s>0}\bigl\{\psi_p(s)\mathcal{E}(f) + \psi_p(s^{-1})\mathcal{E}(g)\bigr\} \quad \text{for any } f,g \in \mathcal{F}. \tag{2.15}$$

(d) *If* $\mathcal{E}$ *satisfies* (2.15), *then* (2.9) *holds.*

Proof. We first recall a key result from [BCL94, Lemma 4]: for any $x,y \in \mathbb{R}$,

$$|x+y|^p + |x-y|^p = \begin{cases} \sup_{s>0}\bigl\{\psi_p(s)\,|x|^p + \psi_p(s^{-1})\,|y|^p\bigr\} & \text{if } p \in (1,2], \\ \inf_{s>0}\bigl\{\psi_p(s)\,|x|^p + \psi_p(s^{-1})\,|y|^p\bigr\} & \text{if } p \in [2,\infty). \end{cases} \tag{2.16}$$

(a): By considering $x+y, x-y$ in (2.16) instead of x,y, we have that for any $s \in (0,\infty)$,

$$2^{-p}\psi_p(s)\,|x+y|^p + 2^{-p}\psi_p(s^{-1})\,|x-y|^p \le |x|^p + |y|^p,$$

which means that T^s satisfies (2.2) with $(q_1,q_2,n_1,n_2) = (p,p,2,2)$. Since $s \in (0,\infty)$ is arbitrary, we obtain (2.14).

(b): Let $f,g \in \mathcal{F}$ with $\mathcal{E}(f) \wedge \mathcal{E}(g) > 0$, set $a \coloneqq \mathcal{E}(f)^{1/(p-1)}$ and $b \coloneqq \mathcal{E}(g)^{1/(p-1)}$. Then,

$$\sup_{s>0}\bigl\{\psi_p(s)\mathcal{E}(f) + \psi_p(s^{-1})\mathcal{E}(g)\bigr\} \ge \psi_p(b/a)a^{p-1} + \psi_p(a/b)b^{p-1} = 2(a+b)^{p-1},$$

which together with (2.14) yields (2.8).

(c): For any $s \in (0,\infty)$, we immediately see from (2.16) that T^s satisfies (2.2). Since $s \in (0,\infty)$ is arbitrary, we obtain (2.15).

(d): Let $f,g \in \mathcal{F}$ with $\mathcal{E}(f) \wedge \mathcal{E}(g) > 0$, set $a \coloneqq \mathcal{E}(f)^{1/(p-1)}$ and $b \coloneqq \mathcal{E}(g)^{1/(p-1)}$. Then,

$$\inf_{s>0}\bigl\{\psi_p(s)\mathcal{E}(f) + \psi_p(s^{-1})\mathcal{E}(g)\bigr\} \le \psi_p(b/a)a^{p-1} + \psi_p(a/b)b^{p-1} = 2(a+b)^{p-1},$$

which together with (2.15) yields (2.9). $\square$

The property $(\mathrm{GC})_p$ is stable under taking suitable limits and some algebraic operations like summations. To state precise results, we recall the following definition on convergences of functionals.

Definition 2.7 ([Dal, Definition 4.1 and Proposition 8.1])**.** Let $\mathcal{X}$ be a topological space, let $F\colon \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ and let $\{F_n\colon \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}\}_{n\in\mathbb{N}}$.

(1) The sequence $\{F_n\}_{n\in\mathbb{N}}$ is said to *converge pointwise* to F if and only if $\lim_{n\to\infty} F_n(x) = F(x)$ for any $x \in \mathcal{X}$.

(2) Assume that $\mathcal{X}$ is a first-countable topological space. The sequence $\{F_n\}_{n\in\mathbb{N}}$ is said to Γ-*converge* to F (with respect to the topology of $\mathcal{X}$) if and only if the following conditions hold for any $x \in \mathcal{X}$:
   (i) If $x_n \to x$ in $\mathcal{X}$, then $F(x) \le \liminf_{n\to\infty} F_n(x_n)$.
   (ii) There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that
$$x_n \to x \text{ in } \mathcal{X} \quad \text{and} \quad \limsup_{n\to\infty} F_n(x_n) \le F(x). \tag{2.17}$$
   A sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying (2.17) is called a *recovery sequence of* $\{F_n\}_{n\in\mathbb{N}}$ *at* x.

We also need the following reverse Minkowski inequality (see, e.g., [AF, Theorem 2.12]).

Proposition 2.8 (Reverse Minkowski inequality)**.** *Let $(Y, \mathcal{A}, \mu)$ be a measure space*[9] *and let $r \in (0, 1]$. Then for any $\mathcal{A}$-measurable functions $f, g\colon Y \to [0, \infty]$,*
$$\left(\int_Y f^r \, d\mu\right)^{1/r} + \left(\int_Y g^r \, d\mu\right)^{1/r} \le \left(\int_Y (f+g)^r \, d\mu\right)^{1/r}. \tag{2.18}$$

In the following definition, we introduce the set of p-homogeneous functionals on $\mathcal{F}$ which satisfies $(\mathrm{GC})_p$.

Definition 2.9. Recall that $\mathcal{F}$ is a linear subspace of $L^0(X, m)$. Define
$$\mathcal{U}_p^{\mathrm{GC}}(\mathcal{F}) \coloneqq \mathcal{U}_p^{\mathrm{GC}} \coloneqq \{\mathcal{E}'\colon \mathcal{F} \to [0,\infty) \mid \mathcal{E}' \text{ is } p\text{-homogeneous, } (\mathcal{E}', \mathcal{F}) \text{ satisfies } (\mathrm{GC})_p\}.$$

Now we can state the desired *stability* of $(\mathrm{GC})_p$.

Proposition 2.10. (a) $a_1\mathcal{E}^{(1)} + a_2\mathcal{E}^{(2)} \in \mathcal{U}_p^{\mathrm{GC}}$ *for any* $\mathcal{E}^{(1)}, \mathcal{E}^{(2)} \in \mathcal{U}_p^{\mathrm{GC}}$ *and any* $a_1, a_2 \in [0, \infty)$.

(b) *Let* $\left\{\mathcal{E}^{(n)}\right\}_{n\in\mathbb{N}} \subseteq \mathcal{U}_p^{\mathrm{GC}}$ *and let* $\mathcal{E}^{(\infty)}\colon \mathcal{F} \to [0,\infty)$*. If* $\{\mathcal{E}^{(n)}\}_{n\in\mathbb{N}}$ *converges pointwise to* $\mathcal{E}^{(\infty)}$*, then* $\mathcal{E}^{(\infty)} \in \mathcal{U}_p^{\mathrm{GC}}$.

(c) *Assume that* $\mathcal{F} \subseteq L^p(X, m)$ *and let us regard* $\mathcal{F}$ *as a topological space equipped with the topology of* $L^p(X, m)$*. Let* $\left\{\mathcal{E}^{(n)} \in \mathcal{U}_p^{\mathrm{GC}}\right\}_{n\in\mathbb{N}}$ *and let* $\mathcal{E}^{(\infty)}\colon \mathcal{F} \to [0,\infty)$*. If* $\{\mathcal{E}^{(n)}\}_{n\in\mathbb{N}}$ Γ*-converges to* $\mathcal{E}^{(\infty)}$*, then* $\mathcal{E}^{(\infty)} \in \mathcal{U}_p^{\mathrm{GC}}$.

[9] In the book [AF], the reverse Minkowski inequality is stated and proved only for the L^r-space over non-empty open subsets of the Euclidean space equipped with the Lebesgue measure, but the same proof works for any measure space.

Proof. The statement (b) is trivial, so we will show (a) and (c). Throughout this proof, we fix $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1, \dots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.2).

(a): Let $\mathcal{E}^{(1)}, \mathcal{E}^{(2)} \in \mathcal{U}_p^{\mathrm{GC}}$. Then $a\mathcal{E}^{(1)} \in \mathcal{U}_p^{\mathrm{GC}}$ is evident for any $a \in [0,\infty)$. Set $E(f) := \mathcal{E}^{(1)}(f) + \mathcal{E}^{(2)}(f)$, $f \in \mathcal{F}$, and let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}^{n_1}$. It suffices to prove $\bigl\| \bigl(E(T_l(\boldsymbol{u}))^{1/p}\bigr)_{l=1}^{n_2} \bigr\|_{\ell^{q_2}} \le \bigl\| \bigl(E(u_k)^{1/p}\bigr)_{k=1}^{n_1} \bigr\|_{\ell^{q_1}}$. For simplicity, we consider the case of $q_2 < \infty$. (The case of $q_2 = \infty$ is similar.) Then we have

$$
\begin{aligned}
&\sum_{l=1}^{n_2} E\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} \\
&= \sum_{l=1}^{n_2} \Bigl[\mathcal{E}^{(1)}\bigl(T_l(\boldsymbol{u})\bigr) + \mathcal{E}^{(2)}\bigl(T_l(\boldsymbol{u})\bigr)\Bigr]^{q_2/p} \\
&\le \left(\sum_{i \in \{1,2\}} \left[\sum_{l=1}^{n_2} \mathcal{E}^{(i)}\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} \right]^{p/q_2} \right)^{q_2/p} \qquad \text{(by the triangle ineq. for } \|\cdot\|_{\ell^{q_2/p}}\text{)} \\
&\overset{(\mathrm{GC})_p}{\le} \left(\left[\sum_{k=1}^{n_1} \mathcal{E}^{(1)}(u_k)^{q_1/p} \right]^{p/q_1} + \left[\sum_{k=1}^{n_1} \mathcal{E}^{(2)}(u_k)^{q_1/p} \right]^{p/q_1} \right)^{q_2/p} \\
&\overset{(2.18)}{\le} \left(\sum_{k=1}^{n_1} \Bigl[\mathcal{E}^{(1)}(u_k) + \mathcal{E}^{(2)}(u_k)\Bigr]^{q_1/p} \right)^{\frac{p}{q_1} \cdot \frac{q_2}{p}} = \left(\sum_{k=1}^{n_1} E(u_k)^{q_1/p} \right)^{q_2/q_1}, \qquad (2.19)
\end{aligned}
$$

which implies $E \in \mathcal{U}_p^{\mathrm{GC}}$.

(c): Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}^{n_1}$, and let $\{\boldsymbol{u}_n = (u_{1,n}, \dots, u_{n_1,n})\}_{n \in \mathbb{N}} \subseteq \mathcal{F}^{n_1}$ be a recovery sequence of $\{\mathcal{E}^{(n)}\}_{n \in \mathbb{N}}$ at $\boldsymbol{u}$. We first show tnat $\|T_l(\boldsymbol{u}) - T_l(\boldsymbol{u}_n)\|_{L^p(X,m)} \to 0$ as $n \to \infty$. Indeed, for any $\boldsymbol{v} = (v_1, \dots, v_{n_1})$ and any $\boldsymbol{z} = (z_1, \dots, z_{n_1}) \in L^p(X,m)^{n_1}$, we see that

$$
\begin{aligned}
\max_{l \in \{1,\dots,n_2\}} \|T_l(\boldsymbol{v}) - T_l(\boldsymbol{z})\|_{L^p(X,m)}^p &\overset{(2.2)}{\le} \int_X \|\boldsymbol{v}(x) - \boldsymbol{z}(x)\|_{\ell^{q_1}}^p \, m(dx) \\
&= \int_X \left(\sum_{k=1}^{n_1} |v_k(x) - z_k(x)|^{p \cdot \frac{q_1}{p}} \right)^{p/q_1} m(dx) \\
&\le n_1^{(p-q_1)/q_1} \sum_{k=1}^{n_1} \|v_k - z_k\|_{L^p(X,m)}^p, \qquad (2.20)
\end{aligned}
$$

where we used Hölder's inequality in the last line. Since $\max_k \|u_k - u_{k,n}\|_{L^p(X,m)} \to 0$ as $n \to \infty$, (2.20) implies the desired convergence $\|T_l(\boldsymbol{u}) - T_l(\boldsymbol{u}_n)\|_{L^p(X,m)} \to 0$.

Now we prove $(\mathrm{GC})_p$ for the Γ-limit $\mathcal{E}^{(\infty)}$ of $\{\mathcal{E}^{(n)}\}_{n \in \mathbb{N}}$ (with respect to the $L^p(X,m)$-topology). It is easy to see that $\mathcal{E}^{(\infty)}$ is p-homogeneous (see, e.g., [Dal, Proposition 11.6]).

We assume that $q_2 < \infty$ since the case $q_2 = \infty$ is similar. Then,

$$\sum_{l=1}^{n_2} \mathcal{E}^{(\infty)}\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} \leq \sum_{l=1}^{n_2} \liminf_{n\to\infty} \mathcal{E}^{(n)}\bigl(T_l(\boldsymbol{u}_n)\bigr)^{q_2/p} \leq \liminf_{n\to\infty} \sum_{l=1}^{n_2} \mathcal{E}^{(n)}\bigl(T_l(\boldsymbol{u}_n)\bigr)^{q_2/p}$$
$$\leq \liminf_{n\to\infty} \left(\sum_{k=1}^{n_1} \mathcal{E}^{(n)}(u_{k,n})^{q_1/p} \right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} = \left(\sum_{k=1}^{n_1} \mathcal{E}^{(\infty)}(u_k)^{q_1/p} \right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}},$$

which proves $\mathcal{E}^{(\infty)} \in \mathcal{U}_p^{\mathrm{GC}}$. □

3 Differentiability of p-energy forms and related results

In this section, we show the existence of the derivative in (1.6) for any p-energy form $(\mathcal{E}, \mathcal{F})$ satisfying p-Clarkson's inequality, (2.8) or (2.9), and establish fundamental properties of the "two-variable version" of $\mathcal{E}$ defined by (1.6).

Throughout this section, we fix $p \in (1, \infty)$, a measure space $(X, \mathcal{B}, m)$, and a *p-energy form* $(\mathcal{E}, \mathcal{F})$ on (X, m) in the following sense:

Definition 3.1 (p-Energy form)**.** Let $\mathcal{F}$ be a linear subspace of $L^0(X, m)$ and let $\mathcal{E} \colon \mathcal{F} \to [0, \infty)$. The pair $(\mathcal{E}, \mathcal{F})$ is said to be a *p-energy form* on (X, m) if and only if $\mathcal{E}^{1/p}$ is a seminorm on $\mathcal{F}$.

Note that the same argument as in the proof of Proposition 2.3-(a) implies that $\mathcal{E}$ is strictly convex on $\mathcal{F}/\mathcal{E}^{-1}(0)$ (see (2.3)).

3.1 p-Clarkson's inequality and differentiability

In this section, we mainly deal with p-energy forms satisfying p-Clarkson's inequality in the following sense.

Definition 3.2 (p-Clarkson's inequality)**.** The pair $(\mathcal{E}, \mathcal{F})$ is said to satisfy *p-Clarkson's inequality*, $(\mathrm{Cla})_p$ for short, if and only if for any $f, g \in \mathcal{F}$,

$$\begin{cases} \mathcal{E}(f+g) + \mathcal{E}(f-g) \geq 2\bigl(\mathcal{E}(f)^{\frac{1}{p-1}} + \mathcal{E}(g)^{\frac{1}{p-1}}\bigr)^{p-1} & \text{if } p \in (1, 2], \\ \mathcal{E}(f+g) + \mathcal{E}(f-g) \leq 2\bigl(\mathcal{E}(f)^{\frac{1}{p-1}} + \mathcal{E}(g)^{\frac{1}{p-1}}\bigr)^{p-1} & \text{if } p \in [2, \infty). \end{cases} \tag{Cla$_p$}$$

Remark 3.3. The following weaker version of p-Clarkson's inequality is also well known: for any $f, g \in \mathcal{F}$,

$$\begin{cases} \mathcal{E}(f+g) + \mathcal{E}(f-g) \leq 2\bigl(\mathcal{E}(f) + \mathcal{E}(g)\bigr) & \text{if } p \in (1, 2], \\ \mathcal{E}(f+g) + \mathcal{E}(f-g) \geq 2\bigl(\mathcal{E}(f) + \mathcal{E}(g)\bigr) & \text{if } p \in [2, \infty). \end{cases} \tag{Cla$'_p$}$$

Since, for any $a, b \in [0, \infty)$, Hölder's inequality yields $\bigl(a^{\frac{1}{p-1}} + b^{\frac{1}{p-1}}\bigr)^{p-1} \geq 2^{p-2}(a+b)$ if $p \in (1, 2]$ and $\bigl(a^{\frac{1}{p-1}} + b^{\frac{1}{p-1}}\bigr)^{p-1} \leq 2^{p-2}(a+b)$ if $p \in [2, \infty)$, $(\mathrm{Cla})_p$ *with* $\frac{f+g}{2}$, $\frac{f-g}{2}$ *in place of* f, g *implies* $(\mathrm{Cla})'_p$. In this paper, we will use this implication without further notice.

To state a consequence of (Cla)$_p$ on the convexity of $\mathcal{E}^{1/p}$, let us recall the notion of uniform convexity. See, e.g., [Cla36, Definition 1]. (The notion of uniform convexity is usually defined for normed spaces in the literature. We present the definition for seminormed spaces because we are mainly interested in $(\mathcal{F}, \mathcal{E}^{1/p})$.)

Definition 3.4 (Uniformly convex seminormed spaces)**.** Let $(\mathcal{X}, |\cdot|)$ be a seminormed space. We say that $(\mathcal{X}, |\cdot|)$ is *uniformly convex* if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ with the property that $|f + g| \leq 2(1 - \delta)$ whenever $f, g \in \mathcal{X}$ satisfy $|f| = |g| = 1$ and $|f - g| > \varepsilon$.

It is well known that (Cla)$_p$ implies the uniform convexity as follows.

Proposition 3.5. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies (Cla)$_p$. Then $(\mathcal{F}, \mathcal{E}^{1/p})$ is uniformly convex.*

Proof. The same argument as in [Cla36, Proof of Corollary of Theorem 2] works. □

Moreover, (Cla)$_p$ provides us the following quantitative estimate for the central difference, which plays a central role in this section.

Proposition 3.6. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies (Cla)$_p$. Then for any $f, g \in \mathcal{F}$,*

$$\mathcal{E}(f+g) + \mathcal{E}(f-g) - 2\mathcal{E}(f) \leq 2\bigl(1 \vee (p-1)\bigr)\Bigl[\mathcal{E}(f)^{\frac{1}{p-1}} + \mathcal{E}(g)^{\frac{1}{p-1}}\Bigr]^{(p-2)^+}\mathcal{E}(g)^{1 \wedge \frac{1}{p-1}}, \tag{3.1}$$

and the function $\mathbb{R} \ni t \mapsto \mathcal{E}(f + tg) \in [0, \infty)$ is differentiable. Moreover, for any $c \in (0, \infty)$,

$$\lim_{\delta \downarrow 0} \sup_{f \in \mathcal{F};\, \mathcal{E}(f) \leq c/(p-2)^+} \sup_{g \in \mathcal{F};\, \mathcal{E}(g) \leq 1} \left| \frac{\mathcal{E}(f + \delta g) - \mathcal{E}(f)}{\delta} - \frac{d}{dt}\mathcal{E}(f + tg)\Big|_{t=0} \right| = 0. \tag{3.2}$$

Proof. Let $f, g \in \mathcal{F}$. If $p \in (1, 2]$, then (3.1) is immediate from (Cla)$'_p$. If $p \in (2, \infty)$, then setting $a \coloneqq \mathcal{E}(f)^{1/(p-1)}$ and $b \coloneqq \mathcal{E}(g)^{1/(p-1)}$, we see from (Cla)$_p$ that

$$\mathcal{E}(f+g) + \mathcal{E}(f-g) - 2\mathcal{E}(f) \leq 2((a+b)^{p-1} - a^{p-1}) = 2(p-1)\int_a^{a+b} s^{p-2}\,ds \leq 2(p-1)(a+b)^{p-2}b,$$

proving (3.1). For the rest of the proof, we first note that by the convexity of $\mathcal{E}$,

$$\text{the limits} \quad \lim_{\delta \downarrow 0} \frac{\mathcal{E}(f + \delta g) - \mathcal{E}(f)}{\delta} \quad \text{and} \quad \lim_{\delta \downarrow 0} \frac{\mathcal{E}(f - \delta g) - \mathcal{E}(f)}{-\delta} \quad \text{exist in } \mathbb{R}, \tag{3.3}$$

and for any $\delta \in (0, \infty)$,

$$D_\delta(f; g) \coloneqq \mathcal{E}(f + \delta g) + \mathcal{E}(f - \delta g) - 2\mathcal{E}(f) \geq 0, \tag{3.4}$$

$$\left| \frac{\mathcal{E}(f + \delta g) - \mathcal{E}(f)}{\delta} - \lim_{s \downarrow 0} \frac{\mathcal{E}(f + sg) - \mathcal{E}(f)}{s} \right| \leq \frac{D_\delta(f; g)}{\delta}. \tag{3.5}$$

On the other hand, we see from (3.1) that for any $\delta \in (0,\infty)$,

$$\frac{D_\delta(f;g)}{\delta} \leq \begin{cases} 2\delta^{p-1}\mathcal{E}(g) & \text{if } p \in (1,2], \\ 2(p-1)\delta^{\frac{1}{p-1}}\left[\mathcal{E}(f)^{\frac{1}{p-1}} + \delta^{\frac{p}{p-1}}\mathcal{E}(g)^{\frac{1}{p-1}}\right]^{p-2}\mathcal{E}(g)^{\frac{1}{p-1}} & \text{if } p \in (2,\infty). \end{cases} \tag{3.6}$$

By (3.4) and (3.6), the limits in (3.3) coincide, so that the function $t \mapsto \mathcal{E}(f+tg)$ is differentiable at 0 and thereby at any $s \in \mathbb{R}$ by replacing f with $f+sg$, and then we obtain (3.2) by combining this differentiability at 0 with (3.5) and (3.6). □

Proposition 3.6, especially (3.2), implies the Fréchet differentiability of $\mathcal{E}$ on $\mathcal{F}/\mathcal{E}^{-1}(0)$. We record this fact and basic properties of these derivatives in the following theorem.

Theorem 3.7. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies* (Cla)$_p$. *Then $\mathcal{E}\colon \mathcal{F}/\mathcal{E}^{-1}(0) \to [0,\infty)$ is Fréchet differentiable on the quotient normed space $\mathcal{F}/\mathcal{E}^{-1}(0)$. In particular, for any $f,g \in \mathcal{F}$,*

$$\text{the derivative} \quad \mathcal{E}(f;g) \coloneqq \frac{1}{p}\frac{d}{dt}\mathcal{E}(f+tg)\bigg|_{t=0} \in \mathbb{R} \quad \text{exists,} \tag{3.7}$$

the map $\mathcal{E}(f;\,\cdot\,)\colon \mathcal{F} \to \mathbb{R}$ is linear, $\mathcal{E}(f;f) = \mathcal{E}(f)$ and $\mathcal{E}(f;h) = 0$ for $h \in \mathcal{E}^{-1}(0)$. Moreover, for any $f, f_1, f_2, g \in \mathcal{F}$ and any $a \in \mathbb{R}$, the following hold:

$$\mathbb{R} \ni t \mapsto \mathcal{E}(f+tg;g) \in \mathbb{R} \text{ is strictly increasing if and only if } \mathcal{E}(g) > 0. \tag{3.8}$$

$$\mathcal{E}(af;g) = \operatorname{sgn}(a)\,|a|^{p-1}\,\mathcal{E}(f;g), \quad \mathcal{E}(f+h;g) = \mathcal{E}(f;g) \quad \text{for } h \in \mathcal{E}^{-1}(0). \tag{3.9}$$

$$|\mathcal{E}(f;g)| \leq \mathcal{E}(f)^{(p-1)/p}\mathcal{E}(g)^{1/p}. \tag{3.10}$$

$$|\mathcal{E}(f_1;g) - \mathcal{E}(f_2;g)| \leq C_p\bigl(\mathcal{E}(f_1) \vee \mathcal{E}(f_2)\bigr)^{(p-1-\alpha_p)/p}\mathcal{E}(f_1-f_2)^{\alpha_p/p}\mathcal{E}(g)^{1/p}, \tag{3.11}$$

where $\alpha_p \coloneqq \frac{1}{p} \wedge \frac{p-1}{p}$ and $C_p \in (0,\infty)$ is a constant determined solely and explicitly by p.

Remark 3.8. The Hölder continuity exponent α_p appearing in (3.11) is not optimal because this exponent can be improved to $(p-1) \wedge 1$ in the case of $\mathcal{E}(f;g) = \int_{\mathbb{R}^n} |\nabla f|^{p-2}\langle \nabla f, \nabla g\rangle\,dx$. However, whether such an improved Hölder continuity holds is unclear even for concrete p-energy forms constructed in the previous works [CGQ22, Kig23, MS25+, Shi24]. We can see the optimal Hölder continuity ((3.11) with $(p-1)\wedge 1$ in place of α_p) for p-energy forms constructed in [KS24+], where a direct construction of p-energy forms based on the Korevaar–Schoen type p-energy forms is presented.

Proof of Theorem 3.7. The existence of $\mathcal{E}(f;g)$ in (3.7) is already proved in Proposition 3.6. The properties $\mathcal{E}(f;ag) = a\mathcal{E}(f;g)$, $\mathcal{E}(af;g) = \operatorname{sgn}(a)\,|a|^{p-1}\,\mathcal{E}(f;g)$ and $\mathcal{E}(f;f) = \mathcal{E}(f)$ are obvious from the definition. The equalities $\mathcal{E}(f+h;g) = \mathcal{E}(f+g)$ and $\mathcal{E}(f;h) = 0$ for any $h \in \mathcal{E}^{-1}(0)$ follow from the triangle inequality for $\mathcal{E}^{1/p}$, so (3.9) holds. The property (3.8) is a consequence of the strict convexity of $\mathcal{E}$ (see (2.3)) and the differentiability in (3.7).

To show that $\mathcal{E}(f;\,\cdot\,)$ is linear, it suffices to prove $\mathcal{E}(f;g_1+g_2) = \mathcal{E}(f;g_1) + \mathcal{E}(f;g_2)$ for any $g_1, g_2 \in \mathcal{F}$. For any $t > 0$, the convexity of $\mathcal{E}$ implies that

$$\frac{\mathcal{E}\bigl(f+t(g_1+g_2)\bigr) - \mathcal{E}(f)}{t} = \frac{\mathcal{E}\bigl(\frac{1}{2}(f+2tg_1) + \frac{1}{2}(f+2tg_2)\bigr) - \mathcal{E}(f)}{t}$$

$$\leq \frac{\mathcal{E}(f+2tg_1)-\mathcal{E}(f)}{2t}+\frac{\mathcal{E}(f+2tg_2)-\mathcal{E}(f)}{2t}. \tag{3.12}$$

Passing to the limit as $t \downarrow 0$, we get $\mathcal{E}(f;g_1+g_2) \leq \mathcal{E}(f;g_1)+\mathcal{E}(f;g_2)$. We obtain the converse inequality by noting that

$$\frac{\mathcal{E}(f-tg)-\mathcal{E}(f)}{t} \to -\left.\frac{d}{dt}\mathcal{E}(f+tg)\right|_{t=0} = -p\mathcal{E}(f;g) \quad \text{as } t \downarrow 0,$$

and by applying (3.12) with $-g_1, -g_2$ in place of g_1, g_2 respectively.

The Hölder-type estimate (3.10) follows from the following elementary estimate:

$$|a^q-b^q| = \left|\int_{a\wedge b}^{a\vee b} qt^{q-1}\,dt\right| \leq q(a^{q-1}\vee b^{q-1})\,|a-b| \quad \text{for } q\in(0,\infty),\ a,b\in[0,\infty). \tag{3.13}$$

Indeed, by (3.13) and the triangle inequality for $\mathcal{E}^{1/p}$, for any $t>0$,

$$\left|\frac{\mathcal{E}(f+tg)-\mathcal{E}(f)}{t}\right| \leq p\bigl(\mathcal{E}(f+tg)^{1/p}\vee\mathcal{E}(f)^{1/p}\bigr)^{p-1}\mathcal{E}(g)^{1/p}. \tag{3.14}$$

We obtain (3.10) by letting $t\downarrow 0$ in (3.14). We conclude that $\mathcal{E}(f;\,\cdot\,)$ is the Fréchet derivative of $\mathcal{E}$ at f by (3.2), the linearity of $\mathcal{E}(f;\,\cdot\,)$ and (3.10).

In the rest of this proof, we prove (3.11). Our proof is partially inspired by an argument due to Šmulian in [Smu40]. In this proof, $C_{p,i}$, $i\in\{1,\dots,5\}$, is a constant depending only on p. We first show an analogue of (3.1) for $\mathcal{E}^{1/p}$. Using (3.13), we can show that there exists $c_*\in(0,2^{-p^3})$ depending only on p such that

$$\sup\left\{\frac{|\mathcal{E}(f)-\mathcal{E}(f+\delta g)|}{\mathcal{E}(f)}\,\middle|\, f,g\in\mathcal{F},\ \delta\in(0,\infty),\ \delta<c_*\mathcal{E}(f)^{1/p},\ \mathcal{E}(g)=1\right\} \leq \frac{1}{10}. \tag{3.15}$$

Define $\psi\colon\mathbb{R}\to\mathbb{R}$ by $\psi(t) := |t|^{1/p}$, and fix $f,g\in\mathcal{F}$ and $\delta\in(0,\infty)$ with $\delta<c_*\mathcal{E}(f)^{1/p}$ and $\mathcal{E}(g)=1$. Then there exist $\theta_1,\theta_2,\theta\in[0,1]$ such that

$$\begin{aligned}
0 &\leq \psi(\mathcal{E}(f+\delta g))+\psi(\mathcal{E}(f-\delta g))-2\psi(\mathcal{E}(f)) \\
&= \psi'(A_{1,\delta})\bigl[\mathcal{E}(f+\delta g)-\mathcal{E}(f)\bigr]-\psi'(A_{2,\delta})\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr] \\
&= \psi'(A_1(\delta))D_\delta(f;g)-\bigl(\psi'(A_{1,\delta})-\psi'(A_{2,\delta})\bigr)\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr] \\
&= \psi'(A_{1,\delta})D_\delta(f;g)-\psi''\bigl(A_{1,\delta}+\theta(A_{2,\delta}-A_{1,\delta})\bigr)(A_{2,\delta}-A_{1,\delta})\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr],
\end{aligned} \tag{3.16}$$

where $D_\delta(f;g)$ is the same as in (3.4) and

$$A_{1,\delta} := \mathcal{E}(f)+\theta_1\bigl[\mathcal{E}(f+\delta g)-\mathcal{E}(f)\bigr], \quad A_{2,\delta} := \mathcal{E}(f-\delta g)+\theta_2\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr].$$

By (3.15), we note that $|A_{1,\delta}|\wedge|A_{1,\delta}+\theta(A_{2,\delta}-A_{1,\delta})| \geq \frac{1}{2}\mathcal{E}(f)$, which together with (3.16) and (3.1) implies that for any $(\delta,f)\in(0,\infty)\times\mathcal{F}$ with $0<\delta<c_*\mathcal{E}(f)^{1/p}$,

$$0 \leq \psi(\mathcal{E}(f+\delta g))+\psi(\mathcal{E}(f-\delta g))-2\psi(\mathcal{E}(f))$$

$$
\begin{aligned}
&\le C_{p,1}\Bigl(\mathcal{E}(f)^{\frac{1}{p}-1+\frac{(p-2)^{+}}{p-1}}\delta^{p\wedge\frac{p}{p-1}}+\mathcal{E}(f)^{\frac{1}{p}-2+\frac{2(p-1)}{p}}\delta^{2}\Bigr)\\
&\le C_{p,1}\delta\cdot\delta^{(p-1)\wedge\frac{1}{p-1}}\Bigl(\mathcal{E}(f)^{\frac{1}{p}-1+\frac{(p-2)^{+}}{p-1}}+\mathcal{E}(f)^{\frac{1}{p}-2+\frac{2(p-1)}{p}}\Bigr).
\end{aligned}
$$

In particular, if $\mathcal{E}(f)=1$, then

$$
\mathcal{E}(f+\delta g)^{1/p}+\mathcal{E}(f-\delta g)^{1/p}\le 2+C_{p,1}\delta^{(p-1)\wedge\frac{1}{p-1}}\delta \quad \text{for any } \delta\in(0,c_{*}). \tag{3.17}
$$

Next let $f_1,f_2\in\mathcal{F}$. Then, by (3.10) and (3.13),

$$
\begin{aligned}
|\mathcal{E}(f_2;f_1)-\mathcal{E}(f_1)| &\le |\mathcal{E}(f_2;f_1)-\mathcal{E}(f_2)|+|\mathcal{E}(f_2)-\mathcal{E}(f_1)|\\
&\le \Bigl(\mathcal{E}(f_2)^{(p-1)/p}+p\Bigl(\mathcal{E}(f_2)^{(p-1)/p}\vee\mathcal{E}(f_1)^{(p-1)/p}\Bigr)\Bigr)\mathcal{E}(f_1-f_2)^{1/p}\\
&\le C_{p,2}\Bigl(\mathcal{E}(f_1)^{(p-1)/p}\vee\mathcal{E}(f_2)^{(p-1)/p}\Bigr)\mathcal{E}(f_1-f_2)^{1/p}.
\end{aligned} \tag{3.18}
$$

Now, for any $f_1,f_2,g\in\mathcal{F}$ with $\mathcal{E}(f_1)=\mathcal{E}(g)=1$ and any $\delta\in(0,c_{*})$, we see that

$$
\begin{aligned}
&\mathcal{E}(f_1;\delta g)-\mathcal{E}(f_2;\delta g)\\
&=\mathcal{E}(f_1;f_1+\delta g)+\mathcal{E}(f_2;f_1-\delta g)-\mathcal{E}(f_1)-\mathcal{E}(f_2;f_1)\\
&\overset{(3.10)}{\le}\Bigl(\mathcal{E}(f_1)^{(p-1)/p}\vee\mathcal{E}(f_2)^{(p-1)/p}\Bigr)\Bigl(\mathcal{E}(f_1+\delta g)^{1/p}+\mathcal{E}(f_1-\delta g)^{1/p}\Bigr)-\mathcal{E}(f_1)-\mathcal{E}(f_2;f_1)\\
&\overset{(3.13),(3.17)}{\le}\Bigl(1+C_{p,3}\mathcal{E}(f_1-f_2)^{1/p}\Bigr)\Bigl(2+C_{p,1}\delta^{(p-1)\wedge\frac{1}{p-1}}\delta\Bigr)-\mathcal{E}(f_1)-\mathcal{E}(f_2;f_1).
\end{aligned}
$$

Similarly, we can show

$$
\begin{aligned}
&\mathcal{E}(f_1;\delta g)-\mathcal{E}(f_2;\delta g)\\
&=-\mathcal{E}(f_1;f_1-\delta g)-\mathcal{E}(f_2;f_1+\delta g)+\mathcal{E}(f_1)+\mathcal{E}(f_2;f_1)\\
&\ge -\Bigl(1+C_{p,3}\mathcal{E}(f_1-f_2)^{1/p}\Bigr)\Bigl(2+C_{p,1}\delta^{(p-1)\wedge\frac{1}{p-1}}\delta\Bigr)+\mathcal{E}(f_1)+\mathcal{E}(f_2;f_1).
\end{aligned}
$$

From these estimates, we have

$$
\begin{aligned}
&|\mathcal{E}(f_1;g)-\mathcal{E}(f_2;g)|=\frac{|\mathcal{E}(f_1;\delta g)-\mathcal{E}(f_2;\delta g)|}{\delta}\\
&\le\Bigl(1+C_{p,3}\mathcal{E}(f_1-f_2)^{1/p}\Bigr)\Bigl(2\delta^{-1}+C_{p,1}\delta^{(p-1)\wedge\frac{1}{p-1}}\Bigr)-\delta^{-1}\mathcal{E}(f_1)-\delta^{-1}\mathcal{E}(f_2;f_1)\\
&=\Bigl(1+C_{p,3}\mathcal{E}(f_1-f_2)^{1/p}\Bigr)\Bigl(2\delta^{-1}+C_{p,1}\delta^{(p-1)\wedge\frac{1}{p-1}}\Bigr)-2\delta^{-1}\mathcal{E}(f_1)+\delta^{-1}\bigl(\mathcal{E}(f_1)-\mathcal{E}(f_2;f_1)\bigr)\\
&\overset{(3.18)}{\le}\Bigl(1+C_{p,3}\mathcal{E}(f_1-f_2)^{1/p}\Bigr)\Bigl(2\delta^{-1}+C_{p,1}\delta^{(p-1)\wedge\frac{1}{p-1}}\Bigr)-2\delta^{-1}+C_{p,2}\delta^{-1}\mathcal{E}(f_1-f_2)^{1/p}\\
&\le C_{p,4}\Bigl(\delta^{(p-1)\wedge\frac{1}{p-1}}+\delta^{-1}\mathcal{E}(f_1-f_2)^{1/p}\Bigr).
\end{aligned}
$$

If $\mathcal{E}(f_1-f_2)<c_{*}^{-p^2/((p-1)\vee 1)}$, then, by choosing $\delta=\mathcal{E}(f_1-f_2)^{((p-1)\vee 1)/p^2}$, we obtain

$$
|\mathcal{E}(f_1;g)-\mathcal{E}(f_2;g)|\le C_{p,5}\mathcal{E}(f_1-f_2)^{((p-1)\wedge 1)/p^2}. \tag{3.19}
$$

The same is clearly true if $\mathcal{E}(f_1 - f_2) \geq c_*^{-p^2/((p-1)\vee 1)}$ since $\mathcal{E}(f_2) \leq 2^{p-1}\bigl(1 + \mathcal{E}(f_1 - f_2)\bigr)$. Finally, for any $f_1, f_2, g \in \mathcal{F}$ with $\mathcal{E}(f_1) \wedge \mathcal{E}(g) > 0$, we have

$$\begin{aligned}
|\mathcal{E}(f_1; g) - \mathcal{E}(f_2; g)| &= \mathcal{E}(f_1)^{(p-1)/p}\mathcal{E}(g)^{1/p}\left|\mathcal{E}\left(\frac{f_1}{\mathcal{E}(f_1)^{1/p}}; \frac{g}{\mathcal{E}(g)^{1/p}}\right) - \mathcal{E}\left(\frac{f_2}{\mathcal{E}(f_1)^{1/p}}; \frac{g}{\mathcal{E}(g)^{1/p}}\right)\right| \\
&\overset{(3.19)}{\leq} C_{p,5}\mathcal{E}(f_1)^{(p-1)/p}\mathcal{E}(g)^{1/p}\mathcal{E}\left(\frac{f_1}{\mathcal{E}(f_1)^{1/p}} - \frac{f_2}{\mathcal{E}(f_1)^{1/p}}\right)^{((p-1)\wedge 1)/p^2} \\
&\overset{(3.19)}{\leq} C_{p,5}\bigl(\mathcal{E}(f_1) \vee \mathcal{E}(f_2)\bigr)^{(p-1-\alpha_p)/p}\mathcal{E}(g)^{1/p}\mathcal{E}(f_1 - f_2)^{\alpha_p/p}.
\end{aligned}$$

The same estimate is clearly true if $\mathcal{E}(f_2) \wedge \mathcal{E}(g) > 0$. Since (3.11) is obvious when $g \in \mathcal{E}^{-1}(0)$ or $\mathcal{E}(f_1) \vee \mathcal{E}(f_2) = 0$, we obtain (3.11). $\square$

The following theorem gives a quantitative continuity for the inverse map of $f \mapsto \mathcal{E}(f; \cdot)$.

Theorem 3.9. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (Cla)$_p$*. Then for any $f, g \in \mathcal{F}$,*

$$\mathcal{E}(f - g) \leq C'_p\bigl[\mathcal{E}(f) \vee \mathcal{E}(g)\bigr]^{\frac{1+(p-1)(2-p)^+}{p}}\left(\sup_{\varphi \in \mathcal{F}; \mathcal{E}(\varphi) \leq 1}|\mathcal{E}(f; \varphi) - \mathcal{E}(g; \varphi)|\right)^{(p-1)\wedge 1}, \tag{3.20}$$

where $C'_p \in (0, \infty)$ is a constant determined solely and explicitly by p.

Proof. For ease of notation, for any linear functional $\Phi\colon \mathcal{F} \to \mathbb{R}$, we set $\|\Phi\|_{\mathcal{F},*} := \sup_{u \in \mathcal{F}; \mathcal{E}(u) \leq 1}|\Phi(u)|$. Clearly, $\|\Phi_1 + \Phi_2\|_{\mathcal{F},*} \leq \|\Phi_1\|_{\mathcal{F},*} + \|\Phi_2\|_{\mathcal{F},*}$ for any linear functionals $\Phi_1, \Phi_2\colon \mathcal{F} \to \mathbb{R}$. Note that $\|\mathcal{E}(f; \cdot\,)\|_{\mathcal{F},*} = \mathcal{E}(f)^{(p-1)/p}$ for any $f \in \mathcal{F}$ by (3.10). In particular, for any $f, g \in \mathcal{F}$,

$$\left|\mathcal{E}(f)^{\frac{p-1}{p}} - \mathcal{E}(g)^{\frac{p-1}{p}}\right| = \left|\|\mathcal{E}(f; \cdot\,)\|_{\mathcal{F},*} - \|\mathcal{E}(g; \cdot\,)\|_{\mathcal{F},*}\right| \leq \|\mathcal{E}(f; \cdot\,) - \mathcal{E}(g; \cdot\,)\|_{\mathcal{F},*},$$

which together with (3.13) with $q = (p-1)/p$ implies that

$$|\mathcal{E}(f) - \mathcal{E}(g)| \leq \frac{p}{p-1}\bigl(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\bigr)\,\|\mathcal{E}(f; \cdot\,) - \mathcal{E}(g; \cdot\,)\|_{\mathcal{F},*}. \tag{3.21}$$

Let us define $\psi\colon \mathbb{R} \to \mathbb{R}$ by $\psi(t) := \frac{1}{p}\mathcal{E}(f + t(g - f))$. Then $\psi \in C^1(\mathbb{R})$ by (3.2) and (3.11); indeed, (3.2) implies that $\psi'(t) = \mathcal{E}(f + t(g - f); g - f)$, which is continuous by (3.11). Now we see that

$$\begin{aligned}
&|\psi'(0)| = |\mathcal{E}(f; g - f)| \leq |\mathcal{E}(f; g) - \mathcal{E}(g)| + |\mathcal{E}(g) - \mathcal{E}(f)| \\
&\overset{(3.21)}{\leq} \|\mathcal{E}(f; \cdot\,) - \mathcal{E}(g; \cdot\,)\|_{\mathcal{F},*}\mathcal{E}(g)^{1/p} + \frac{p}{p-1}\bigl(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\bigr)\,\|\mathcal{E}(f; \cdot\,) - \mathcal{E}(g; \cdot\,)\|_{\mathcal{F},*} \\
&\leq \left(1 + \frac{p}{p-1}\right)\bigl(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\bigr)\,\|\mathcal{E}(f; \cdot\,) - \mathcal{E}(g; \cdot\,)\|_{\mathcal{F},*}.
\end{aligned}$$

Similarly,

$$|\psi'(1)| = |\mathcal{E}(g; g-f)| \leq \left(1 + \frac{p}{p-1}\right) \left(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\right) \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}.$$

Since ψ is C^1-convex, we obtain

$$\begin{aligned} \left| -\mathcal{E}(f) + \mathcal{E}\left(\frac{f+g}{2}\right) \right| &= p\,|\psi(1/2) - \psi(0)| \leq \frac{p}{2}\big(|\psi'(0)| \vee |\psi'(1)|\big) \\ &\leq c_p\big(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\big) \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}, \end{aligned}$$

where we put $c_p \coloneqq \frac{p}{2}\big(1 + \frac{p}{p-1}\big)$. Similarly,

$$\left| -\mathcal{E}(g) + \mathcal{E}\left(\frac{f+g}{2}\right) \right| = p\,|\psi(1/2) - \psi(1)| \leq c_p\big(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\big) \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}.$$

Therefore, it follows that

$$\mathcal{E}\left(\frac{f+g}{2}\right) \geq \left(\mathcal{E}(f) \vee \mathcal{E}(g) - c_p\big(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\big) \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}\right)^+. \tag{3.22}$$

Next we derive an estimate on $\mathcal{E}(\frac{f-g}{2})$ by using $(\mathrm{Cla})_p$ and (3.22). Set $a \coloneqq \mathcal{E}(f) \vee \mathcal{E}(g)$ for simplicity. If $p \in [2,\infty)$, then

$$\begin{aligned} \mathcal{E}\left(\frac{f-g}{2}\right) &\overset{(\mathrm{Cla})_p}{\leq} 2^{1-p}\big(\mathcal{E}(f)^{1/(p-1)} + \mathcal{E}(g)^{1/(p-1)}\big)^{p-1} - \mathcal{E}\left(\frac{f+g}{2}\right) \\ &\overset{(3.22)}{\leq} a - \left(a - c_p a^{1/p} \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}\right)^+ \\ &\leq c_p a^{1/p} \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}. \end{aligned}$$

In the rest of the proof, we assume that $p \in (1,2]$. We see that

$$\begin{aligned} \mathcal{E}\left(\frac{f-g}{2}\right)^{1/(p-1)} &\overset{(\mathrm{Cla})_p}{\leq} \left(\frac{\mathcal{E}(f) + \mathcal{E}(g)}{2}\right)^{1/(p-1)} - \mathcal{E}\left(\frac{f+g}{2}\right)^{1/(p-1)} \\ &\overset{(3.22)}{\leq} a^{1/(p-1)} - \left[\left(a - c_p a^{1/p} \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}\right)^+\right]^{1/(p-1)}. \end{aligned} \tag{3.23}$$

In the case of $a \leq c_p a^{1/p} \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}$, we have

$$\mathcal{E}\left(\frac{f-g}{2}\right) \leq a = a^{(2-p)+(p-1)} \leq c_p^{p-1} a^{2-p+\frac{p-1}{p}} \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}^{p-1}.$$

Let us consider the remaining case $a > c_p a^{1/p} \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}$. Then we have from (3.13) with $q = 1/(p-1)$ that

$$\begin{aligned} \mathcal{E}\left(\frac{f-g}{2}\right)^{1/(p-1)} &= a^{1/(p-1)} - \left(a - c_p a^{1/p} \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}\right)^{1/(p-1)} \\ &\leq \frac{c_p}{p-1} a^{\frac{2-p}{p-1} + \frac{1}{p}} \left\| \mathcal{E}(f; \,\cdot\,) - \mathcal{E}(g; \,\cdot\,) \right\|_{\mathcal{F},*}. \end{aligned}$$

Hence we obtain the desired estimate (3.20). □

The following proposition is a kind of monotonicity on values of p-Laplacian. This result will play important roles in Subsection 6.4 later and in the subsequent works [KS+a, KS+b].

Proposition 3.10. *Assume that* $(\mathcal{E},\mathcal{F})$ *satisfies* (Cla)$_p$ *and the strong subadditivity* (2.6). *Let* $u_1,u_2,v\in\mathcal{F}$ *satisfy* $((u_2-u_1)\wedge v)(x)=0$ *for* m*-a.e.* $x\in X$. *Then* $\mathcal{E}(u_1;v)\geq\mathcal{E}(u_2;v)$.

Proof. Let $t>0$. Define $f,g\in\mathcal{F}$ by $f\coloneqq u_1+tv$ and $g\coloneqq u_2$. Then we easily see that $f\vee g=u_2+tv$ and $f\wedge g=u_1$. By (2.6), we have $\mathcal{E}(u_2+tv)+\mathcal{E}(u_1)\leq\mathcal{E}(u_1+tv)+\mathcal{E}(u_2)$, which implies that

$$\frac{\mathcal{E}(u_2+tv)-\mathcal{E}(u_2)}{t}\leq\frac{\mathcal{E}(u_1+tv)-\mathcal{E}(u_1)}{t}.$$

Letting $t\downarrow 0$, we get $\mathcal{E}(u_2;v)\leq\mathcal{E}(u_1;v)$. □

We conclude this subsection by viewing typical examples of p-energy forms.

Example 3.11. (1) Let $D\in\mathbb{N}$, let Ω be an open subset of $\mathbb{R}^D$, let $\mathcal{B}\coloneqq\mathcal{B}(\Omega)$, let m be the D-dimensional Lebesgue measure on Ω and let $\mathcal{F}=W^{1,p}(\Omega)$ be the usual $(1,p)$-Sobolev space on Ω (see [AF, p. 60] for example). Define $\mathcal{E}(f)\coloneqq\|\nabla f\|_{L^p(\Omega,m)}^p$, $f\in\mathcal{F}$, where the gradient operator ∇ is regarded in the distribution sense. Then, by following a similar argument as in the proof of Theorem A.19, one can show that $(\mathcal{E},\mathcal{F})$ is a p-energy form on (Ω,m) satisfying (GC)$_p$. In this case, we have

$$\mathcal{E}(f;g)=\int_\Omega|\nabla f(x)|^{p-2}\langle\nabla f(x),\nabla g(x)\rangle_{\mathbb{R}^D}\,dx,\quad f,g\in\mathcal{F},$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}^D}$ denotes the inner product on $\mathbb{R}^D$.

(2) In the recent work [Kig23, MS25+], a p-energy form $(\mathcal{E},\mathcal{F})$ on a compact metrizable space is constructed via discrete approximations under some analytic and geometric assumptions. See [CGQ22, HPS04] for constructions of p-energy forms on post-critically finite self-similar sets. The construction in [CGQ22] can be seen as a generalization of that in [HPS04]. As will be seen in more detail later in Section 8, we can prove that p-energy forms constructed in [CGQ22, Kig23, MS25+] satisfy (GC)$_p$ while even (Cla)$_p$ is not mentioned in [CGQ22, Kig23]. Furthermore, very recently, Kuwae [Kuw24] introduced a p-energy form $(\mathscr{E}^p,H^{1,p})$ based on a strongly local Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ on $L^2(X,m)$. It is shown that $(\mathscr{E}^p,H^{1,p})$ satisfies (Cla)$_p$ in [Kuw24, Theorem 1.7]. We can also verify (GC)$_p$ for $(\mathscr{E}^p,H^{1,p})$ by using some good estimates due to the bilinearity (Theorem A.19). See Appendix A for details.

(3) There are various ways to define $(1,p)$-Sobolev spaces in the field of analysis on metric spaces (see, e.g., [HKST, Chapter 10]). In these definitions, roughly speaking, we find a counterpart of $|\nabla u|$, e.g., the minimal p-weak upper gradient $g_u\geq 0$ (see, e.g., [HKST, Chapter 6] for details), and consider a p-energy form $(\widetilde{\mathcal{E}},\mathcal{F})$ on (X,m) given by $\widetilde{\mathcal{E}}(u)\coloneqq\int_X g_u^p\,dm$ and $\mathcal{F}\coloneqq\{u\in L^p(X,m)\mid g_u\in L^p(X,m)\}$. Unfortunately, this p-energy form may not satisfy (Cla)$_p$ because the map $u\mapsto g_u$ is not linear in general (see, e.g., [HKST, (6.3.18)]). However, in a suitable setting, we can construct a

functional which is equivalent to $\widetilde{\mathcal{E}}$ and satisfies (Cla)$_p$; see the p-energy form denoted by $(\mathcal{F}_p, W^{1,p})$ in [ACD15, Theorem 40]. Moreover, we can verify (GC)$_p$ for $(\mathcal{F}_p, W^{1,p})$ since $(\mathcal{F}_{\delta_k,p}, W^{1,p})$ defined in [ACD15, (7.3)] satisfies (GC)$_p$ and $\mathcal{F}_p$ is defined as a Γ-limit point of $\mathcal{F}_{\delta_k,p}$ as $k \to \infty$. (See also the proof of Theorem 8.19.)

3.2 p-Clarkson's inequality and approximations in p-energy forms

In this subsection, in addition to the setting specified at the beginning of this section, by considering $\mathcal{F} \cap L^p(X,m)$ instead of $\mathcal{F}$ if necessary, we also assume for simplicity that $\mathcal{F} \subseteq L^p(X,m)$.

We introduce a family of natural norms on $\mathcal{F}$ in the following definition.

Definition 3.12 ($(\mathcal{E},\alpha)$-norm)**.** Let $\alpha \in (0,\infty)$. We define the norm $\|\cdot\|_{\mathcal{E},\alpha}$ on $\mathcal{F}$ by

$$\|f\|_{\mathcal{E},\alpha} := \Bigl(\mathcal{E}(f) + \alpha \, \|f\|_{L^p(X,m)}^p\Bigr)^{1/p}, \quad f \in \mathcal{F} \tag{3.24}$$

We call $\|\cdot\|_{\mathcal{E},\alpha}$ the $(\mathcal{E},\alpha)$-*norm* on $\mathcal{F}$.

Clearly, for any $\alpha, \alpha' \in (0,\infty)$, $\|\cdot\|_{\mathcal{E},\alpha}$ and $\|\cdot\|_{\mathcal{E},\alpha'}$ are equivalent to each other.

The following proposition states on the convexity of $\|\cdot\|_{\mathcal{E},\alpha}$.

Proposition 3.13. *Let $\alpha \in (0,\infty)$ and assume that $(\mathcal{E},\mathcal{F})$ satisfies* (Cla)$_p$*. Then $(\|\cdot\|_{\mathcal{E},\alpha}^p, \mathcal{F})$ is a p-energy form on (X,m) satisfying* (Cla)$_p$*, and $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is uniformly convex. Moreover, if $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is a Banach space in addition, then it is reflexive.*

Proof. We have (Cla)$_p$ for the p-energy form $(\|\cdot\|_{\mathcal{E},\alpha}^p, \mathcal{F})$ on (X,m) by applying (2.19) to $T\colon \mathbb{R}^2 \to \mathbb{R}$ given in Proposition 2.3-(e),(f). The uniform convexity $\|\cdot\|_{\mathcal{E},\alpha}$ follows from [Cla36, Proof of Corollary of Theorem 2].

Assume that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is a Banach space. Then $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is reflexive by the Milman–Pettis theorem (see, e.g., [Yos, Theorem 2 in Section V.2]) since $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is uniformly convex. □

We will frequently use the following Mazur's lemma, which is an elementary fact in the theory of Banach spaces.

Lemma 3.14 (Mazur's lemma; see, e.g., [Yos, Theorem 2 in Section V.1])**.** *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in a normed space V converging weakly to some element $v \in V$. Then there exist $N_k \in \mathbb{N}$ with $N_k \geq k$ and $\{\lambda_{k,l}\}_{k \leq l \leq N_k} \subseteq [0,1]$ with $\sum_{l=k}^{N_k} \lambda_{k,l} = 1$ for each $k \in \mathbb{N}$ such that $\lim_{k\to\infty} \sum_{l=k}^{N_k} \lambda_{k,l} v_l = v$ in norm in V.*

We also prepare the following two lemmas.

Lemma 3.15. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies* (Cla)$_p$ *and that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space. For $v \in L^{\frac{p}{p-1}}(X,m)$, we define a bounded linear map $\Psi_v\colon L^p(X,m) \to \mathbb{R}$ by $\Psi_v(u) := \int_X uv\,dm$. Then $\{\Psi_v|_{\mathcal{F}} \mid v \in L^{\frac{p}{p-1}}(X,m)\}$ is dense in $\mathcal{F}^*$, and the map $L^{\frac{p}{p-1}}(X,m) \ni v \mapsto \Psi_v|_{\mathcal{F}} \in \mathcal{F}^*$ is a bounded linear map with operator norm at most 1.*

Proof. Set $M := \{\Psi_v|_{\mathcal{F}} \mid v \in L^{\frac{p}{p-1}}(X,m)\}$. Then $M \subseteq \mathcal{F}^*$ since $\|u\|_{L^p(X,m)} \leq \|u\|_{\mathcal{E},1}$ for any $u \in \mathcal{F}$. Suppose that $\overline{M}^{\mathcal{F}^*} \neq \mathcal{F}^*$. Let $\varphi \in \mathcal{F}^* \setminus \overline{M}^{\mathcal{F}^*}$. By the Hahn–Banach theorem, there exists $\Phi \in \mathcal{F}^{**}$ such that $\Phi(\varphi) \neq 0$ and $\Phi|_{\overline{M}^{\mathcal{F}^*}} = 0$. Since $\mathcal{F}$ is reflexive by Proposition 3.13, there exists $u \in \mathcal{F}$ such that $\Phi(\psi) = \psi(u)$ for any $\psi \in \mathcal{F}^*$. Then for any $\psi \in M$ we have $\psi(u) = \Phi(\psi) = 0$, which implies that $u = 0$. This contradicts $\varphi(u) = \Phi(\varphi) \neq 0$ and hence we obtain $\overline{M}^{\mathcal{F}^*} = \mathcal{F}^*$. The map $L^{\frac{p}{p-1}}(X,m) \ni v \mapsto \Psi_v|_{\mathcal{F}} \in \mathcal{F}^*$ is obviously linear, and is easily seen to have operator norm at most 1 by Hölder's inequality and the fact that $\|u\|_{L^p(X,m)} \leq \|u\|_{\mathcal{E},1}$ for any $u \in \mathcal{F}$. □

Corollary 3.16. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies* (Cla)$_p$ *and that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space. If $L^p(X,m)$ is separable, then $\mathcal{F}$ and $\mathcal{F}^*$ are separable.*

Proof. Since $L^{\frac{p}{p-1}}(X,m)$ is separable by the separability of $L^{\frac{p}{p-1}}(X,m)^* = L^p(X,m)$ and [Yos, Lemma in Section V.2], it follows from Lemma 3.15 that $\mathcal{F}^*$ is separable, which in turn implies by [Yos, Lemma in Section V.2] that $\mathcal{F}$ is separable. □

Lemma 3.17. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies* (Cla)$_p$ *and that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ converges in norm in $L^p(X,m)$ to $u \in L^p(X,m)$ and $\sup_{n\in\mathbb{N}} \mathcal{E}(u_n) < \infty$, then $u \in \mathcal{F}$ and $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ to u.*

Proof. Since $\mathcal{F}$ is reflexive and $\sup_{n\in\mathbb{N}} \|u_n\|_{\mathcal{E},1} < \infty$, some subsequence of $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ to some $f \in \mathcal{F}$ by [Yos, Theorem 1 in Section V.2] and hence weakly in $L^p(X,m)$ to both u and f by the continuity of the inclusion map of $\mathcal{F}$ into $L^p(X,m)$, and thus $u = f \in \mathcal{F}$. For any $\varphi \in \mathcal{F}^*$ and any $\varepsilon > 0$, by Lemma 3.15, there exists $v \in L^{\frac{p}{p-1}}(X,m)$ such that $\|\varphi - \Psi_v|_{\mathcal{F}}\|_{\mathcal{F}^*} < \varepsilon$. Then we easily see that

$$\begin{aligned} |\varphi(u) - \varphi(u_n)| &\leq |\varphi(u) - \Psi_v(u)| + |\Psi_v(u) - \Psi_v(u_n)| + |\varphi(u_n) - \Psi_v(u_n)| \\ &\leq \varepsilon\biggl(\|u\|_{\mathcal{E},1} + \sup_{n\in\mathbb{N}} \|u_n\|_{\mathcal{E},1}\biggr) + |\Psi_v(u) - \Psi_v(u_n)|, \end{aligned}$$

whence $\limsup_{n\to\infty} |\varphi(u) - \varphi(u_n)| \leq \varepsilon\bigl(\|u\|_{\mathcal{E},1} + \sup_{n\in\mathbb{N}} \|u_n\|_{\mathcal{E},1}\bigr)$. Since $\varepsilon > 0$ is arbitrary, we obtain $\lim_{n\to\infty} \varphi(u_n) = \varphi(u)$. This completes the proof. □

We collect some useful results on convergence in $\mathcal{E}$ in the following proposition.

Proposition 3.18. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies* (Cla)$_p$ *and that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is a Banach space.*

(a) *If $\{u_n\}_{n\in\mathbb{N}} \subseteq L^p(X,m)$ converges in norm in $L^p(X,m)$ to $u \in L^p(X,m)$, then $\mathcal{E}(u) \leq \liminf_{n\to\infty} \mathcal{E}(u_n)$, where we set $\mathcal{E}(f) := \infty$ for $f \in L^p(X,m) \setminus \mathcal{F}$.*
(b) *If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ converges in norm in $L^p(X,m)$ to $u \in \mathcal{F}$ and $\lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}(u)$, then $\lim_{n\to\infty} \|u - u_n\|_{\mathcal{E},1} = 0$.*

Proof. (a): If $\liminf_{n\to\infty} \mathcal{E}(u_n) = \infty$, then the desired statement clearly holds. So, we assume that $\liminf_{n\to\infty} \mathcal{E}(u_n) < \infty$. Pick a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mathcal{E}(u_{n_k}) = \liminf_{n\to\infty} \mathcal{E}(u_n)$. Then $\{u_{n_k}\}_{k\in\mathbb{N}}$ is a bounded sequence in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ converging in norm in $L^p(X,m)$ to u and hence Lemma 3.17 implies that $u \in \mathcal{F}$ and that $\{u_{n_k}\}_{k\in\mathbb{N}}$ converges weakly in $\mathcal{F}$ to u. Since $\|\cdot\|_{\mathcal{E},1}$ is lower semicontinuous with respect to the weak topology of $\mathcal{F}$, we have from $\lim_{k\to\infty}\|u_{n_k}\|_{L^p(X,m)} = \|u\|_{L^p(X,m)}$ that $\mathcal{E}(u)^{1/p} \le \liminf_{n\to\infty}\mathcal{E}(u_n)^{1/p}$.

(b): If $u \in \mathcal{E}^{-1}(0)$, then $\mathcal{E}(u-u_n) = \mathcal{E}(u_n) \to \mathcal{E}(u) = 0$. It suffices to consider the case of $\mathcal{E}(u) = 1$. Since $u + u_n$ converges in $L^p(X,m)$ to $2u$ as $n \to \infty$, by (a),

$$\begin{aligned} 2 = \mathcal{E}(2u)^{1/p} \le \liminf_{n\to\infty}\mathcal{E}\bigl(u+u_n\bigr)^{1/p} &\le \limsup_{n\to\infty}\mathcal{E}\bigl(u+u_n\bigr)^{1/p} \\ &\le \lim_{n\to\infty}\mathcal{E}(u_n)^{1/p} + \mathcal{E}(u)^{1/p} = 2, \end{aligned}$$

i.e., $\lim_{n\to\infty}\mathcal{E}(u+u_n) = 2^p$. By $(\mathrm{Cla})_p$, if $p \le 2$, then

$$\begin{aligned} \lim_{n\to\infty}\mathcal{E}(u-u_n)^{1/(p-1)} &\le 2\Bigl(\mathcal{E}(u) + \lim_{n\to\infty}\mathcal{E}(u_n)\Bigr)^{1/(p-1)} - \lim_{n\to\infty}\mathcal{E}(u+u_n)^{1/(p-1)} \\ &= 2\cdot 2^{1/(p-1)} - 2^{p/(p-1)} = 0. \end{aligned}$$

If $p \ge 2$, then

$$\lim_{n\to\infty}\mathcal{E}(u-u_n) \le 2^{p-1}\Bigl(\mathcal{E}(u) + \lim_{n\to\infty}\mathcal{E}(u_n)\Bigr) - \lim_{n\to\infty}\mathcal{E}(u+u_n) = 2^{p-1}\cdot 2 - 2^p = 0.$$

Since $\{u_n\}_{n\in\mathbb{N}}$ converges in norm in $L^p(X,m)$ to u, we obtain the desired convergence. □

The following convergences in $\mathcal{E}$ are also useful. These are analogues of [FOT, Theorem 1.4.2-(iii),(iv),(v)].

Corollary 3.19. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies (2.4) and $(\mathrm{Cla})_p$ and that $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ is a Banach space.*

(a) *Let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C(\mathbb{R})$ satisfy $\lim_{n\to\infty}\varphi_n(t) = t$, $\varphi_n(0) = 0$ and $|\varphi_n(t)-\varphi_n(s)| \le |t-s|$ for any $n \in \mathbb{N}$, $s,t \in \mathbb{R}$. Then $\{\varphi_n(u)\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\lim_{n\to\infty}\mathcal{E}(u-\varphi_n(u)) = 0$ for any $u \in \mathcal{F}$.*

(b) *Let $u \in \mathcal{F}$, $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\varphi \in C(\mathbb{R})$ satisfy $\lim_{n\to\infty}\|u-u_n\|_{\mathcal{E},1} = 0$, $\varphi(0) = 0$, $|\varphi(t)-\varphi(s)| \le |t-s|$ for any $s,t \in \mathbb{R}$ and $\varphi(u) = u$. Then $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\lim_{n\to\infty}\mathcal{E}(u-\varphi(u_n)) = 0$.*

Remark 3.20. Let us make the same remark as [KS23+, Remark 2.21] for the reader's convenience. Typical choices of $\{\varphi_n\}_{n\in\mathbb{N}} \subset C(\mathbb{R})$ in Corollary 3.19-(a) are $\varphi_n(t) = (-n)\vee(t\wedge n)$ and $\varphi_n(t) = t - (-\frac{1}{n})\vee(t\wedge\frac{1}{n})$. A typical use of Corollary 3.19-(b) is to obtain a sequence of I-valued functions converging to u in $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ when $I \subset \mathbb{R}$ is a closed interval and $u \in \mathcal{F}$ is I-valued, by considering $\varphi \in C(\mathbb{R})$ given by $\varphi(t) \coloneqq (\inf I)\vee(t\wedge\sup I)$.

Proof of Corollary 3.19. (a): It is immediate from the dominated convergence theorem that $\{\varphi_n(u)\}_{n\in\mathbb{N}}$ converges in norm in $L^p(X,m)$ to u. Since $\varphi_n(u) \in \mathcal{F}$ and $\mathcal{E}(\varphi_n(u)) \le \mathcal{E}(u)$ for any $n \in \mathbb{N}$ by (2.4), we see from Proposition 3.18-(a) that

$$\mathcal{E}(u) \le \liminf_{n\to\infty}\mathcal{E}(\varphi_n(u)) \le \limsup_{n\to\infty}\mathcal{E}(\varphi_n(u)) \le \mathcal{E}(u).$$

Thus $\lim_{n\to\infty}\mathcal{E}(\varphi_n(u)) = \mathcal{E}(u)$, and $\lim_{n\to\infty}\mathcal{E}(u-\varphi_n(u)) = 0$ by Proposition 3.18-(b).

(b): By (2.4) we have $\varphi(u_n) \in \mathcal{F}$ and $\mathcal{E}(\varphi(u_n)) \le \mathcal{E}(u_n)$ for any $n \in \mathbb{N}$, and $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ converges in norm in $L^p(X,m)$ to $\varphi(u) = u$ since $|\varphi(u) - \varphi(u_n)| \le |u - u_n|$ on X. We therefore see from Proposition 3.18-(a) that

$$\mathcal{E}(u) \le \liminf_{n\to\infty}\mathcal{E}(\varphi(u_n)) \le \limsup_{n\to\infty}\mathcal{E}(\varphi(u_n)) \le \lim_{n\to\infty}\mathcal{E}(u_n) = \mathcal{E}(u).$$

Thus $\lim_{n\to\infty}\mathcal{E}(\varphi(u_n)) = \mathcal{E}(u)$, and $\lim_{n\to\infty}\mathcal{E}(u-\varphi(u_n)) = 0$ by Proposition 3.18-(b). □

3.3 Fréchet derivative as a homeomorphism to the dual space

In many practical situations, the quotient normed space $\mathcal{F}/\mathcal{E}^{-1}(0)$ (equipped with the norm $\mathcal{E}^{1/p}$) becomes a Banach space (see Subsection 6.2). To state some basic properties of this Banach space, we recall the notion of *uniformly smoothness.*

Definition 3.21 (Uniformly smooth normed space)**.** Let $(\mathcal{X}, \|\cdot\|)$ be a normed space. The normed space $\mathcal{X}$ is said to be *uniformly smooth* if and only if

$$\lim_{\tau\to 0}\tau^{-1}\sup\left\{\frac{\|u+v\|+\|u-v\|}{2} - 1 \;\middle|\; \|u\| = 1, \|v\| = \tau\right\} = 0.$$

The following duality between uniform convexity and uniform smoothness is well known. (See also [BCL94, Lemma 5] for a quantitative version of this theorem.)

Theorem 3.22 (Day's duality theorem; see, e.g., [LT, Proposition 1.e.2])**.** *Let $\mathcal{X}$ be a Banach space. Then $\mathcal{X}$ is uniformly convex if and only if its dual space $\mathcal{X}^*$ is uniformly smooth.*

We also recall the notion of duality mapping and fundamental results on it in the following proposition (see, e.g., [Miya, Definition 2.1, Lemmas 2.1 and 2.2]).

Proposition 3.23 (Duality mapping)**.** *Let $\mathcal{X}$ be a Banach space and let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$. Let $\|\cdot\|_W$ be the norm of W for each $W \in \{\mathcal{X}, \mathcal{X}^*\}$. For $(x,f) \in \mathcal{X}\times\mathcal{X}^*$, we set $\langle x, f\rangle := f(x)$. For $x \in \mathcal{X}$, define $F\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ by*

$$F(x) := \left\{f \in \mathcal{X}^* \;\middle|\; \langle x, f\rangle = \|x\|_{\mathcal{X}}^2 = \|f\|_{\mathcal{X}^*}^2\right\},$$

which is called the duality mapping *of $\mathcal{X}$. Then the following properties hold:*

(a) *$F(x) \neq \emptyset$ for any $x \in \mathcal{X}$.*
(b) *If $\mathcal{X}$ is reflexive, then $\bigcup_{x\in\mathcal{X}} F(x) = \mathcal{X}^*$.*
(c) *If $\mathcal{X}$ is strictly convex, i.e., $\|\lambda x + (1-\lambda)y\|_{\mathcal{X}} < \lambda\|x\|_{\mathcal{X}} + (1-\lambda)\|y\|_{\mathcal{X}}$ for any $\lambda \in (0,1)$ and any $x, y \in \mathcal{X}\setminus\{0\}$, then $\#(F(x)) = 1$ for any $x \in \mathcal{X}$.*

Now we can state a result on the dual space of $\mathcal{F}/\mathcal{E}^{-1}(0)$.

Theorem 3.24. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies (Cla)_p and that $\mathcal{F}/\mathcal{E}^{-1}(0)$ is a Banach space.*

(a) *The Banach space $\mathcal{F}/\mathcal{E}^{-1}(0)$ is uniformly convex and uniformly smooth. In particular, it is reflexive and its dual Banach spaces $\bigl(\mathcal{F}/\mathcal{E}^{-1}(0)\bigr)^{*}$ is also uniformly convex and uniformly smooth.*

(b) *The map $f \mapsto \mathcal{E}(f;\cdot)$ is a homeomorphism from $\mathcal{F}/\mathcal{E}^{-1}(0)$ to $\bigl(\mathcal{F}/\mathcal{E}^{-1}(0)\bigr)^{*}$. In particular, $\bigl(\mathcal{F}/\mathcal{E}^{-1}(0)\bigr)^{*} = \{\mathcal{E}(f;\,\cdot\,) \mid f \in \mathcal{F}\}$.*

Proof. For ease of notation, set $\mathcal{X} := \mathcal{F}/\mathcal{E}^{-1}(0)$ and $\|u\|_{\mathcal{X}} := \mathcal{E}(u)^{1/p}$ for any $u \in \mathcal{X}$.

(a): The uniform convexity of $\mathcal{X}$ is immediate from Proposition 3.5, whence $\mathcal{X}$ is reflexive by the Milman–Pettis theorem. Also, we easily see from (3.17) that $\mathcal{X}$ is uniformly smooth. The same properties for $\mathcal{X}^{*}$ follow from Theorem 3.22.

(b): Let $u \in \mathcal{X}$ and define $\mathcal{A}(u) := \mathcal{E}(u)^{\frac{2}{p}-1}\mathcal{E}(u;\,\cdot\,) \in \mathcal{X}^{*}$. (We define $\mathcal{A}(u) = 0$ if $\mathcal{E}(u) = 0$.) We will show that $\mathcal{A}\colon \mathcal{X} \to \mathcal{X}^{*}$ is a bijection. By (3.10), we have

$$\|\mathcal{A}(u)\|_{\mathcal{X}^{*}} = \mathcal{E}(u)^{\frac{2}{p}-1}\,\|\mathcal{E}(u;\,\cdot\,)\|_{\mathcal{X}^{*}} = \mathcal{E}(u)^{\frac{2}{p}-1+\frac{p-1}{p}} = \|u\|_{\mathcal{X}}\,.$$

Then $\langle u, \mathcal{A}(u)\rangle = \mathcal{E}(u)^{\frac{2}{p}} = \|u\|_{\mathcal{X}}^{2} = \|\mathcal{A}(u)\|_{\mathcal{X}^{*}}^{2}$ and hence

$$\mathcal{A}(u) \in \{f \in \mathcal{X}^{*} \mid \langle u, f\rangle = \|u\|_{\mathcal{X}}^{2} = \|f\|_{\mathcal{X}^{*}}^{2}\} = F(u),$$

where $F\colon \mathcal{X} \to \mathcal{X}^{*}$ is the duality mapping. We see from Proposition 3.23 and (a) that $\mathcal{A}\colon \mathcal{X} \to \mathcal{X}^{*}$ is a surjection. Note that the mapping $F^{-1}\colon \mathcal{X}^{*} \to \mathcal{X}^{**} = \mathcal{X}$ defined by $F^{-1}(f) = \{u \in \mathcal{X} \mid \langle u, f\rangle = \|u\|_{\mathcal{X}}^{2} = \|f\|_{\mathcal{X}^{*}}^{2}\}$ for $f \in \mathcal{X}^{*}$ is the duality mapping from $\mathcal{X}^{*}$ to $\mathcal{X}$. By Proposition 3.23 and (a) again, we conclude that $\mathcal{A}$ is injective. The map $f \mapsto \mathcal{E}(f;\cdot)$ and its inverse are continuous by (3.11) and (3.20), respectively. □

We also present a similar statement for $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$.

Corollary 3.25. *Let $\alpha \in (0,\infty)$. Assume that $\mathcal{F} \subseteq L^{p}(X,m)$, that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{Cla})_{p}$ and that $\mathcal{X}_{\alpha} := (\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is a Banach space.*

(a) *The Banach space $\mathcal{X}_{\alpha}$ is uniformly convex and uniformly smooth. In particular, it is reflexive and its dual space $\mathcal{X}_{\alpha}^{*}$ is also uniformly convex and uniformly smooth.*

(b) *For each $f \in \mathcal{F}$, define a linear map $\Psi_{p,\alpha}^{f}\colon \mathcal{F} \to \mathbb{R}$ by*

$$\Psi_{p,\alpha}^{f}(g) := \mathcal{E}(f;g) + \alpha \int_{X} \operatorname{sgn}(f)\,|f|^{p-1}\,g\,dm, \quad g \in \mathcal{F}. \tag{3.25}$$

Then the map $f \mapsto \Psi_{p,\alpha}^{f}$ is a homeomorphism from $\mathcal{X}_{\alpha}$ to $\mathcal{X}_{\alpha}^{*}$. In particular, $\mathcal{X}_{\alpha}^{*} = \{\Psi_{p,\alpha}^{f} \mid f \in \mathcal{F}\}$.

Proof. We define $\mathcal{E}_{\alpha}\colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ by

$$\mathcal{E}_{\alpha}(u;v) := \mathcal{E}(u;v) + \alpha \int_{X} \operatorname{sgn}(u)\,|u|^{p-1}\,v\,dm, \quad u, v \in \mathcal{F}.$$

and set $\mathcal{E}_{\alpha}(u) := \mathcal{E}_{\alpha}(u;u) = \|u\|_{\mathcal{E},\alpha}^{p}$. Then $(\mathcal{E}_{\alpha},\mathcal{F})$ is a p-energy form on (X,m) and it satisfies $(\mathrm{Cla})_{p}$ by Proposition 3.13. We have the desired result by applying Theorem 3.24 to $(\mathcal{E}_{\alpha},\mathcal{F})$. □

3.4 Regularity and strong locality

In this subsection, in addition to the setting specified at the beginning of this section, we make the same topological assumptions as [FOT, (1.1.7)], i.e.,

$$X \text{ is a locally compact separable metrizable topological space,} \tag{3.26}$$

$$m \text{ is a (positive) Radon measure on } X \text{ with } \operatorname{supp}_X[m] = X \tag{3.27}$$

(it is implicit in (3.27) that the σ-algebra $\mathcal{B}$ which X is equipped with is assumed to be the Borel σ-algebra $\mathcal{B}(X)$ of X). Here, as usual, by a *(positive) Radon measure* on X we mean a Borel measure on X which is finite on any compact subset of X. Under this setting, the map from $C(X)$ to $L^0(X,m) = L^0(X,\mathcal{B}(X),m)$ defined by taking $u \in C(X)$ to its m-equivalence class is injective and hence gives a canonical embedding of $C(X)$ into $L^0(X,m)$ as a subalgebra, and we will consider $C(X)$ as a subset of $L^0(X,m)$ through this embedding without further notice.

The following definitions are analogues of the notions in the theory of regular symmetric Dirichlet forms (see, e.g., [FOT, p. 6]).

Definition 3.26 (Core)**.** Let $\mathscr{C}$ be a subset of $\mathcal{F} \cap C_c(X)$.

(1) $\mathscr{C}$ is said to be a *core* of $(\mathcal{E},\mathcal{F})$ if and only if $\mathscr{C}$ is dense both in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ and in $(C_c(X), \|\cdot\|_{\sup})$.

(2) A core $\mathscr{C}$ is said to be *special* if and only if $\mathscr{C}$ is a linear subspace of $\mathcal{F} \cap C_c(X)$, $\mathscr{C}$ is a dense subalgebra of $(C_c(X), \|\cdot\|_{\sup})$, and for any compact subset K of X and any relatively compact open subset G of X with $K \subseteq G$, there exists $\varphi \in \mathscr{C}$ such that $\varphi \geq 0$, $\varphi = 1$ on K and $\varphi = 0$ on $X \setminus G$.

Definition 3.27 (Regularity)**.** We say that $(\mathcal{E},\mathcal{F})$ is *regular* if and only if there exists a core $\mathscr{C}$ of $(\mathcal{E},\mathcal{F})$.

We can show the following result on regular p-energy forms, which is an analogue of [FOT, Exercise 1.4.1].

Proposition 3.28. *Assume that $(\mathcal{E},\mathcal{F})$ is regular and that $\mathcal{F}$ has the following properties:*

$$u^+ \wedge 1 \in \mathcal{F} \quad \textit{for any } u \in \mathcal{F}, \tag{3.28}$$

$$uv \in \mathcal{F} \quad \textit{for any } u, v \in \mathcal{F} \cap C_b(X). \tag{3.29}$$

Then $\mathcal{F} \cap C_c(X)$ is a special core of $(\mathcal{E},\mathcal{F})$.

Proof. It is clear that $\mathcal{F} \cap C_c(X)$ is a core of $(\mathcal{E},\mathcal{F})$. By (3.29), $\mathcal{F} \cap C_c(X)$ is a subalgebra of $C_c(X)$. Let K be a compact subset of X and G be a relatively compact open subset G of X with $K \subseteq G$. By Urysohn's lemma, there exists $\varphi_0 \in C_c(X)$ such that $\varphi_0 = 2$ on K and $\varphi_0 = 0$ on $X \setminus G$. Let $\varepsilon \in (0, 1/2)$. Fix $\psi \in \mathcal{F} \cap C_c(X)$ satisfying $\psi = 1$ on $\overline{G}^X$, which exists by the regularity of $(\mathcal{E},\mathcal{F})$, the locally compactness of X and (3.28). Since $\mathcal{F} \cap C_c(X)$ is a core of $(\mathcal{E},\mathcal{F})$, there exists $\widetilde{\varphi} \in \mathcal{F} \cap C_c(X)$ such that $\|\varphi_0 - \widetilde{\varphi}\|_{\sup} < \varepsilon$. Now we define $\varphi \in C_c(X)$ by $\varphi := (\widetilde{\varphi} - \varepsilon\psi)^+ \wedge 1$. (Note that $\mathrm{supp}_X[\varphi]$ is compact since $\overline{G}^X$ is compact.) Then $\varphi \in \mathcal{F} \cap C_c(X)$ by (3.28). Clearly, $\varphi = 1$ on K and $\varphi = 0$ on $X \setminus G$, so the proof is completed. □

The proposition above ensures when there exist *cutoff* functions in $\mathcal{F}$. We also introduce the following condition stating the existence of cutoff functions in a weaker sense.

Definition 3.29. We say that a p-energy form $(\mathcal{E}, \mathcal{F})$ on (X, m) satisfies the property $(\mathrm{CF})_m$ if and only if, for any open subset U of X and any compact subset K of U, there exists $\varphi \in \mathcal{F} \cap L^\infty(X, m)$ such that $\varphi(x) = 1$ for m-a.e. $x \in K$ and $\varphi(x) = 0$ for m-a.e. $x \in X \setminus U$.

We could consider variants of $(\mathrm{CF})_m$ such as one requiring $\varphi \in \mathcal{F} \cap C(K)$ in addition, but we do not discuss those in this paper. Note that $(\mathrm{CF})_m$ holds if $(\mathcal{E}, \mathcal{F})$ admits a special core.

Next we introduce two formulations of the notion of strong locality for $(\mathcal{E}, \mathcal{F})$.

Definition 3.30 (Strong locality)**.** (1) We say that $(\mathcal{E}, \mathcal{F})$ has the strong local property (SL1) if and only if, for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - \alpha_1]$ or $\mathrm{supp}_m[f_2 - \alpha_2]$ compact and $\mathrm{supp}_m[f_1 - \alpha_1] \cap \mathrm{supp}_m[f_2 - \alpha_2] = \emptyset$ for some $\alpha_1, \alpha_2 \in \mathcal{E}^{-1}(0)$,

$$\mathcal{E}(f_1 + f_2 + g) + \mathcal{E}(g) = \mathcal{E}(f_1 + g) + \mathcal{E}(f_2 + g). \tag{3.30}$$

(2) Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{Cla})_p$. We say that $(\mathcal{E}, \mathcal{F})$ has the strong local property (SL2) if and only if, for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - f_2 - \alpha]$ or $\mathrm{supp}_m[g - \beta]$ compact and $\mathrm{supp}_m[f_1 - f_2 - \alpha] \cap \mathrm{supp}_m[g - \beta] = \emptyset$ for some $\alpha, \beta \in \mathcal{E}^{-1}(0)$,

$$\mathcal{E}(f_1; g) = \mathcal{E}(f_2; g). \tag{3.31}$$

In the following propositions, we collect basic results about (SL1) and (SL2).

Proposition 3.31. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{Cla})_p$.*

(a) *If $(\mathcal{E}, \mathcal{F})$ satisfies (SL1), then for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - \alpha_1]$ or $\mathrm{supp}_m[f_2 - \alpha_2]$ compact and $\mathrm{supp}_m[f_1 - \alpha_1] \cap \mathrm{supp}_m[f_2 - \alpha_2] = \emptyset$ for some $\alpha_1, \alpha_2 \in \mathcal{E}^{-1}(0)$,*

$$\mathcal{E}(f_1 + f_2; g) = \mathcal{E}(f_1; g) + \mathcal{E}(f_2; g). \tag{3.32}$$

(b) *If $(\mathcal{E}, \mathcal{F})$ satisfies (SL2), then for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - f_2 - \alpha]$ or $\mathrm{supp}_m[g - \beta]$ compact and $\mathrm{supp}_m[f_1 - f_2 - \alpha] \cap \mathrm{supp}_m[g - \beta] = \emptyset$ for some $\alpha, \beta \in \mathcal{E}^{-1}(0)$,*

$$\mathcal{E}(g; f_1) = \mathcal{E}(g; f_2). \tag{3.33}$$

Proof. (a): Note that (3.30) with $g = 0$ implies that $\mathcal{E}(f_1 + f_2) = \mathcal{E}(f_1) + \mathcal{E}(f_2)$. For any $t \in (0, \infty)$, we have from (3.30) that

$$\frac{\mathcal{E}(f_1 + f_2 + tg) - \mathcal{E}(f_1 + f_2)}{t} + t^{p-1}\mathcal{E}(g) = \frac{\mathcal{E}(f_1 + tg) - \mathcal{E}(f_1)}{t} + \frac{\mathcal{E}(f_2 + tg) - \mathcal{E}(f_2)}{t}.$$

We obtain (3.32) by letting $t \downarrow 0$ in this equality.

(b): Since $\mathcal{E}(g;\,\cdot\,)$ is linear by Theorem 3.7, it suffices to prove $\mathcal{E}(g; f_1 - f_2) = 0$, which follows from (3.31) with $g, 0, f_1 - f_2$ in place of f_1, f_2, g. □

Proposition 3.32. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (Cla)$_p$.

(a) *If $(\mathcal{E}, \mathcal{F})$ satisfies* (SL1)*, then $(\mathcal{E}, \mathcal{F})$ also satisfies* (SL2)*.*

(b) *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (SL2) *and the following three conditions:*

$$uv \in \mathcal{F} \text{ for any } u, v \in \mathcal{F} \cap L^\infty(X, m). \tag{3.34}$$

$$\text{For any } u \in \mathcal{F},\ \{(-n) \vee (u \wedge n)\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \text{ and } \lim_{n\to\infty} \mathcal{E}\bigl(u - (-n) \vee (u \wedge n)\bigr) = 0. \tag{3.35}$$

$$(\mathcal{E}, \mathcal{F}) \text{ satisfies } (\mathrm{CF})_m. \tag{3.36}$$

Then $(\mathcal{E}, \mathcal{F})$ satisfies (SL1)*.*

Proof. (a): Let $f_1, f_2, g \in \mathcal{F}$, $\alpha_1, \alpha_2 \in \mathcal{E}^{-1}(0)$ and $t \in \mathbb{R} \setminus \{0\}$, and assume that either $\mathrm{supp}_m[f_1 - f_2 - \alpha]$ or $\mathrm{supp}_m[g - \beta]$ is compact and that $\mathrm{supp}_m[f_1 - f_2 - \alpha] \cap \mathrm{supp}_m[g - \beta] = \emptyset$. By (3.30) with $f_2 - f_1, tg, f_1$ in place of f_1, f_2, g we have

$$\mathcal{E}\bigl((f_2 - f_1) + tg + f_1\bigr) + \mathcal{E}(f_1) = \mathcal{E}\bigl((f_2 - f_1) + f_1\bigr) + \mathcal{E}(tg + f_1),$$

whence

$$\mathcal{E}(f_1; g) = \frac{1}{p} \lim_{t\to 0} \frac{\mathcal{E}(f_1 + tg) - \mathcal{E}(f_1)}{t} = \frac{1}{p} \lim_{t\to 0} \frac{\mathcal{E}(f_2 + tg) - \mathcal{E}(f_2)}{t} = \mathcal{E}(f_2; g),$$

proving (SL2).

(b): We first consider the case $g \in \mathcal{F} \cap L^\infty(X, m)$. Let $f_1, f_2 \in \mathcal{F}$ and $\alpha_1, \alpha_2 \in \mathcal{E}^{-1}(0)$, and assume that $\mathrm{supp}_m[f_1 - \alpha_1]$ is compact and that $\mathrm{supp}_m[f_1 - \alpha_1] \cap \mathrm{supp}_m[f_2 - \alpha_2] = \emptyset$. Let U be an open neighborhood of $\mathrm{supp}_m[f_1 - \alpha_1]$ such that $U \subseteq X \setminus \mathrm{supp}_m[f_2 - \alpha_2]$. By (3.36) and the locally compactness of K, there exists $\varphi \in \mathcal{F} \in L^\infty(X, m)$ such that $\varphi(x) = 1$ for m-a.e. $x \in U$, $\mathrm{supp}_m[\varphi]$ is compact and $\mathrm{supp}_m[\varphi] \cap \mathrm{supp}_m[f_2 - \alpha_2] = \emptyset$. Note that $\varphi g \in \mathcal{F}$ by (3.34). Then we see from (SL2) that

$$\begin{aligned}
\mathcal{E}(f_1 + f_2 + g) + \mathcal{E}(g) &= \mathcal{E}(f_1 + f_2 + g; f_1) + \mathcal{E}(f_1 + f_2 + g; f_2) + \mathcal{E}(f_1 + f_2 + g; g) + \mathcal{E}(g) \\
&\overset{\text{(SL2)}}{=} \mathcal{E}(f_1 + g; f_1) + \mathcal{E}(f_2 + g; f_2) + \mathcal{E}(f_1 + f_2 + g; g) + \mathcal{E}(g) \\
&= \mathcal{E}(f_1 + g; f_1) + \mathcal{E}(f_2 + g; f_2) \\
&\quad + \mathcal{E}(f_1 + f_2 + g; (1 - \varphi)g) + \mathcal{E}(f_1 + f_2 + g; \varphi g) + \mathcal{E}(g).
\end{aligned} \tag{3.37}$$

Since $\mathrm{supp}_m[\varphi g]$ and $\mathrm{supp}_m[f_1 - \alpha_1]$ are compact, $\mathrm{supp}_m[f_1 - \alpha_1] \cap \mathrm{supp}_m[(1 - \varphi)g] = \emptyset$ and $\mathrm{supp}_m[f_2 - \alpha_2] \cap \mathrm{supp}_m[\varphi g] = \emptyset$, we have the following equalities by (SL2):

$$\mathcal{E}(f_1 + f_2 + g; (1 - \varphi)g) = \mathcal{E}(f_2 + g; (1 - \varphi)g).$$

$$\mathcal{E}(f_1 + f_2 + g; \varphi g) = \mathcal{E}(f_1 + g; \varphi g).$$

$$\mathcal{E}(g) = \mathcal{E}(g; (1 - \varphi)g) + \mathcal{E}(g; \varphi g) = \mathcal{E}(f_1 + g; (1 - \varphi)g) + \mathcal{E}(f_2 + g; \varphi g).$$

By combining these equalities and (3.37), we obtain

$$\begin{aligned}\mathcal{E}(f_1 + f_2 + g) + \mathcal{E}(g) &= \mathcal{E}(f_1 + g; f_1) + \mathcal{E}(f_2 + g; f_2) + \mathcal{E}(f_1 + g; g) + \mathcal{E}(f_2 + g; g)\\ &= \mathcal{E}(f_1 + g) + \mathcal{E}(f_2 + g).\end{aligned}$$

The proof for the case where $\operatorname{supp}_m[f_2-\alpha_2]$ instead of $\operatorname{supp}_m[f_1-\alpha_1]$ is compact is similar, so (SL1) holds if $g \in \mathcal{F} \cap L^\infty(X,m)$.

Lastly, we prove (SL1) without assuming the boundedness of g. Let $g \in \mathcal{F}$ and set $g_n \coloneqq (-n) \vee (g \wedge n)$, $n \in \mathbb{N}$. Then $g_n \in \mathcal{F}$ by (3.35), and the statement proved in the previous paragraph yields that

$$\mathcal{E}(f_1 + f_2 + g_n) + \mathcal{E}(g_n) = \mathcal{E}(f_1 + g_n) + \mathcal{E}(f_2 + g_n)$$

for any $n \in \mathbb{N}$. Thanks to (3.35) and the triangle inequality for $\mathcal{E}^{1/p}$, we obtain the desired equality (3.31) by letting $n \to \infty$ in the equality above. □

4 p-Energy measures and their basic properties

In this section, we discuss p-energy measures dominated by a p-energy form. Similar to the case of p-energy forms, we introduce the two-variable version of p-energy measures and prove their basic properties.

As in the previous section, throughout this section we fix $p \in (1,\infty)$, a measure space $(X,\mathcal{B},m)$ and a p-energy form $(\mathcal{E},\mathcal{F})$ on (X,m).

4.1 p-Energy measures and p-Clarkson's inequality

The following definition specifies the class of families of measures which we call p-energy measures and consider in this section.

Definition 4.1 (p-Energy measures dominated by a p-energy form)**.** Let $\mathcal{B}_0$ be a σ-algebra in X,[10] and let $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ be a family of measures on $(X,\mathcal{B}_0)$. We say that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ is a *family of p-energy measures on* $(X,\mathcal{B}_0)$ *dominated by* $(\mathcal{E},\mathcal{F})$ if and only if the following hold:

$(\mathrm{EM1})_p$ $\Gamma\langle f\rangle(X) \le \mathcal{E}(f)$ for any $f \in \mathcal{F}$.

$(\mathrm{EM2})_p$ $\Gamma\langle\,\cdot\,\rangle(A)^{1/p}$ is a seminorm on $\mathcal{F}$ for any $A \in \mathcal{B}_0$.

We then see that $(\Gamma\langle\,\cdot\,\rangle(A),\mathcal{F})$ is a p-energy form on (X,m) for each $A \in \mathcal{B}_0$ by $(\mathrm{EM2})_p$.

We say that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies p-Clarkson's inequality, $(\mathrm{Cla})_p$ for short, if and only if $(\Gamma\langle\,\cdot\,\rangle(A),\mathcal{F})$ satisfies $(\mathrm{Cla})_p$ for any $A \in \mathcal{B}_0$, i.e., for any $f,g \in \mathcal{F}$,

$$\begin{cases}\Gamma\langle f+g\rangle(A) + \Gamma\langle f-g\rangle(A) \ge 2\bigl(\Gamma\langle f\rangle(A)^{\frac{1}{p-1}} + \Gamma\langle g\rangle(A)^{\frac{1}{p-1}}\bigr)^{p-1} & \text{if } p \in (1,2],\\ \Gamma\langle f+g\rangle(A) + \Gamma\langle f-g\rangle(A) \le 2\bigl(\Gamma\langle f\rangle(A)^{\frac{1}{p-1}} + \Gamma\langle g\rangle(A)^{\frac{1}{p-1}}\bigr)^{p-1} & \text{if } p \in [2,\infty).\end{cases} \tag{Cla_p}$$

[10] While we typically take $\mathcal{B}_0 = \mathcal{B} = \mathcal{B}(X)$ for a prescribed topology on X, we allow $\mathcal{B}_0 \ne \mathcal{B}$ here. This formulation is suitable in the setting of a p-resistance form on X considered in Section 6 and later, where we choose $(\mathcal{B},m)$ to be the pair of 2^X and the counting measure on X as mentioned in Remark 2.1 but may take $\mathcal{B}_0 = \mathcal{B}(X)$ for the topology on X induced by the associated p-resistance metric.

We also say that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies the generalized p-contraction property, $(\mathrm{GC})_p$ for short, if and only if $(\Gamma\langle\,\cdot\,\rangle(A),\mathcal{F})$ satisfies $(\mathrm{GC})_p$ for any $A\in\mathcal{B}_0$.

Example 4.2. (1) Consider the same setting as in Example 3.11-(1). Then the family $\{\Gamma\langle f\rangle\}_{f\in W^{1,p}(\Omega)}$ of Borel measures on Ω given by

$$\Gamma\langle f\rangle(A) \coloneqq \int_A |\nabla f(x)|^p\,dx \quad \text{for } f\in W^{1,p}(\Omega) \text{ and } A\in\mathcal{B}(\Omega),$$

is easily seen to be a family of p-energy measures on $(\Omega,\mathcal{B}(\Omega))$ dominated by the p-energy form $(\mathcal{E},W^{1,p}(\Omega))$ given by $\mathcal{E}(f)\coloneqq\int_\Omega|\nabla f(x)|^p\,dx$. Similar to Example 3.11-(1), one can show $(\mathrm{GC})_p$ for $\{\Gamma\langle f\rangle\}_{f\in W^{1,p}(\Omega)}$ by following an argument in the proof of Theorem A.19. Recall that $\mathcal{E}(f;g)=\int_\Omega|\nabla f(x)|^{p-2}\langle\nabla f(x),\nabla g(x)\rangle_{\mathbb{R}^D}\,dx$. Then we can see that, by the Leibniz and the chain rule for ∇, for any $u,\varphi\in W^{1,p}(\Omega)\cap C^1(\Omega)$,

$$\int_\Omega \varphi\,d\Gamma\langle u\rangle = \mathcal{E}(u;u\varphi) - \left(\frac{p-1}{p}\right)^{p-1}\mathcal{E}\bigl(|u|^{\frac{p}{p-1}};\varphi\bigr). \tag{4.1}$$

(2) Although p-energy forms have been constructed on compact metric spaces [Kig23, MS25+], we do not know how to construct the associated p-energy measures because of the lack of the density "$|\nabla u(x)|^p$". (As described in (3) below, the theory of Dirichlet forms gives 2-energy measures $\{\mu_{\langle u\rangle}\}_{u\in\mathcal{F}_2}$ associated with a given nice Dirichlet form $(\mathcal{E}_2,\mathcal{F}_2)$. On a large class of self-similar sets, however, it is known that $\mu_{\langle u\rangle}$ is singular with respect to the natural Hausdorff measure on the underlying fractal [Hin05, KM20].) In the case of self-similar sets, under suitable assumptions, *self-similar p-energy forms* are constructed in [CGQ22, Kig23, MS25+, Shi24], and we can introduce p-energy measures satisfying $(\mathrm{EM1})_p$, $(\mathrm{EM2})_p$ and $(\mathrm{GC})_p$ by using the self-similarity of p-energy forms. See Section 5 for details.
In [KS24+], under the assumption called the *weak monotonicity condition*, the authors construct a good p-energy form $\mathcal{E}_p^{\mathrm{KS}}$, which is called a *Korevaar–Shoen p-energy form*, on a locally compact separable metric space (X,d) equipped with a σ-finite Borel measure m with full topological support. As an advantage of $\mathcal{E}_p^{\mathrm{KS}}$, the right-hand side of (4.1) with $\mathcal{E}_p^{\mathrm{KS}}$ in place of $\mathcal{E}$ can be extended to a bounded positive linear functional in $\varphi\in C_c(X)$ and the p-energy measure $\Gamma_p^{\mathrm{KS}}\langle u\rangle$ associated with $\mathcal{E}_p^{\mathrm{KS}}$ is constructed as the unique Radon measure corresponding to this functional through the Riesz–Markov–Kakutani representation theorem. A notable fact is that this approach does not rely on the self-similarity of the underlying space or of the p-energy form. In [KS24+, Sections 3 and 4], basic properties for $\Gamma_p^{\mathrm{KS}}\langle\,\cdot\,\rangle$ like $(\mathrm{EM1})_p$, $(\mathrm{EM2})_p$ and $(\mathrm{GC})_p$ are also shown.

(3) The case of $p=2$ is very special thanks to the theory of symmetric Dirichlet forms. If $(\mathscr{E},D(\mathscr{E}))$ is a strongly local regular symmetric Dirichlet form on $L^2(X,m)$, where X and m are as specified in (3.26) and (3.27), then $\mathscr{E}(u)\coloneqq\mathscr{E}(u,u)$ is a 2-energy form on (X,m) and satisfies $(\mathrm{GC})_2$ (see Proposition A.2). In addition, the Dirichlet form theory provides us with a Borel measure $\mu_{\langle u\rangle}$ on X, called the *$\mathscr{E}$-energy measure* of

$u \in D(\mathscr{E})$ associated with $(\mathscr{E}, D(\mathscr{E}))$, through the following formula[11]:

$$\int_X \varphi \, d\mu_{\langle u \rangle} = \mathscr{E}(u, u\varphi) - \frac{1}{2}\mathscr{E}(u^2, \varphi) \quad \text{for any } \varphi \in D(\mathscr{E}) \cap C_c(X) \tag{4.2}$$

(recall (4.1), and see [FOT, Section 3.2] for details on energy measures associated with regular symmetric Dirichlet forms). We easily see that $\{\mu_{\langle u \rangle}\}_{u \in D(\mathscr{E})}$ satisfies $(\mathrm{EM1})_2$ and the parallelogram law, which implies $(\mathrm{EM2})_2$ and $(\mathrm{Cla})_2$. We can also verify $(\mathrm{GC})_2$ for $\{\mu_{\langle u \rangle}\}_{u \in D(\mathscr{E})}$ (Proposition A.14). As discussed in [Kuw24], under the additional assumption of a suitable closability in $L^p(X, m)$ formulated as (A.21) in Definition A.17, we can introduce a family of p-energy measures on $(X, \mathcal{B}(X))$ satisfying $(\mathrm{EM1})_p$, $(\mathrm{EM2})_p$ and $(\mathrm{GC})_p$ by setting $\Gamma\langle u \rangle(A) \coloneqq \int_A \Gamma_\mu(u)^{\frac{p}{2}} \, d\mu$, where μ is an $\mathscr{E}$-dominant measure (i.e., $\mu_{\langle u \rangle} \ll \mu$ for any $u \in D(\mathscr{E})$) and $\Gamma_\mu(u) \coloneqq d\mu_{\langle u \rangle}/d\mu$; see Theorem A.19 for the details of this family of p-energy measures.

(4) Let (X, d) be a separable metric space and m a Borel measure on X such that $m(X) > 0$ and $m(B_d(x, r)) < \infty$ for some $r \in (0, \infty)$ for any $x \in X$. Let g_u be the minimal p-weak upper gradient of $u \in N^{1,p}(X, m)$, where $N^{1,p}(X, m) \coloneqq \{u \in L^p(X, m) \mid g_u \in L^p(X, m)\}$ is the Newton-Sobolev space (see [HKST, Section 7.1]). Then $\Gamma\langle u \rangle(A) \coloneqq \int_A g_u^p \, dm$ defines p-energy measures satisfying $(\mathrm{EM1})_p$ and $(\mathrm{EM2})_p$. Indeed, we have $(\mathrm{EM2})_p$ by [HKST, (6.3.18)]. However, $(\mathrm{Cla})_p$ for these measures is unclear because the map $u \mapsto g_u$ is not linear in general.

In the rest of this subsection, we assume that $\mathcal{B}_0$ is a σ-algebra in X and that $\{\Gamma\langle f \rangle\}_{f \in \mathcal{F}}$ is a family of p-energy measures on $(X, \mathcal{B}_0)$ dominated by $(\mathcal{E}, \mathcal{F})$. The same argument as in the proof of Proposition 3.6 yields the following result.

Proposition 4.3. *Assume that $\{\Gamma\langle f \rangle\}_{f \in \mathcal{F}}$ satisfies $(\mathrm{Cla})_p$. Then for any $f, g \in \mathcal{F}$ and any $A \in \mathcal{B}_0$,*

$$\begin{aligned} &\Gamma\langle f + g \rangle(A) + \Gamma\langle f - g \rangle(A) - 2\Gamma\langle f \rangle(A) \\ &\leq 2\bigl(1 \vee (p-1)\bigr)\Bigl[\Gamma\langle f \rangle(A)^{\frac{1}{p-1}} + \Gamma\langle g \rangle(A)^{\frac{1}{p-1}}\Bigr]^{(p-2)^+} \Gamma\langle f \rangle(A)^{1 \wedge \frac{1}{p-1}}, \end{aligned} \tag{4.3}$$

and the function $\mathbb{R} \ni t \mapsto \Gamma\langle f + tg \rangle(A) \in [0, \infty)$ is differentiable. Moreover, for any $c \in (0, \infty)$,

$$\lim_{\delta \downarrow 0} \sup_{A \in \mathcal{B}_0,\, f, g \in \mathcal{F};\, \mathcal{E}(f) \leq c/(p-2)^+,\, \mathcal{E}(g) \leq 1} \left| \frac{\Gamma\langle f + \delta g \rangle(A) - \Gamma\langle f \rangle(A)}{\delta} - \frac{d}{dt}\Gamma\langle f + tg \rangle(A)\Big|_{t=0} \right| = 0. \tag{4.4}$$

Definition 4.4. Assume that $\{\Gamma\langle f \rangle\}_{f \in \mathcal{F}}$ satisfies $(\mathrm{Cla})_p$. For each $f, g \in \mathcal{F}$, we define $\Gamma\langle f; g \rangle \colon \mathcal{B}_0 \to \mathbb{R}$ by

$$\Gamma\langle f; g \rangle(A) \coloneqq \frac{1}{p} \frac{d}{dt}\Gamma\langle f + tg \rangle(A)\Big|_{t=0}, \qquad A \in \mathcal{B}_0, \tag{4.5}$$

which exists by Proposition 4.3.

[11] To be precise, the definition of $\mu_{\langle u \rangle}$ through (4.2) is valid only for $u \in D(\mathscr{E}) \cap L^\infty(X, m)$. We can still define $\mu_{\langle u \rangle}$ for any $u \in D(\mathscr{E})$ by considering the limit of $\mu_{\langle (-n) \vee (u \wedge n) \rangle}$ as $n \to \infty$.

The following properties of $\Gamma\langle f; g\rangle$ can be shown in a similar way as Theorem 3.7.

Theorem 4.5. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies $(\mathrm{Cla})_p$. Let $A \in \mathcal{B}_0$. Then $\Gamma\langle f; \cdot\rangle(A)$ is the Fréchet derivative of $\Gamma\langle\,\cdot\,\rangle(A)\colon \mathcal{F}/\mathcal{E}^{-1}(0) \to [0,\infty)$ at $f \in \mathcal{F}$. In particular, the map $\Gamma\langle f; \cdot\rangle(A)\colon \mathcal{F} \to \mathbb{R}$ is linear, $\Gamma\langle f; f\rangle(A) = \Gamma\langle f\rangle(A)$ and $\Gamma\langle f; h\rangle(A) = 0$ if $h \in \mathcal{F}$ satisfies $\Gamma\langle h\rangle(A) = 0$. Moreover, for any $f, f_1, f_2, g \in \mathcal{F}$ and any $a \in \mathbb{R}$, the following hold:*

$$\mathbb{R} \ni t \mapsto \Gamma\langle f + tg; g\rangle(A) \in \mathbb{R} \text{ is strictly increasing if and only if } \Gamma\langle g\rangle(A) > 0. \tag{4.6}$$

$$\Gamma\langle af; g\rangle(A) = \operatorname{sgn}(a)\,|a|^{p-1}\,\Gamma\langle f; g\rangle(A), \quad \Gamma\langle f + h; g\rangle(A) = \Gamma\langle f; g\rangle(A) \text{ if } \Gamma\langle h\rangle(A) = 0. \tag{4.7}$$

$$|\Gamma\langle f; g\rangle(A)| \le \Gamma\langle f\rangle(A)^{(p-1)/p}\Gamma\langle g\rangle(A)^{1/p}. \tag{4.8}$$

$$|\Gamma\langle f_1; g\rangle(A) - \Gamma\langle f_2; g\rangle(A)| \le C_p\bigl(\Gamma\langle f_1\rangle(A) \vee \Gamma\langle f_2\rangle(A)\bigr)^{\frac{p-1-\alpha_p}{p}}\Gamma\langle f_1 - f_2\rangle(A)^{\frac{\alpha_p}{p}}\Gamma\langle g\rangle(A)^{\frac{1}{p}}, \tag{4.9}$$

where α_p, C_p are the same as in Theorem 3.7.

The set function $\Gamma\langle f; g\rangle$ is a signed measure as shown in the following theorem.

Theorem 4.6. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies $(\mathrm{Cla})_p$. Then for any $f, g \in \mathcal{F}$, the set function $\Gamma\langle f; g\rangle$ is a signed measure on $(X, \mathcal{B}_0)$. Moreover, for any $\mathcal{B}_0$-measurable function $\varphi\colon X \to [0,\infty)$ with $\|\varphi\|_{\sup} < \infty$, $\int_X \varphi\, d\Gamma\langle\,\cdot\,\rangle\colon \mathcal{F}/\mathcal{E}^{-1}(0) \to \mathbb{R}$ is Fréchet differentiable and has the same properties as those of $\Gamma\langle\,\cdot\,\rangle(A)$ in Theorem 4.5 with "$\Gamma\langle g\rangle(A) > 0$" in (4.6) replaced by "$\int_X \varphi\, d\Gamma\langle g\rangle > 0$", and for any $f, g \in \mathcal{F}$,*

$$\int_X \varphi\, d\Gamma\langle f; g\rangle = \frac{1}{p}\frac{d}{dt}\int_X \varphi\, d\Gamma\langle f + tg\rangle\bigg|_{t=0}. \tag{4.10}$$

Proof. The equalities $\Gamma\langle f; g\rangle(\emptyset) = 0$ and $|\Gamma\langle f; g\rangle(X)| = |\mathcal{E}(f; g)| < \infty$ are clear from the definition. We will show the countable additivity of $\Gamma\langle f; g\rangle$. The finite additivity of $\Gamma\langle f; g\rangle$ is obvious. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}_0$ be a family of disjoint measurable sets. Set $B_N \coloneqq \bigcup_{n=N+1}^{\infty} A_n$ for each $N \in \mathbb{N}$. Then we see that

$$\begin{aligned}\left|\Gamma\langle f; g\rangle\left(\bigcup_{n\in\mathbb{N}} A_n\right) - \sum_{n=1}^{N}\Gamma\langle f; g\rangle(A_n)\right| &= |\Gamma\langle f; g\rangle(B_N)| \\ &\overset{(4.8)}{\le} \Gamma\langle f\rangle(B_N)^{(p-1)/p}\Gamma\langle g\rangle(B_N)^{1/p} \xrightarrow[N\to\infty]{} 0,\end{aligned}$$

which shows that $\Gamma\langle f; g\rangle$ is a signed measure on $(X, \mathcal{B}_0)$.

The other properties except for (4.10) can be proved by following the arguments in the proof of Theorem 3.7, so we shall prove (4.10). By the finite additivity of $\int_X \varphi\, d\Gamma\langle f; g\rangle$ and $\frac{1}{p}\frac{d}{dt}\int_X \varphi\, d\Gamma\langle f + tg\rangle\big|_{t=0}$ in φ, we can assume that $\varphi \ge 0$. Let $s_n = \sum_{k=1}^{l_n} a_k \mathbb{1}_{A_k}$ with $a_k \ge 0$ and $A_k \in \mathcal{B}_0$ be a sequence of simple functions so that $s_n \uparrow \varphi$ m-a.e. as $n \to \infty$. Then we immediately have (4.10) with $\varphi = s_n$. Since $\lim_{n\to\infty}\int_X s_n\, d\Gamma\langle f; g\rangle = \int_X \varphi\, d\Gamma\langle f; g\rangle$ by the dominated convergence theorem, it suffices to prove

$$\lim_{n\to\infty}\frac{d}{dt}\int_X s_n\, d\Gamma\langle f + tg\rangle\bigg|_{t=0} = \frac{d}{dt}\int_X \varphi\, d\Gamma\langle f + tg\rangle\bigg|_{t=0}. \tag{4.11}$$

Since (3.14) with $\int_X \varphi \, d\Gamma\langle\,\cdot\,\rangle$ in place of $\mathcal{E}$ holds by the fact that $(\int_X \varphi \, d\Gamma\langle\,\cdot\,\rangle, \mathcal{F})$ is a p-energy form on (X, m), we know that for any $\mathcal{B}_0$-measurable function $\psi \colon X \to [0,\infty)$ with $\|\psi\|_{\sup} < \infty$,

$$\left| \frac{d}{dt} \int_X \psi \, d\Gamma\langle f + tg\rangle \Big|_{t=0} \right| \leq \left(\int_X \psi \, d\Gamma\langle f\rangle \right)^{(p-1)/p} \left(\int_X \psi \, d\Gamma\langle g\rangle \right)^{1/p}. \tag{4.12}$$

By (4.12) with $\psi = \varphi - s_n$ and the dominated convergence theorem, we obtain (4.11). □

Remark 4.7. As mentioned in the introduction, a signed measure corresponding to $\Gamma\langle f; g\rangle$ is discussed in [BV05, Section 5] under some non-trivial assumptions, which have not been verified for fractals like the Sierpiński gasket and the Sierpiński carpet in the literature.

The following proposition gives a natural Hölder-type inequality for the total variation measure $|\Gamma\langle f; g\rangle|$ of $\Gamma\langle f; g\rangle$.

Proposition 4.8. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies (Cla)$_p$. Then for any $f, g \in \mathcal{F}$ and any $\mathcal{B}_0$-measurable functions $\varphi, \psi \colon X \to [0,\infty]$,*

$$\int_X \varphi\psi \, d\left|\Gamma\langle f; g\rangle\right| \leq \left(\int_X \varphi^{\frac{p}{p-1}} \, d\Gamma\langle f\rangle \right)^{(p-1)/p} \left(\int_X \psi^p \, d\Gamma\langle g\rangle \right)^{1/p}. \tag{4.13}$$

Proof. Let $X = \mathcal{P} \sqcup \mathcal{N}$ be the Hahn decomposition with respect to $\Gamma\langle f; g\rangle$, i.e., $\mathcal{P}, \mathcal{N} \in \mathcal{B}_0$, $\Gamma\langle f; g\rangle(A \cap \mathcal{P}) \geq 0$ and $\Gamma\langle f; g\rangle(A \cap \mathcal{N}) \leq 0$ for any $A \in \mathcal{B}_0$. Then the total variation measure $|\Gamma\langle f; g\rangle|$ is given by

$$\left|\Gamma\langle f; g\rangle\right|(A) = \Gamma\langle f; g\rangle(\mathcal{P} \cap A) - \Gamma\langle f; g\rangle(\mathcal{N} \cap A), \quad A \in \mathcal{B}_0.$$

Therefore, by (4.8),

$$\begin{aligned} \left|\Gamma\langle f; g\rangle\right|(A) &\leq \Gamma\langle f\rangle(\mathcal{P} \cap A)^{(p-1)/p}\Gamma\langle g\rangle(\mathcal{P} \cap A)^{1/p} + \Gamma\langle f\rangle(\mathcal{N} \cap A)^{(p-1)/p}\Gamma\langle g\rangle(\mathcal{N} \cap A)^{1/p} \\ &\leq \bigl(\Gamma\langle f\rangle(\mathcal{P} \cap A) + \Gamma\langle f\rangle(\mathcal{N} \cap A)\bigr)^{(p-1)/p}\bigl(\Gamma\langle g\rangle(\mathcal{P} \cap A) + \Gamma\langle g\rangle(\mathcal{N} \cap A)\bigr)^{1/p} \\ &= \Gamma\langle f\rangle(A)^{(p-1)/p}\Gamma\langle g\rangle(A)^{1/p}, \end{aligned} \tag{4.14}$$

where we used Hölder's inequality in the second inequality.

Now we prove (4.13). First, we consider the case where φ and ψ are given by

$$\varphi = \sum_{k=1}^{N_1} \widetilde{a}_k \mathbb{1}_{A_k}, \quad \psi = \sum_{k=1}^{N_2} \widetilde{b}_k \mathbb{1}_{B_k}, \text{ where } \widetilde{a}_k, \widetilde{b}_k \in [0,\infty) \text{ and } A_k, B_k \in \mathcal{B}_0.$$

Then we can assume that there exist $N \in \mathbb{N}$, $\{a_k\}_{k=1}^N, \{b_k\}_{k=1}^N \subseteq [0,\infty)$ and a disjoint family of measurable sets $\{E_k\}_{k=1}^N \subseteq \mathcal{B}_0$ such that $\varphi = \sum_{k=1}^N a_k \mathbb{1}_{E_k}$ and $\psi = \sum_{k=1}^N b_k \mathbb{1}_{E_k}$. Since $\varphi\psi = \sum_{k=1}^N a_k b_k \mathbb{1}_{E_k}$, combining (4.14) and Hölder's inequality yields

$$\int_X \varphi\psi \, d\left|\Gamma\langle f; g\rangle\right| = \sum_{k=1}^N a_k b_k \left|\Gamma\langle f; g\rangle(E_k)\right|$$

$$\leq \left(\sum_{k=1}^{N} a_k^{p/(p-1)} \Gamma\langle f\rangle(E_k)\right)^{(p-1)/p} \left(\sum_{k=1}^{N} b_k^{p} \Gamma\langle g\rangle(E_k)\right)^{1/p}$$
$$= \left(\int_X \varphi^{p/(p-1)}\, d\Gamma\langle f\rangle\right)^{(p-1)/p} \left(\int_X \psi^{p}\, d\Gamma\langle g\rangle\right)^{1/p}. \tag{4.15}$$

Next, assume that φ and ψ are $[0,\infty]$-valued $\mathcal{B}_0$-measurable functions, and for each $w \in \{\varphi,\psi\}$ let $\{s_{n,w}\}_{n\in\mathbb{N}}$ be a non-decreasing sequence of non-negative $\mathcal{B}_0$-measurable simple functions such that $\lim_{n\to\infty} s_{n,w}(x) = w(x)$ for any $x \in X$. Then by (4.15) we have (4.13) with $s_{n,\varphi}, s_{n,\psi}$ in place of φ, ψ for any $n \in \mathbb{N}$, and letting $n \to \infty$ yields (4.13) by the monotone convergence theorem. □

In the following proposition, we show that integrals of non-negative bounded $\mathcal{B}_0$-measurable functions with respect to p-energy measures satisfying $(\mathrm{GC})_p$ give p-energy forms on (X,m) that satisfy $(\mathrm{GC})_p$.

Proposition 4.9. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies $(\mathrm{GC})_p$. Then for any $\mathcal{B}_0$-measurable function $\varphi\colon X \to [0,\infty)$ with $\|\varphi\|_{\sup} < \infty$, $(\int_X \varphi\, d\Gamma\langle\,\cdot\,\rangle, \mathcal{F})$ is a p-energy form on (X,m) satisfying $(\mathrm{GC})_p$.*

Proof. Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2), and let $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathcal{F}^{n_1}$. Similar to (2.19), by using the triangle inequality for the $\ell^{q_2/p}$-norm and the reverse Minkowski inequality (Proposition 2.8) for the $\ell^{q_1/p}$-norm, we see that for any non-negative $\mathcal{B}_0$-measurable simple function φ on X,

$$\left\| \left(\left(\int_X \varphi\, d\Gamma\langle T_l(\boldsymbol{u})\rangle \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left(\left(\int_X \varphi\, d\Gamma\langle u_k\rangle \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{4.16}$$

We can extend (4.16) to any $\mathcal{B}_0$-measurable function $\varphi\colon X \to [0,\infty]$ by taking a non-decreasing sequence of non-negative $\mathcal{B}_0$-measurable simple functions converging pointwise to φ and applying the monotone convergence theorem, which completes the proof. □

The following Fatou type result is useful.

Proposition 4.10. *Assume that $\mathcal{F} \subseteq L^p(X,m)$. Let $\varphi\colon X \to [0,\infty)$ be $\mathcal{B}_0$-measurable and satisfy $\|\varphi\|_{\sup} < \infty$. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ converges weakly in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ to $u \in \mathcal{F}$, then*

$$\int_X \varphi\, d\Gamma\langle u\rangle \leq \liminf_{n\to\infty} \int_X \varphi\, d\Gamma\langle u_n\rangle. \tag{4.17}$$

Proof. Let $\{u_{n_k}\}_k$ be a subsequence with $\lim_{k\to\infty} \int_X \varphi\, d\Gamma\langle u_{n_k}\rangle = \liminf_{n\to\infty} \int_X \varphi\, d\Gamma\langle u_n\rangle$. By Mazur's lemma (Lemma 3.14), there exist $N(l) \in \mathbb{N}$ and $\{\alpha_{l,k}\}_{k=l}^{N(l)} \subseteq [0,1]$ such that $N(l) > l$, $\sum_{k=l}^{N(l)} \alpha_{l,k} = 1$ and $v_l := \sum_{k=l}^{N(l)} \alpha_{l,k} u_{n_k}$ converges to u in $\mathcal{F}$ as $l \to \infty$. We see from the triangle inequality for $\left(\int_X \varphi\, d\Gamma\langle\,\cdot\,\rangle\right)^{1/p}$ that

$$\left(\int_X \varphi\, d\Gamma\langle v_l\rangle\right)^{1/p} \leq \sum_{k=l}^{N(l)} \alpha_{l,k} \left(\int_X \varphi\, d\Gamma\langle u_{n_k}\rangle\right)^{1/p},$$

which implies (4.17) by letting $l \to \infty$. □

4.2 Extensions of p-energy measures

Throughout this subsection, we fix a linear subspace $\mathcal{D}$ of $\mathcal{F}$ and assume that $\mathcal{B}_0$ is a σ-algebra in X and that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ is a family of p-energy measures on $(X,\mathcal{B}_0)$ dominated by $(\mathcal{E},\mathcal{D})$. In the following proposition, we extend $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ to $f\in\mathcal{D}^{\#}$, where $\mathcal{D}^{\#}$ is a linear subspace of $\mathcal{F}$ defined as

$$\mathcal{D}^{\#} := \Bigl\{ u\in\mathcal{F} \Bigm| \lim_{n\to\infty}\mathcal{E}(u-u_n)=0 \text{ for some } \{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{D} \Bigr\}; \tag{4.18}$$

note that, if $\mathcal{F}\subseteq L^p(X,m)$ and $\mathcal{F}$ is equipped with the norm $\|\cdot\|_{\mathcal{E},1}$, then $\overline{\mathcal{D}}^{\mathcal{F}}\subseteq\mathcal{D}^{\#}$, where the inclusion can be strict in general.

Proposition 4.11. *For any $u\in\mathcal{D}^{\#}$, there exists a unique measure $\Gamma\langle u\rangle$ on $(X,\mathcal{B}_0)$ such that for any $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{D}$ with $\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$ and any $\mathcal{B}_0$-measurable function $\varphi\colon X\to[0,\infty)$ with $\|\varphi\|_{\sup}<\infty$,*

$$\int_X \varphi\,d\Gamma\langle u\rangle = \lim_{n\to\infty}\int_X\varphi\,d\Gamma\langle u_n\rangle, \tag{4.19}$$

and $\Gamma\langle u\rangle$ further satisfies $\Gamma\langle u\rangle(X)\le\mathcal{E}(u)$. Moreover, for each such φ, $(\int_X\varphi\,d\Gamma\langle\,\cdot\,\rangle,\mathcal{D}^{\#})$ is a p-energy form on (X,m).

Proof. By (EM2)_p and the monotone convergence theorem, for any $\mathcal{B}_0$-measurable function $\varphi\colon X\to[0,\infty]$ and any $u,v\in\mathcal{D}$,

$$\left(\int_X\varphi\,d\Gamma\langle u+v\rangle\right)^{1/p} \le \left(\int_X\varphi\,d\Gamma\langle u\rangle\right)^{1/p} + \left(\int_X\varphi\,d\Gamma\langle v\rangle\right)^{1/p}. \tag{4.20}$$

In the rest of this proof, let $\varphi\colon X\to[0,\infty)$ be $\mathcal{B}_0$-measurable and satisfy $\|\varphi\|_{\sup}<\infty$. Let $u\in\mathcal{D}^{\#}$ and $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{D}$ satisfy $\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$. By (4.20), $\bigl\{\int_X\varphi\,d\Gamma\langle u_n\rangle\bigr\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $[0,\infty)$ and $\lim_{n\to\infty}\int_X\varphi\,d\Gamma\langle u_n\rangle =: I_u(\varphi)$ is independent of the choice of $\{u_n\}_n$. In addition, we have that

$$\left|\left(\int_X\varphi\,d\Gamma\langle u_n\rangle\right)^{1/p} - I_u(\varphi)^{1/p}\right| \le \|\varphi\|_{\sup}^{1/p}\,\mathcal{E}(u_n-u)^{1/p}, \tag{4.21}$$

that $0\le I_u(\varphi)\le\|\varphi\|_{\sup}\,\mathcal{E}(u)$ and that I_n is linear in the sense that $I_u\bigl(\sum_{k=1}^N a_k\varphi_k\bigr)=\sum_{k=1}^N a_k I_u(\varphi_k)$ for any $N\in\mathbb{N}$, $(a_k)_{k=1}^N\subseteq[0,\infty)$ and $\mathcal{B}_0$-measurable functions $\varphi_k\colon X\to[0,\infty)$ with $\|\varphi_k\|_{\sup}<\infty$, $k\in\{1,\dots,N\}$. Now we define $\Gamma\langle u\rangle(A):=I_u(\mathbb{1}_A)\in[0,\infty)$ for $A\in\mathcal{B}_0$, and show that $\Gamma\langle u\rangle$ is a finite measure on $(X,\mathcal{B}_0)$. Clearly, $\Gamma\langle u\rangle$ is finitely additive and $\Gamma\langle u\rangle(X)\le\mathcal{E}(u)<\infty$. Let us show the countable additivity of $\Gamma\langle u\rangle$. By (4.21), for any $\varepsilon>0$ there exists $N_0\in\mathbb{N}$ such that $\sup_{A\in\mathcal{B}_0}\bigl|\Gamma\langle u\rangle(A)^{1/p}-\Gamma\langle u_n\rangle(A)^{1/p}\bigr|<\varepsilon$ for any $n\ge N_0$. Let $\{A_k\}_{k\in\mathbb{N}}\subseteq\mathcal{B}_0$ be a sequence of disjoint measurable sets, and set $B_N:=\bigcup_{k=N+1}^\infty A_k$ for each $N\in\mathbb{N}$. Then we see that for any $N\in\mathbb{N}$ and any $n\ge N_0$,

$$\left|\Gamma\langle u\rangle\left(\bigcup_{k\in\mathbb{N}}A_k\right)-\sum_{k=1}^N\Gamma\langle u\rangle(A_k)\right|^{1/p} = \Gamma\langle u\rangle(B_N)^{1/p} \le \varepsilon+\Gamma\langle u_n\rangle(B_N)^{1/p},$$

whence $\lim_{N\to\infty}\left|\Gamma\langle u\rangle\left(\bigcup_{k\in\mathbb{N}}A_k\right)-\sum_{k=1}^{N}\Gamma\langle u\rangle(A_k)\right|=0$, proving the desired countable additivity.

Note that $I_{u+v}(\varphi)^{1/p}\leq I_u(\varphi)^{1/p}+I_v(\varphi)^{1/p}$ for any $u,v\in\mathcal{D}^{\#}$ by (4.20) and the definition of $I_{\bullet}(\varphi)$. This together with the monotone convergence theorem implies the triangle inequality for $\left(\int_X\varphi\,d\Gamma\langle\,\cdot\,\rangle\right)^{1/p}$ on $\mathcal{D}^{\#}$; in particular, $(\int_X\varphi\,d\Gamma\langle\,\cdot\,\rangle,\mathcal{D}^{\#})$ is a p-energy form on (X,m). Next we show (4.19). Let $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{D}$ be a sequence satisfying $\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$. By the triangle inequality for $(\int_X\varphi\,d\Gamma\langle\,\cdot\,\rangle,\mathcal{D}^{\#})$,

$$\left|\left(\int_X\varphi\,d\Gamma\langle u\rangle\right)^{1/p}-\left(\int_X\varphi\,d\Gamma\langle u_n\rangle\right)^{1/p}\right|\leq\left(\int_X\varphi\,d\Gamma\langle u-u_n\rangle\right)^{1/p}\leq\|\varphi\|_{\sup}^{1/p}\,\mathcal{E}(u-u_n)^{1/p},$$

which together with (4.21) implies (4.19); indeed,

$$\begin{aligned}&\left|I_u(\varphi)^{1/p}-\left(\int_X\varphi\,d\Gamma\langle u\rangle\right)^{1/p}\right|\\&\leq\left|I_u(\varphi)^{1/p}-\left(\int_X\varphi\,d\Gamma\langle u_n\rangle\right)^{1/p}\right|+\left|\left(\int_X\varphi\,d\Gamma\langle u_n\rangle\right)^{1/p}-\left(\int_X\varphi\,d\Gamma\langle u\rangle\right)^{1/p}\right|\\&\leq 2\,\|\varphi\|_{\sup}^{1/p}\,\mathcal{E}(u-u_n)^{1/p}\xrightarrow[n\to\infty]{}0.\end{aligned}$$

□

If, in addition, $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies $(\mathrm{Cla})_p$, then we can easily see that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}^{\#}}$ also satisfies $(\mathrm{Cla})_p$. We record this fact in the following proposition.

Proposition 4.12. *If $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies $(\mathrm{Cla})_p$, then so does $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}^{\#}}$.*

Proof. This is immediate from (4.19). □

If $\mathcal{F}\subseteq L^p(X,m)$ and $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space, then $\overline{\mathcal{D}}^{\mathcal{F}}\subseteq\mathcal{D}^{\#}$ as remarked after (4.18), and $(\mathrm{GC})_p$ for $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ also extends to $\{\Gamma\langle f\rangle\}_{f\in\overline{\mathcal{D}}^{\mathcal{F}}}$ as follows.

Proposition 4.13. *Assume that $\mathcal{F}\subseteq L^p(X,m)$, that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space, and that both $(\mathcal{E},\mathcal{D})$ and $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfy $(\mathrm{GC})_p$. Then for any $\mathcal{B}_0$-measurable function $\varphi\colon X\to[0,\infty)$ with $\|\varphi\|_{\sup}<\infty$, $\left(\int_X\varphi\,d\Gamma\langle\,\cdot\,\rangle,\overline{\mathcal{D}}^{\mathcal{F}}\right)$ is a p-energy form on (X,m) satisfying $(\mathrm{GC})_p$.*

Proof. Since $(\mathcal{E},\mathcal{D})$ satisfies $(\mathrm{Cla})_p$ by $(\mathrm{GC})_p$ for $(\mathcal{E},\mathcal{D})$ and Proposition 2.3-(e),(f), so does $\left(\mathcal{E},\overline{\mathcal{D}}^{\mathcal{F}}\right)$, which together with the completeness of $\left(\overline{\mathcal{D}}^{\mathcal{F}},\|\cdot\|_{\mathcal{E},1}\right)$ guarantees that Lemma 3.17 is applicable to $\left(\mathcal{E},\overline{\mathcal{D}}^{\mathcal{F}}\right)$.

Let $n_1,n_2\in\mathbb{N}$, $q_1\in(0,p]$, $q_2\in[p,\infty]$ and $T=(T_1,\ldots,T_{n_2})\colon\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ satisfy (2.2). Let $\boldsymbol{u}=(u_1,\ldots,u_{n_1})\in\left(\overline{\mathcal{D}}^{\mathcal{F}}\right)^{n_1}$. For each $k\in\{1,\ldots,n_1\}$, choose $\{u_{k,n}\}_{n\in\mathbb{N}}\subseteq\mathcal{D}$ so that $\lim_{n\to\infty}\|u_k-u_{k,n}\|_{\mathcal{F},1}=0$, set $\boldsymbol{u}_n:=(u_{1,n},\ldots,u_{n_1,n})$, and let $l\in\{1,\ldots,n_2\}$. Then by $(\mathrm{GC})_p$ for $(\mathcal{E},\mathcal{D})$ and (2.2) we have $\{T_l(\boldsymbol{u}_n)\}_{n\in\mathbb{N}}\subseteq\mathcal{D}$, $\sup_{n\in\mathbb{N}}\mathcal{E}(T_l(\boldsymbol{u}_n))<\infty$ and $\lim_{n\to\infty}\|T_l(\boldsymbol{u}_n)-T_l(\boldsymbol{u})\|_{L^p(X,m)}=0$, and therefore Lemma 3.17 applied to $\left(\mathcal{E},\overline{\mathcal{D}}^{\mathcal{F}}\right)$

implies that $T_l(\boldsymbol{u}) \in \overline{\mathcal{D}}^{\mathcal{F}}$ and that $\{T_l(\boldsymbol{u}_n)\}_{n\in\mathbb{N}}$ converges weakly in $\bigl(\overline{\mathcal{D}}^{\mathcal{F}}, \|\cdot\|_{\mathcal{E},1}\bigr)$ to $T_l(\boldsymbol{u})$. If $q_2 < \infty$, then we see from Proposition 4.10, which is applicable to $\{\Gamma\langle f\rangle\}_{f\in\overline{\mathcal{D}}^{\mathcal{F}}}$ by Proposition 4.11, and (GC)$_p$ for $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ that

$$\begin{aligned}
\left\| \left(\left(\int_X \varphi \, d\Gamma\langle T_l(\boldsymbol{u})\rangle \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} &\overset{(4.17)}{\leq} \left(\sum_{l=1}^{n_2} \liminf_{n\to\infty} \left(\int_X \varphi \, d\Gamma\langle T_l(\boldsymbol{u}_n)\rangle \right)^{q_2/p} \right)^{1/q_2} \\
&\leq \liminf_{n\to\infty} \left(\sum_{l=1}^{n_2} \left(\int_X \varphi \, d\Gamma\langle T_l(\boldsymbol{u}_n)\rangle \right)^{q_2/p} \right)^{1/q_2} \\
&\overset{(\mathrm{GC})_p}{\leq} \liminf_{n\to\infty} \left(\sum_{k=1}^{n_1} \left(\int_X \varphi \, d\Gamma\langle u_{k,n}\rangle \right)^{q_1/p} \right)^{1/q_1} \\
&\overset{(4.19)}{=} \left\| \left(\left(\int_X \varphi \, d\Gamma\langle u_k\rangle \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \qquad (4.22)
\end{aligned}$$

The case of $q_2 = \infty$ is similar, so $\bigl(\int_X \varphi \, d\Gamma\langle\,\cdot\,\rangle, \overline{\mathcal{D}}^{\mathcal{F}}\bigr)$ satisfies (GC)$_p$. □

4.3 Chain rule and strong locality of p-energy measures

In this subsection, we see that strong local properties for p-energy measures hold if p-energy measures satisfy a chain rule (see Definition 4.14 below). In this subsection, we assume that (X, m) satisfies (3.26) and (3.27), that $\mathcal{D}$ is a linear subspace of $\mathcal{F} \cap C(X)$, that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ is a family of p-energy measures on $(X, \mathcal{B}(X))$ dominated by $(\mathcal{E}, \mathcal{D})$, and that $\mathcal{F} \subseteq L^p(X, m)$, and we equip $\mathcal{F}$ with the norm $\|\cdot\|_{\mathcal{E},1}$.

Definition 4.14 (Chain rules for p-energy measures)**.** (1) We say that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies the chain rule (CL1) if and only if for any $u \in \mathcal{D}$ and any $\Phi \in C^1(\mathbb{R})$, we have $\Phi(u) \in \mathcal{D}$ and

$$d\Gamma\langle\Phi(u)\rangle = |\Phi'(u)|^p \, d\Gamma\langle u\rangle. \qquad (4.23)$$

(2) Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies (Cla)$_p$. We say that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies the chain rule (CL2) if and only if for any $n \in \mathbb{N}$, $u \in \mathcal{D}$, $\boldsymbol{v} = (v_1, \dots, v_n) \in \mathcal{D}^n$, $\Phi \in C^1(\mathbb{R})$ and $\Psi \in C^1(\mathbb{R}^n)$, we have $\Phi(u), \Psi(\boldsymbol{v}) \in \mathcal{D}$ and

$$d\Gamma\langle\Phi(u); \Psi(\boldsymbol{v})\rangle = \sum_{k=1}^{n} \operatorname{sgn}\bigl(\Phi'(u)\bigr) \, |\Phi'(u)|^{p-1} \, \partial_k\Psi(\boldsymbol{v}) \, d\Gamma\langle u; v_k\rangle. \qquad (4.24)$$

Theorem 4.15. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies* (Cla)$_p$ *and* (CL2)*.*

(a) *$\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies* (CL1)*.*

(b) (Leibniz rule) *For any $u, v, w \in \mathcal{D}$, we have $vw \in \mathcal{D}$ and*

$$d\Gamma\langle u; vw\rangle = v \, d\Gamma\langle u; w\rangle + w \, d\Gamma\langle u; v\rangle. \qquad (4.25)$$

(c) ($(\mathrm{GC})_p$ for $\mathbb{R}$-valued T) *Let $n \in \mathbb{N}$, $q \in (0,p]$ and $T\colon \mathbb{R}^n \to \mathbb{R}$ satisfy $T(0)=0$ and $|T(x)-T(y)| \leq \|x-y\|_{\ell^q}$ for any $x,y \in \mathbb{R}^n$, and let $\varphi\colon X \to [0,\infty]$ be Borel measurable.*

(1) *If $T \in C^1(\mathbb{R}^n)$, then for any $\boldsymbol{u} = (u_1,\dots,u_n) \in \mathcal{D}^n$,* ($T(\boldsymbol{u}) \in \mathcal{D}$ by (CL2), and)

$$\left(\int_X \varphi\, d\Gamma\langle T(\boldsymbol{u})\rangle\right)^{1/p} \leq \left\| \left(\int_X \varphi\, d\Gamma\langle u_k\rangle\right)^{1/p} \right\|_{\ell^q}. \tag{4.26}$$

(2) *If $\Gamma\langle u\rangle(X) = \mathcal{E}(u)$ for any $u \in \mathcal{D}$, and if $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space, then for any $\boldsymbol{u} = (u_1,\dots,u_n) \in \left(\overline{\mathcal{D}}^{\mathcal{F}}\right)^n$, $T(\boldsymbol{u}) \in \overline{\mathcal{D}}^{\mathcal{F}}$ and (4.26) holds.*

(d) *Let $\psi\colon \mathbb{R} \to \mathbb{R}$ satisfy $\psi(0)=0$ and $0 \leq \psi(t)-\psi(s) \leq t-s$ for any $s,t \in \mathbb{R}$ with $s \leq t$, and let $\varphi\colon X \to [0,\infty]$ be Borel measurable.*

(1) *If $\psi \in C^1(\mathbb{R})$, then for any $u,v \in \mathcal{D}$,* ($\psi(u-v) \in \mathcal{D}$ by (CL1) from (a), and)

$$\int_X \varphi\, d\Gamma\langle u-\psi(u-v)\rangle + \int_X \varphi\, d\Gamma\langle v+\psi(u-v)\rangle \leq \int_X \varphi\, d\Gamma\langle u\rangle + \int_X \varphi\, d\Gamma\langle v\rangle. \tag{4.27}$$

(2) *If $\Gamma\langle u\rangle(X) = \mathcal{E}(u)$ for any $u \in \mathcal{D}$, and if $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space, then for any $u,v \in \overline{\mathcal{D}}^{\mathcal{F}}$, $\psi(u-v) \in \overline{\mathcal{D}}^{\mathcal{F}}$ and (4.27) holds.*

Proof. (a),(b): These are immediate from (CL2).

(c)-(1): Assume $T \in C^1(\mathbb{R}^n)$, and let $\boldsymbol{u} = (u_1,\dots,u_n) \in \mathcal{D}^n$. It suffices to prove that

$$\Gamma\langle T(\boldsymbol{u})\rangle(A)^{1/p} \leq \left\| \left(\Gamma\langle u_k\rangle(A)^{1/p}\right)_{k=1}^n \right\|_{\ell^q} \quad \text{for any } A \in \mathcal{B}(X); \tag{4.28}$$

indeed, it is routine to extend (4.28) to (4.26) (see the proof of Proposition 4.9). To show (4.28), we first construct a good μ-version of $\Upsilon\langle v_1;v_2\rangle := \frac{d\Gamma\langle v_1;v_2\rangle}{d\mu}$ for each $v_1,v_2 \in \{T(\boldsymbol{u}),u_1,\dots,u_n\}$, where $\mu := \Gamma\langle T(\boldsymbol{u})\rangle + \sum_{k=1}^n \Gamma\langle u_k\rangle$. Let $\{A_k\}_{k\in\mathbb{N}}$ be a countable open base for the topology of X. Set $A_k^0 := X \setminus A_k$ and $A_k^1 := A_k$ for each $k \in \mathbb{N}$, and define

$$\mathcal{A}_k := \left\{ \bigcup_{\alpha\in\mathcal{I}} A_k^\alpha \,\middle|\, \mathcal{I} \subseteq \{0,1\}^k \right\}, \quad k \in \mathbb{N}, \tag{4.29}$$

where $A_k^\alpha := \bigcap_{i=1}^k A_k^{\alpha_i}$ for $\alpha = (\alpha_i)_{i=1}^k \in \{0,1\}^k$. Note that $\bigcup_{\alpha\in\mathcal{I}} A_k^\alpha = \emptyset$ if $\mathcal{I} = \emptyset$. Then $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ is a non-decreasing sequence of σ-algebras on X with $\bigcup_{k\in\mathbb{N}} \mathcal{A}_k$ generating $\mathcal{B}(X)$. Note that $\bigcup_{\alpha\in\{0,1\}^k} A_k^\alpha = X$ and that $A_k^\alpha \cap A_k^\beta = \emptyset$ for $\alpha,\beta \in \{0,1\}^k$ with $\alpha \neq \beta$. For $v_1,v_2 \in \{T(\boldsymbol{u}),u_1,\dots,u_n\}$, $k \in \mathbb{N}$, $\alpha \in \{0,1\}^k$, define $\Upsilon_k\langle v_1;v_2\rangle\colon X \to [0,\infty)$ by, for $x \in A_k^\alpha$,

$$\Upsilon_k\langle v_1;v_2\rangle(x) := \mu(A_k^\alpha)^{-1}\Gamma\langle v_1;v_2\rangle(A_k^\alpha). \tag{4.30}$$

Then $\mathbb{E}_\mu[\Upsilon\langle v_1;v_2\rangle \mid \mathcal{A}_k] = \Upsilon_k\langle v_1;v_2\rangle$ μ-a.e. on X and hence $\lim_{k\to\infty} \Upsilon_k\langle v_1;v_2\rangle = \Upsilon\langle v_1;v_2\rangle$ μ-a.e. on X by the martingale convergence theorem (see, e.g., [Dud, Theorem 10.5.1]) and the fact that $\bigcup_{k\in\mathbb{N}}\mathcal{A}_k$ generates $\mathcal{B}(X)$. From this convergence together with (4.30) and (4.8), we obtain

$$\left|\frac{d\Gamma\langle v_1;v_2\rangle}{d\mu}\right| \le \left(\frac{d\Gamma\langle v_1\rangle}{d\mu}\right)^{\frac{p-1}{p}}\left(\frac{d\Gamma\langle v_2\rangle}{d\mu}\right)^{\frac{1}{p}} \quad \mu\text{-a.e. on } X. \tag{4.31}$$

Now we prove (4.28) on the basis of (CL2) and (4.31). Recalling that we have assumed $T\in C^1(\mathbb{R}^n)$, we see from the assumption on T that for any $x,y=(y_1,\dots,y_n)\in\mathbb{R}^n$,

$$\left|\sum_{k=1}^{n}\partial_k T(x)y_k\right| = \lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\left|T(x)-T(x+\varepsilon y)\right| \le \|y\|_{\ell^q}. \tag{4.32}$$

Then for any $A\in\mathcal{B}(X)$, from (CL2), (4.31), (4.32), Hölder's inequality, and the triangle inequality for the $L^{p/q}(A,\mu|_A)$-norm, we obtain

$$\begin{aligned}
\Gamma\langle T(\boldsymbol{u})\rangle(A) &\overset{\text{(CL2)}}{=} \int_A \sum_{k=1}^{n}\partial_k T(\boldsymbol{u}(x))\frac{\Gamma\langle T(\boldsymbol{u});u_k\rangle}{d\mu}(x)\,\mu(dx)\\
&\overset{(4.31)}{\le} \int_A \sum_{k=1}^{n}\left|\partial_k T(\boldsymbol{u}(x))\right|\left(\frac{d\Gamma\langle T(\boldsymbol{u})\rangle}{d\mu}(x)\right)^{\frac{p-1}{p}}\left(\frac{d\Gamma\langle u_k\rangle}{d\mu}(x)\right)^{\frac{1}{p}}\mu(dx)\\
&\overset{(4.32)}{\le} \int_A\left(\frac{d\Gamma\langle T(\boldsymbol{u})\rangle}{d\mu}(x)\right)^{\frac{p-1}{p}}\left\|\left(\operatorname{sgn}(\partial_k T(\boldsymbol{u}(x)))\left(\frac{d\Gamma\langle u_k\rangle}{d\mu}(x)\right)^{\frac{1}{p}}\right)_{k=1}^{n}\right\|_{\ell^q}\mu(dx)\\
&\le \left(\int_A \frac{d\Gamma\langle T(\boldsymbol{u})\rangle}{d\mu}\,d\mu\right)^{\frac{p-1}{p}}\left(\int_A\left[\sum_{k=1}^{n}\left(\frac{d\Gamma\langle u_k\rangle}{d\mu}\right)^{\frac{q}{p}}\right]^{\frac{p}{q}}d\mu\right)^{\frac{1}{p}}\\
&\le \Gamma\langle T(\boldsymbol{u})\rangle(A)^{\frac{p-1}{p}}\left[\sum_{k=1}^{n}\left(\int_A\frac{d\Gamma\langle u_k\rangle}{d\mu}\,d\mu\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}\\
&= \Gamma\langle T(\boldsymbol{u})\rangle(A)^{\frac{p-1}{p}}\left\|\left(\Gamma\langle u_k\rangle(A)^{1/p}\right)_{k=1}^{n}\right\|_{\ell^q},
\end{aligned}$$

proving (4.28) and thereby (c)-(1).

(c)-(2): Recall that $\{\Gamma\langle f\rangle\}_{f\in\overline{\mathcal{D}}^{\mathcal{F}}}$ is uniquely defined through (4.19) by Proposition 4.11, and note that the equality $\Gamma\langle u\rangle(X)=\mathcal{E}(u)$ extends from $u\in\mathcal{D}$ to $u\in\overline{\mathcal{D}}^{\mathcal{F}}$ by (4.19) and the triangle inequality for $\mathcal{E}^{1/p}$, and hence that $\{\Gamma\langle f\rangle\}_{f\in\overline{\mathcal{D}}^{\mathcal{F}}}$ and $\left(\mathcal{E},\overline{\mathcal{D}}^{\mathcal{F}}\right)$ satisfy $(\mathrm{Cla})_p$ by Proposition 4.12. In particular, in view of the completeness of $\left(\overline{\mathcal{D}}^{\mathcal{F}},\|\cdot\|_{\mathcal{E},1}\right)$, Lemma 3.17 is applicable to $\left(\mathcal{E},\overline{\mathcal{D}}^{\mathcal{F}}\right)$. Now, to see $T(\boldsymbol{u})\in\overline{\mathcal{D}}^{\mathcal{F}}$ and (4.28) for $\boldsymbol{u}=(u_1,\dots,u_n)\in\mathcal{D}^n$, define $j\colon\mathbb{R}^n\to\mathbb{R}$ by $j(x):=\exp\left(-\frac{1}{1-\|x\|^2}\right)$ for $\|x\|\le 1$ and $j(x):=0$ for $\|x\|>1$, and set $j_l(x):=l^n j(lx)$ for each $l\in\mathbb{N}$ (see [Kuw24, p. 10]). We further define $T_l(x):=\int_{\mathbb{R}^n}(j_l(x-y)-j_l(y))T(y)\,dy=\int_{\mathbb{R}^n}j_l(y)(T(x-y)-T(y))\,dy$ so that $T_l\in C^\infty(\mathbb{R}^n)$, $T_l(0)=0$ and $\lim_{l\to\infty}T_l(x)=T(x)$ for any $x\in\mathbb{R}^n$. Moreover, for any $x,y\in\mathbb{R}^n$,

$$\left|T_l(x)-T_l(y)\right| = \left|\int_{\mathbb{R}^n} j_l(z)(T(x-z)-T(y-z))\,dz\right| \le \|x-y\|_{\ell^q}. \tag{4.33}$$

Therefore, letting $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathcal{D}^n$, by (c)-(1) we have (4.28) with T_l in place of T, which together with $\mathcal{E}(T_l(\boldsymbol{u})) = \Gamma\langle T_l(\boldsymbol{u})\rangle(X)$ implies that $\sup_{l\in\mathbb{N}} \mathcal{E}(T_l(\boldsymbol{u})) < \infty$. Since $\{T_l(\boldsymbol{u})\}_{l\in\mathbb{N}}$ converges in $L^p(X,m)$ to $T(\boldsymbol{u})$ as $l \to \infty$ by $T_l(0) = 0$, (4.33) and the dominated convergence theorem, we conclude from Lemma 3.17 that $T(\boldsymbol{u}) \in \overline{\mathcal{D}}^{\mathcal{F}}$ and that $\{T_l(\boldsymbol{u})\}_{l\in\mathbb{N}}$ converges weakly in $\bigl(\overline{\mathcal{D}}^{\mathcal{F}}, \|\cdot\|_{\mathcal{E},1}\bigr)$ to $T(\boldsymbol{u})$. Now we obtain (4.28) by combining Proposition 4.10 applied to $\{\Gamma\langle f\rangle\}_{f\in\overline{\mathcal{D}}^{\mathcal{F}}}$ and (4.28) with T_l in place of T.

Lastly, let $\boldsymbol{u} = (u_1, \dots, u_n) \in \bigl(\overline{\mathcal{D}}^{\mathcal{F}}\bigr)^n$, and choose $\bigl\{\boldsymbol{u}^{(l)} = \bigl(u_1^{(l)}, \dots, u_n^{(l)}\bigr)\bigr\}_{l\in\mathbb{N}} \subseteq \mathcal{D}^n$ so that $\bigl\{u_k^{(l)}\bigr\}_{l\in\mathbb{N}}$ converges in norm in $\mathcal{F}$ to u_k for any $k \in \{1, \dots, n\}$. Then by the result of the previous paragraph we have $\{T(\boldsymbol{u}^{(l)})\}_{l\in\mathbb{N}} \subseteq \overline{\mathcal{D}}^{\mathcal{F}}$ and (4.28) with $\boldsymbol{u}^{(l)}$ in place of $\boldsymbol{u}$, which together with $\mathcal{E}(T(\boldsymbol{u}^{(l)})) = \Gamma\langle T(\boldsymbol{u}^{(l)})\rangle(X)$ and the assumption on T implies that $\{T(\boldsymbol{u}^{(l)})\}_{l\in\mathbb{N}}$ is a bounded sequence in $\bigl(\overline{\mathcal{D}}^{\mathcal{F}}, \|\cdot\|_{\mathcal{E},1}\bigr)$ converging in norm in $L^p(X,m)$ to $T(\boldsymbol{u})$. Thus $T(\boldsymbol{u}) \in \overline{\mathcal{D}}^{\mathcal{F}}$ and $\{T(\boldsymbol{u}^{(l)})\}_{l\in\mathbb{N}}$ converges weakly in $\bigl(\overline{\mathcal{D}}^{\mathcal{F}}, \|\cdot\|_{\mathcal{E},1}\bigr)$ to $T(\boldsymbol{u})$ by Lemma 3.17, and hence combining Proposition 4.10 applied to $\{\Gamma\langle f\rangle\}_{f\in\overline{\mathcal{D}}^{\mathcal{F}}}$ and (4.28) with $\boldsymbol{u}^{(l)}$ in place of $\boldsymbol{u}$ yields (4.28) for $\boldsymbol{u} = (u_1, \dots, u_n) \in \bigl(\overline{\mathcal{D}}^{\mathcal{F}}\bigr)^n$, proving (c)-(2).

(d)-(1): Assume $\psi \in C^1(\mathbb{R})$, and let $u, v \in \mathcal{D}$. Again, in view of the proof of Proposition 4.9 it suffices to show that

$$\Gamma\langle u-\psi(u-v)\rangle(A)+\Gamma\langle v+\psi(u-v)\rangle(A) \le \Gamma\langle u\rangle(A)+\Gamma\langle v\rangle(A) \quad \text{for any } A \in \mathcal{B}(X). \tag{4.34}$$

Indeed, since $\int_X \varphi\, d\Gamma\langle f; \cdot\rangle$ is linear for any $f \in \mathcal{D}$ by Theorem 4.6 if $\|\varphi\|_{\sup} < \infty$, we see from (CL2), Proposition 4.8, $0 \le \psi' \le 1$ on $\mathbb{R}$, and Hölder's inequality that for any $A \in \mathcal{B}(X)$,

$$\begin{aligned}
&\Gamma\langle u-\psi(u-v)\rangle(A)+\Gamma\langle v+\psi(u-v)\rangle(A)\\
&\overset{\text{(CL2)}}{=} \Gamma\langle u-\psi(u-v); u\rangle(A) - \int_A \psi'(u-v)\, d\Gamma\langle u-\psi(u-v); u-v\rangle\\
&\qquad + \Gamma\langle v+\psi(u-v); v\rangle(A) + \int_A \psi'(u-v)\, d\Gamma\langle v+\psi(u-v); u-v\rangle\\
&= \int_A (1-\psi'(u-v))\, d\Gamma\langle u-\psi(u-v); u\rangle + \int_A \psi'(u-v)\, d\Gamma\langle u-\psi(u-v); v\rangle\\
&\qquad + \int_A (1-\psi'(u-v))\, d\Gamma\langle v+\psi(u-v); v\rangle + \int_A \psi'(u-v)\, d\Gamma\langle v+\psi(u-v); u\rangle\\
&\overset{(4.13)}{\le} \biggl(\int_A (1-\psi'(u-v))\, d\Gamma\langle u-\psi(u-v)\rangle\biggr)^{\frac{p-1}{p}} \biggl(\int_A (1-\psi'(u-v))\, d\Gamma\langle u\rangle\biggr)^{\frac{1}{p}}\\
&\qquad + \biggl(\int_A \psi'(u-v)\, d\Gamma\langle u-\psi(u-v)\rangle\biggr)^{\frac{p-1}{p}} \biggl(\int_A \psi'(u-v)\, d\Gamma\langle v\rangle\biggr)^{\frac{1}{p}}\\
&\qquad + \biggl(\int_A (1-\psi'(u-v))\, d\Gamma\langle v+\psi(u-v)\rangle\biggr)^{\frac{p-1}{p}} \biggl(\int_A (1-\psi'(u-v))\, d\Gamma\langle v\rangle\biggr)^{\frac{1}{p}}\\
&\qquad + \biggl(\int_A \psi'(u-v)\, d\Gamma\langle v+\psi(u-v)\rangle\biggr)^{\frac{p-1}{p}} \biggl(\int_A \psi'(u-v)\, d\Gamma\langle u\rangle\biggr)^{\frac{1}{p}}
\end{aligned}$$

$$\overset{\text{Hölder}}{\leq} \bigl(\Gamma\langle u-\psi(u-v)\rangle(A)+\Gamma\langle v+\psi(u-v)\rangle(A)\bigr)^{\frac{p-1}{p}}\bigl(\Gamma\langle u\rangle(A)+\Gamma\langle v\rangle(A)\bigr)^{\frac{1}{p}},$$

proving (4.34) and thereby (d)-(1).

(d)-(2): This is proved by following closely the above proof of (c)-(2) on the basis of (d)-(1) and arguing as in (4.22) upon applying Proposition 4.10 to conclude (4.34). □

We also have the following representation formula (see also [Cap03, Theorem 4.1]).

Proposition 4.16 (Representation formula)**.** *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies $(\mathrm{Cla})_p$ and (CL2) and that $\Gamma\langle f\rangle(X)=\mathcal{E}(f)$ for any $f\in\mathcal{D}$. Then for any $u,\varphi\in\mathcal{D}$,*

$$\int_X \varphi\,d\Gamma\langle u\rangle=\mathcal{E}(u;u\varphi)-\left(\frac{p-1}{p}\right)^{p-1}\mathcal{E}\bigl(|u|^{\frac{p}{p-1}};\varphi\bigr). \tag{4.35}$$

Proof. Note that $(\mathcal{E},\mathcal{D})$ satisfies $(\mathrm{Cla})_p$ by $\Gamma\langle f\rangle(X)=\mathcal{E}(f)$. Define $\Phi\in C^1(\mathbb{R})$ by $\Phi(x):=|x|^{\frac{p}{p-1}}$. Note that $\Phi'(x)=\frac{p}{p-1}\operatorname{sgn}(x)\,|x|^{\frac{1}{p-1}}$. By (4.25) and (CL2), we see that

$$\begin{aligned}
&\mathcal{E}(u;u\varphi)-\left(\frac{p-1}{p}\right)^{p-1}\mathcal{E}(\Phi(u);\varphi)\\
&=\int_X u\,d\Gamma\langle u;\varphi\rangle+\int_X\varphi\,d\Gamma\langle u\rangle-\left(\frac{p-1}{p}\right)^{p-1}\int_X \operatorname{sgn}\bigl(\Phi'(u)\bigr)\,|\Phi'(u)|^{p-1}\,d\Gamma\langle u;\varphi\rangle\\
&=\int_X u\,d\Gamma\langle u;\varphi\rangle+\int_X\varphi\,d\Gamma\langle u\rangle-\left(\frac{p-1}{p}\right)^{p-1}\left(\frac{p}{p-1}\right)^{p-1}\int_X \operatorname{sgn}(u)\,|u|\,d\Gamma\langle u;\varphi\rangle\\
&=\int_X\varphi\,d\Gamma\langle u\rangle,
\end{aligned} \tag{4.36}$$

proving (4.35). □

We have the following theorem as a consequence of (CL1).

Theorem 4.17 (Image density property)**.** *Assume that $(\mathcal{E},\mathcal{D})$ satisfies (2.4) and $(\mathrm{Cla})_p$, that $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ is a Banach space, and that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies (CL1). Then for any $u\in\mathcal{D}$, the Borel measure $\Gamma\langle u\rangle\circ u^{-1}$ on $\mathbb{R}$ defined by $(\Gamma\langle u\rangle\circ u^{-1})(A):=\Gamma\langle u\rangle(u^{-1}(A))$, $A\in\mathcal{B}(\mathbb{R})$, is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. This is proved, on the basis of (4.23), in exactly the same way as [Shi24, Proposition 7.6], which is a simple adaptation of [CF, Theorem 4.3.8], but we present the details because in [Shi24] the underlying topological space X is assumed to be a generalized Sierpiński carpet. It suffices to prove that $(\Gamma\langle u\rangle\circ u^{-1})(F)=0$ for any $u\in\mathcal{D}$ and any compact subset F of $\mathbb{R}$ such that $\mathscr{L}^1(F)=0$, where $\mathscr{L}^1$ denotes the 1-dimensional Lebesgue measure on $\mathbb{R}$. Let $\{\varphi_n\}_{n\in\mathbb{N}}\subseteq C_c(\mathbb{R})$ satisfy $|\varphi_n|\leq 1$, $\lim_{n\to\infty}\varphi_n(x)=\mathbb{1}_F(x)$ for any $x\in\mathbb{R}$ and

$$\int_0^\infty \varphi_n(t)\,dt=\int_{-\infty}^0\varphi_n(t)\,dt=0\quad\text{for any } n\in\mathbb{N}.$$

We define $\Phi_n(x) := \int_0^x \varphi_n(t)\,dt$, $x \in \mathbb{R}$, and $u_n := \Phi_n \circ u$ for any $n \in \mathbb{N}$. Then we easily see that $\Phi_n \in C^1(\mathbb{R}) \cap C_c(\mathbb{R})$, $\Phi_n(0) = 0$, and $\Phi_n' = \varphi_n$ for any $n \in \mathbb{N}$. Also, $\{u_n\}_{n\in\mathbb{N}}$ converges in norm in $L^p(X,m)$ to 0 by the dominated convergence theorem, and by (2.4) for $(\mathcal{E},\mathcal{D})$ we have $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}$ and $\sup_{n\in\mathbb{N}} \mathcal{E}(u_n) < \infty$. Since $(\mathrm{Cla})_p$ for $(\mathcal{E},\mathcal{D})$ yields $(\mathrm{Cla})_p$ for $\bigl(\mathcal{E},\overline{\mathcal{D}}^{\mathcal{F}}\bigr)$ and $\bigl(\overline{\mathcal{D}}^{\mathcal{F}}, \|\cdot\|_{\mathcal{E},1}\bigr)$ is complete, Lemma 3.17 is applicable to $\bigl(\mathcal{E},\overline{\mathcal{D}}^{\mathcal{F}}\bigr)$ and implies that $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $\bigl(\overline{\mathcal{D}}^{\mathcal{F}}, \|\cdot\|_{\mathcal{E},1}\bigr)$ to 0. By Mazur's lemma (Lemma 3.14), there exist $N(l) \in \mathbb{N}$ and $\{a_{l,k}\}_{k=l}^{N(l)} \subseteq [0,1]$ such that $N(l) > l$, $\sum_{k=l}^{N(l)} a_{l,k} = 1$ and $\sum_{k=l}^{N(l)} a_{l,k} u_{n_k}$ converges in norm in $\mathcal{F}$ to 0 as $l \to \infty$. Let us define $\Psi_l \in C^1(\mathbb{R})$ by $\Psi_l := \sum_{k=l}^{N(l)} a_{l,k}\Phi_{n_k}$. Then $\Psi_l(0) = 0$ and $\lim_{l\to\infty} \Psi_l'(x) = \mathbb{1}_F(x)$ for any $x \in \mathbb{R}$. Furthermore, by Fatou's lemma, (4.23) and $(\mathrm{EM1})_p$,

$$
\begin{aligned}
(\Gamma\langle u\rangle \circ u^{-1})(F) &= \int_{\mathbb{R}} \lim_{l\to\infty} |\Psi_l'(t)|^p \,(\Gamma\langle u\rangle \circ u^{-1})(dt) \\
&\le \liminf_{l\to\infty} \int_X |\Psi_l'(u(x))|^p \,\Gamma\langle u\rangle(dx) \\
&= \liminf_{l\to\infty} \Gamma\langle \Psi_l(u)\rangle(X) \le \liminf_{l\to\infty} \mathcal{E}(\Psi_l(u)) = 0,
\end{aligned}
$$

which completes the proof. □

The following theorem gives arguably the strongest possible forms of the strong locality of p-energy measures.

Theorem 4.18 (Strong locality of energy measures)**.** *Assume that $(\mathcal{E},\mathcal{D})$ satisfies (2.4) and $(\mathrm{Cla})_p$, that $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ is a Banach space, and that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies (CL1). Let $u, u_1, u_2, v \in \mathcal{D}$, $a, a_1, a_2, b \in \mathbb{R}$ and $A \in \mathcal{B}(X)$.*

(a) *If $A \subseteq u^{-1}(a)$, then $\Gamma\langle u\rangle(A) = 0$.*

(b) *If $A \subseteq (u-v)^{-1}(a)$, then $\Gamma\langle u\rangle(A) = \Gamma\langle v\rangle(A)$.*

(c) *If $A \subseteq u_1^{-1}(a_1) \cup u_2^{-1}(a_2)$, then*

$$
\Gamma_{\mathcal{E}}\langle u_1 + u_2 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A). \tag{4.37}
$$

If $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies $(\mathrm{Cla})_p$ and $A \subseteq u_1^{-1}(a_1) \cup u_2^{-1}(a_2)$, then

$$
\Gamma_{\mathcal{E}}\langle u_1 + u_2; v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_1; v\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2; v\rangle(A). \tag{4.38}
$$

(d) *If $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies $(\mathrm{Cla})_p$ and $A \subseteq (u_1 - u_2)^{-1}(a) \cup v^{-1}(b)$, then*

$$
\Gamma_{\mathcal{E}}\langle u_1; v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_2; v\rangle(A) \quad \textit{and} \quad \Gamma_{\mathcal{E}}\langle v; u_1\rangle(A) = \Gamma_{\mathcal{E}}\langle v; u_2\rangle(A). \tag{4.39}
$$

Proof. (a): This is immediate from Theorem 4.17.

(b): This follows from (a) and the triangle inequality for $\Gamma_{\mathcal{E}}\langle\,\cdot\,\rangle(A)^{1/p}$.

(c): Set $A_i := A \cap u_i^{-1}(a_i)$, $i \in \{1,2\}$. We see from (b) that

$$
\Gamma_{\mathcal{E}}\langle u_1 + u_2 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle v\rangle(A)
$$

$$
\begin{aligned}
&= \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A_1) + \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A_2) + \Gamma_{\mathcal{E}}\langle v\rangle(A) \\
&= \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A_1) + \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A_2) + \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A_1) + \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A_2) \\
&= \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A),
\end{aligned}
$$

which proves (4.37). Note that $\Gamma_{\mathcal{E}}\langle u_1 + u_2\rangle(A) = \Gamma_{\mathcal{E}}\langle u_1\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2\rangle(A)$ by (4.37) in the case $v = 0$. Next assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies (Cla)_p. By using this equality and applying (4.37) with v replaced by tv for $t \in (0,\infty)$, we have

$$
\begin{aligned}
&\frac{\Gamma_{\mathcal{E}}\langle u_1 + u_2 + tv\rangle(A) - \Gamma_{\mathcal{E}}\langle u_1 + u_2\rangle(A)}{t} + t^{p-1}\Gamma_{\mathcal{E}}\langle v\rangle(A) \\
&= \frac{\Gamma_{\mathcal{E}}\langle u_1 + tv\rangle(A) - \Gamma_{\mathcal{E}}\langle u_1\rangle(A)}{t} + \frac{\Gamma_{\mathcal{E}}\langle u_2 + tv\rangle(A) - \Gamma_{\mathcal{E}}\langle u_2\rangle(A)}{t},
\end{aligned}
$$

which implies (4.38) by letting $t \downarrow 0$.

(d): The proof will be very similar to that of Proposition 3.32-(a). By applying (4.37) with $u_2 - u_1, tv, u_1$ for $t \in (0,\infty)$ in place of u_1, u_2, v, we have

$$
\frac{\Gamma_{\mathcal{E}}\langle u_1 + tv\rangle(A) - \Gamma_{\mathcal{E}}\langle u_1\rangle(A)}{t} = \frac{\Gamma_{\mathcal{E}}\langle u_2 + tv\rangle(A) - \Gamma_{\mathcal{E}}\langle u_2\rangle(A)}{t},
$$

which implies the former equality in (4.39) by letting $t \downarrow 0$. This equality in turn with $v, 0, u_1 - u_2$ in place of u_1, u_2, v yields the latter equality in (4.39) by the linearity of $\Gamma_{\mathcal{E}}\langle v; \cdot\rangle(A)$. $\square$

5 p-Energy measures associated with self-similar p-energy forms

In this section, we focus on the self-similar case. We will introduce the self-similarity of p-energy forms and construct p-energy measures with respect to self-similar p-energy forms. Some fundamental properties of p-energy measures will be shown.

5.1 Self-similar structure and related notions

We first recall standard notation and terminology on self-similar structures (see [Kig01, Chapter 1] for example). Throughout this section, we fix a compact metrizable space K, a finite set S with $\#S \geq 2$ and a continuous injective map $F_i \colon K \to K$ for each $i \in S$. We set $\mathcal{L} \coloneqq (K, S, \{F_i\}_{i\in S})$.

Definition 5.1. (1) Let $W_0 \coloneqq \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_n \coloneqq S^n = \{w_1 \dots w_n \mid w_i \in S \text{ for } i \in \{1,\dots,n\}\}$ for $n \in \mathbb{N}$ and let $W_* \coloneqq \bigcup_{n\in\mathbb{N}\cup\{0\}} W_n$. For $w \in W_*$, the unique $n \in \mathbb{N}\cup\{0\}$ with $w \in W_n$ is denoted by $|w|$ and called the *length of* w. For $w, v \in W_*$, $w = w_1 \dots w_{n_1}$, $v = v_1 \dots v_{n_2}$, we define $wv \in W_*$ by $wv \coloneqq w_1 \dots w_{n_1} v_1 \dots v_{n_2}$ ($w\emptyset \coloneqq w, \emptyset v \coloneqq v$).

(2) We set $\Sigma := S^{\mathbb{N}} = \{\omega_1\omega_2\omega_3\dots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, which is always equipped with the product topology of the discrete topology on S, and define the *shift map* $\sigma\colon \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\omega_3\dots) := \omega_2\omega_3\omega_4\dots$. For $i \in S$ we define $\sigma_i\colon \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\omega_3\dots) := i\omega_1\omega_2\omega_3\dots$. For $\omega = \omega_1\omega_2\omega_3\dots \in \Sigma$ and $n \in \mathbb{N}\cup\{0\}$, we write $[\omega]_n := \omega_1\dots\omega_n \in W_n$.

(3) For $w = w_1\dots w_n \in W_*$, we set $F_w := F_{w_1} \circ \dots \circ F_{w_n}$ ($F_\emptyset := \mathrm{id}_K$), $K_w := F_w(K)$, $\sigma_w := \sigma_{w_1} \circ \dots \circ \sigma_{w_n}$ ($\sigma_\emptyset := \mathrm{id}_\Sigma$) and $\Sigma_w := \sigma_w(\Sigma)$.

(4) A finite subset Λ of W_* is called a *partition* of Σ if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Lambda$ with $w \neq v$ and $\Sigma = \bigcup_{w\in\Lambda} \Sigma_w$.

Definition 5.2. $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is called a *self-similar structure* if and only if there exists a continuous surjective map $\chi\colon \Sigma \to K$ such that $F_i \circ \chi = \chi \circ \sigma_i$ for any $i \in S$. Note that such χ, if it exists, is unique and satisfies $\{\chi(\omega)\} = \bigcap_{n\in\mathbb{N}} K_{[\omega]_n}$ for any $\omega \in \Sigma$.

In the following definition, we recall the definition of *post-critically finite self-similar structures* introduced by Kigami in [Kig93], which is mainly dealt with in Subsection 8.3.

Definition 5.3. Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure.

(1) We define the *critical set* $\mathcal{C}_\mathcal{L}$ and the *post-critical set* $\mathcal{P}_\mathcal{L}$ of $\mathcal{L}$ by

$$\mathcal{C}_\mathcal{L} := \chi^{-1}\bigl(\textstyle\bigcup_{i,j\in S,\, i\neq j} K_i \cap K_j\bigr) \qquad \text{and} \qquad \mathcal{P}_\mathcal{L} := \textstyle\bigcup_{n\in\mathbb{N}} \sigma^n(\mathcal{C}_\mathcal{L}). \tag{5.1}$$

$\mathcal{L}$ is called *post-critically finite*, or *p.-c.f.* for short, if and only if $\mathcal{P}_\mathcal{L}$ is a finite set.

(2) We set $V_0 := \chi(\mathcal{P}_\mathcal{L})$, $V_n := \bigcup_{w\in W_n} F_w(V_0)$ for $n \in \mathbb{N}$ and $V_* := \bigcup_{n\in\mathbb{N}\cup\{0\}} V_n$.

The set V_0 should be considered as the *"boundary"* of the self-similar set K; indeed, by [Kig01, Proposition 1.3.5-(2)], we have

$$K_w \cap K_v = F_w(V_0) \cap F_v(V_0) \text{ for any } w, v \in W_* \text{ with } \Sigma_w \cap \Sigma_v = \emptyset. \tag{5.2}$$

According to [Kig01, Lemma 1.3.11], $V_{n-1} \subseteq V_n$ for any $n \in \mathbb{N}$, and V_* is dense in K if $V_0 \neq \emptyset$.

The family of cells $\{K_w\}_{w\in W_*}$ describes the local topology of a self-similar structure. Indeed, $\{K_{n,x}\}_{n\geq 0}$, where $K_{n,x} := \bigcup_{w\in W_n; x\in K_w} K_w$, forms a fundamental system of neighborhoods of $x \in K$ [Kig01, Proposition 1.3.6]. Moreover, the proof of [Kig01, Proposition 1.3.6] implies that any metric d on K giving the original topology of K satisfies

$$\lim_{n\to\infty} \max_{w\in W_n} \mathrm{diam}(K_w, d) = 0. \tag{5.3}$$

Let us recall the notion of self-similar measure.

Definition 5.4 (Self-similar measures)**.** Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure and let $(\theta_i)_{i\in S} \in (0,1)^S$ satisfy $\sum_{i\in S} \theta_i = 1$. A Borel probability measure m on K is said to be a *self-similar measure on* $\mathcal{L}$ *with weight* $(\theta_i)_{i\in S}$ if and only if the following equality (of Borel measures on K) holds:

$$m = \sum_{i\in S} \theta_i (m \circ F_i^{-1}), \tag{5.4}$$

where $m \circ F_i^{-1}$ denotes the image measure of m by F_i, i.e., $(m \circ F_i^{-1})(A) := m(F_i^{-1}(A))$ for $A \in \mathcal{B}(K)$.

Remark 5.5. Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure, m a self-similar measure on $\mathcal{L}$, and $w \in W_*$. We then easily see from (5.4) that $u \circ F_w = v \circ F_w$ m-a.e. on K for any Borel measurable functions $u, v\colon K \to [-\infty, \infty]$ satisfying $u = v$ m-a.e. on K, thereby that we can define a map $F_w^*\colon L^0(K, m) \to L^0(K, m)$ by setting $F_w^* u := u \circ F_w$, and further that $F_w^*\colon L^p(K, m) \to L^p(K, m)$ is a bounded linear operator for any $p \in [1, \infty]$.

Proposition 5.6 ([Kig01, Section 1.4], [Kig09, Theorem 1.2.7])**.** *Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure and let $(\theta_i)_{i\in S} \in (0, 1)^S$ satisfy $\sum_{i\in S} \theta_i = 1$. Then there exists a self-similar measure m on $\mathcal{L}$ with weight $(\theta_i)_{i\in S}$. Moreover, if $K \neq \overline{V_0}^K$, then such m is unique and satisfies $m(K_w) = \theta_w$ and $m(F_w(\overline{V_0}^K)) = 0$ for any $w \in W_*$, where $\theta_w := \theta_{w_1} \cdots \theta_{w_n}$ for $w = w_1 \dots w_n \in W_*$ ($\theta_\emptyset := 1$).*

5.2 Self-similar p-energy forms and p-energy measures

In this subsection, we introduce the notion of self-similar p-energy form and define the p-energy measures associated with a given self-similar p-energy form. Throughout this subsection, we fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ with K connected, a σ-algebra $\mathcal{B}$ in K including $\mathcal{B}(K)$, a measure m on $(K, \mathcal{B})$ with $\mathrm{supp}_K[m] = K$, $p \in (1, \infty)$, and a p-energy form $(\mathcal{E}, \mathcal{F})$ on (K, m).

Definition 5.7 (Self-similar p-energy form)**.** Let $\boldsymbol{\rho} = (\rho_i)_{i\in S} \in (0, \infty)^S$. A p-energy form $(\mathcal{E}, \mathcal{F})$ on (K, m) is said to be *self-similar on* $(\mathcal{L}, m)$ *with weight* $\boldsymbol{\rho}$ if and only if the following hold:

$$\mathcal{F} \cap C(K) = \{f \in C(K) \mid f \circ F_i \in \mathcal{F} \text{ for any } i \in S\}, \tag{5.5}$$

$$\mathcal{E}(f) = \sum_{i\in S} \rho_i \mathcal{E}(f \circ F_i) \quad \text{for any } f \in \mathcal{F} \cap C(K). \tag{5.6}$$

Note that for any partition Λ of Σ, (5.6) implies

$$\mathcal{E}(f) = \sum_{w\in\Lambda} \rho_w \mathcal{E}(f \circ F_w), \quad f \in \mathcal{F} \cap C(K), \tag{5.7}$$

where $\rho_w := \rho_{w_1} \cdots \rho_{w_n}$ for $w = w_1 \dots w_n \in W_*$. Indeed, (5.7) follows from an induction with respect to $\max_{w\in\Lambda} |w|$.

In the rest of this subsection, we assume that $(\mathcal{E}, \mathcal{F})$ is a self-similar p-energy form on $\mathcal{L}$ with weight $\boldsymbol{\rho} = (\rho_i)_{i\in S}$. We can see that the two-variable version $\mathcal{E}(f; g)$ also has the following self-similarity.

Proposition 5.8. *Assume that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies* (Cla)$_p$*. Then*

$$\mathcal{E}(f; g) = \sum_{i\in S} \rho_i \mathcal{E}(f \circ F_i; g \circ F_i) \quad \textit{for any } f, g \in \mathcal{F} \cap C(K). \tag{5.8}$$

Proof. For any $f, g \in \mathcal{F} \cap C(K)$ and any $t > 0$, we have

$$\frac{\mathcal{E}(f+tg) - \mathcal{E}(f)}{t} = \sum_{i \in S} \rho_i \frac{\mathcal{E}\bigl(f \circ F_i + t(g \circ F_i)\bigr) - \mathcal{E}(f \circ F_i)}{t}.$$

Letting $t \downarrow 0$ yields (5.8). □

Next we see that p-energy measures are naturally introduced by virtue of the self-similarity of $(\mathcal{E}, \mathcal{F})$ (see also [Hin05, MS25+]). For $f \in \mathcal{F} \cap C(K)$, we define a finite measure $\mathfrak{m}_{\mathcal{E}}^{(n)}\langle f \rangle$ on $W_n = S^n$ by putting $\mathfrak{m}_{\mathcal{E}}^{(n)}\langle f \rangle(\{w\}) := \rho_w \mathcal{E}(f \circ F_w)$ for each $w \in W_n$. Then $\{\mathfrak{m}_{\mathcal{E}}^{(n)}\langle f \rangle\}_{n \geq 0}$ satisfies the consistency condition by (5.7), and hence Kolmogorov's extension theorem (see, e.g., [Dud, Theorem 12.1.2]) guarantees that there exists a unique Borel measure $\mathfrak{m}_{\mathcal{E}}\langle f \rangle$ on $\Sigma = S^{\mathbb{N}}$ such that $\mathfrak{m}_{\mathcal{E}}\langle f \rangle(\Sigma_w) = \rho_w \mathcal{E}(f \circ F_w)$ for any $w \in W_*$. In particular, $\mathfrak{m}_{\mathcal{E}}\langle f \rangle(\Sigma) = \mathcal{E}(f)$. Basic properties of $\mathfrak{m}_{\mathcal{E}}\langle \,\cdot\, \rangle$ are collected in the following proposition.

Proposition 5.9. (a) *Assume that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies* (GC)$_p$*. Then for any $A \in \mathcal{B}(\Sigma)$, $(\mathfrak{m}_{\mathcal{E}}\langle \,\cdot\, \rangle(A), \mathcal{F} \cap C(K))$ is a p-energy form on (K, m) satisfying* (GC)$_p$*.*

(b) *Assume that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies* (Cla)$_p$*. Then for any $A \in \mathcal{B}(\Sigma)$, $(\mathfrak{m}_{\mathcal{E}}\langle \,\cdot\, \rangle(A), \mathcal{F} \cap C(K))$ is a p-energy form on (K, m) satisfying* (Cla)$_p$*, and in particular, for any $f, g \in \mathcal{F} \cap C(K)$, the following derivative exists in $\mathbb{R}$:*

$$\mathfrak{m}_{\mathcal{E}}\langle f; g \rangle(A) := \frac{1}{p} \frac{d}{dt} \mathfrak{m}_{\mathcal{E}}\langle f + tg \rangle(A) \biggr|_{t=0}, \tag{5.9}$$

Moreover, $\mathfrak{m}_{\mathcal{E}}\langle f; g \rangle$ is a signed Borel measure on Σ.

Proof. (a): Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$ and $T = (T_1, \ldots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2), and let $\boldsymbol{u} = (u_1, \ldots, u_{n_1}) \in \bigl(\mathcal{F} \cap C(K)\bigr)^{n_1}$. We are to show that

$$\Bigl\| \bigl(\mathfrak{m}_{\mathcal{E}}\langle T_l(\boldsymbol{u}) \rangle(A)^{1/p} \bigr)_{l=1}^{n_2} \Bigr\|_{\ell^{q_2}} \leq \Bigl\| \bigl(\mathfrak{m}_{\mathcal{E}}\langle u_k \rangle(A)^{1/p} \bigr)_{k=1}^{n_1} \Bigr\|_{\ell^{q_1}}, \quad A \in \mathcal{B}(\Sigma). \tag{5.10}$$

If $A = \Sigma_w$ for some $w \in W_*$, then (5.10) is clearly true by (GC)$_p$ for $(\mathcal{E}, \mathcal{F})$. By a similar argument using the reverse Minkowski inequality on $\ell^{q_1/p}$ and the Minkowski inequality on $\ell^{q_2/p}$ as in (2.19), we obtain (5.10) for any A belonging to the algebra in Σ generated by $\{\Sigma_w\}_{w \in W_*}$. Hence the monotone class theorem (see, e.g., [Dud, Theorem 4.4.2]) implies that (5.10) holds for any $A \in \mathcal{B}(\Sigma)$.

(b): Note that a special case of (5.10) proves (Cla)$_p$ for $(\mathfrak{m}_{\mathcal{E}}\langle \,\cdot\, \rangle(A), \mathcal{F} \cap C(K))$; see also Proposition 2.3-(e),(f). Then the derivative in (5.9) exists by Proposition 3.6 and (5.10). In addition, $\mathfrak{m}_{\mathcal{E}}\langle f; g \rangle$ turns out to be a signed Borel measure on Σ by Theorem 4.6. (Even if $(\mathcal{E}, \mathcal{F})$ does not satisfy (GC)$_p$, the above proof of (a) together with the triangle inequality for $\mathcal{E}^{1/p}$ shows (5.10) with $(n_1, n_2, q_1, q_2) = (2, 1, p, p)$ and $T_1(x, y) = x + y$, namely the triangle inequality on $\mathcal{F} \cap C(K)$ for $\mathfrak{m}_{\mathcal{E}}\langle \,\cdot\, \rangle(A)^{1/p}$.) □

We now define a family $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ of finite Borel measures on K by

$$\Gamma_{\mathcal{E}}\langle f\rangle(A) \coloneqq (\mathfrak{m}_{\mathcal{E}}\langle f\rangle\circ\chi^{-1})(A) \coloneqq \mathfrak{m}_{\mathcal{E}}\langle f\rangle(\chi^{-1}(A)), \quad A\in\mathcal{B}(K) \tag{5.11}$$

for $f\in\mathcal{F}\cap C(K)$, where $\chi\colon\Sigma\to K$ is the same map as in Definition 5.2. The following proposition states basic properties and the self-similarity of $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$.

Proposition 5.10. (a) *$\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ satisfies $\Gamma_{\mathcal{E}}\langle f\rangle(K)=\mathcal{E}(f)$ for any $f\in\mathcal{F}\cap C(K)$, in particular (EM1)$_p$, and (EM2)$_p$.*

(b) *For any $f\in\mathcal{F}\cap C(K)$, any $w\in W_*$ and any $n\in\mathbb{N}\cup\{0\}$,*

$$\rho_w\mathcal{E}(f\circ F_w)\le\Gamma_{\mathcal{E}}\langle f\rangle(K_w)\le\sum_{v\in W_n;K_v\cap K_w\neq\emptyset}\rho_v\mathcal{E}(f\circ F_v). \tag{5.12}$$

(c) *Assume that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies (GC)$_p$, let $\varphi\colon K\to[0,\infty]$ be Borel measurable, and let $n_1,n_2\in\mathbb{N}$, $q_1\in(0,p]$, $q_2\in[p,\infty]$ and $T=(T_1,\dots,T_{n_2})\colon\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ satisfy (2.2). Then for any $\boldsymbol{u}=(u_1,\dots,u_{n_1})\in\bigl(\mathcal{F}\cap C(K)\bigr)^{n_1}$,*

$$\left\|\left(\left(\int_K\varphi\,d\Gamma_{\mathcal{E}}\langle T_l(\boldsymbol{u})\rangle\right)^{1/p}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}}\le\left\|\left(\left(\int_K\varphi\,d\Gamma_{\mathcal{E}}\langle u_k\rangle\right)^{1/p}\right)_{k=1}^{n_1}\right\|_{\ell^{q_1}}. \tag{5.13}$$

In particular Proposition 2.3 with $(\int_K\varphi\,d\Gamma_{\mathcal{E}}\langle\,\cdot\,\rangle,\mathcal{F}\cap C(K))$ in place of $(\mathcal{E},\mathcal{F})$ holds provided $\|\varphi\|_{\sup}<\infty$.

(d) *The following equality (of Borel measures on K) holds:*

$$\Gamma_{\mathcal{E}}\langle f\rangle=\sum_{i\in S}\rho_i(\Gamma_{\mathcal{E}}\langle f\circ F_i\rangle\circ F_i^{-1}) \quad \text{for any } f\in\mathcal{F}\cap C(K). \tag{5.14}$$

(e) *Assume that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies (Cla)$_p$. Then $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ also satisfies (Cla)$_p$, and the following equality (of signed Borel measures on K) holds:*

$$\Gamma_{\mathcal{E}}\langle f;g\rangle=\sum_{i\in S}\rho_i(\Gamma_{\mathcal{E}}\langle f\circ F_i;g\circ F_i\rangle\circ F_i^{-1}) \quad \text{for any } f,g\in\mathcal{F}\cap C(K). \tag{5.15}$$

(f) *Assume that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies (Cla)$_p$. Then $\mathfrak{m}_{\mathcal{E}}\langle f;g\rangle\circ\chi^{-1}=\Gamma_{\mathcal{E}}\langle f;g\rangle$ for any $f,g\in\mathcal{F}\cap C(K)$.*

Proof. (a): We easily have $\Gamma_{\mathcal{E}}(K)=\mathfrak{m}_{\mathcal{E}}\langle f\rangle(\chi^{-1}(K))=\mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma)=\mathcal{E}(f)$. The proof of (EM2)$_p$ is included in the proof of (c) below.

(b): This statement is the same as [MS23+, Lemma 9.15], which is easily proved by noting that $\Sigma_w\subseteq\chi^{-1}(K_w)\subseteq\bigcup_{v\in W_n;K_v\cap K_w\neq\emptyset}\Sigma_v$.

(c): Assume that $(\mathcal{E},\mathcal{F})$ satisfies (GC)$_p$. Let us fix $T=(T_1,\dots,T_{n_2})\colon\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ satisfying (2.2) and $\boldsymbol{u}=(u_1,\dots,u_{n_1})\in\bigl(\mathcal{F}\cap C(K)\bigr)^{n_1}$. For any $B\in\mathcal{B}(K)$, by (GC)$_p$ for $(\mathfrak{m}_{\mathcal{E}}\langle\,\cdot\,\rangle(\chi^{-1}(B)),\mathcal{F}\cap C(K))$ (see Proposition 5.9-(a)), we obtain

$$\left\|\left(\Gamma_{\mathcal{E}}\langle T_l(\boldsymbol{u})\rangle(B)^{1/p}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}}\le\left\|\left(\Gamma_{\mathcal{E}}\langle u_k\rangle(B)^{1/p}\right)_{k=1}^{n_1}\right\|_{\ell^{q_1}}. \tag{5.16}$$

Again by a similar argument as in (2.19), we see that (5.13) holds for any non-negative Borel measurable simple function φ on K. We get the desired extension, (5.13) for any Borel measurable function $\varphi\colon K \to [0,\infty]$, by the monotone convergence theorem.

(d): The proof is very similar to [Shi24, Proof of Theorem 7.5]. Let $k \in \mathbb{N}$, $w = w_1 \dots w_k \in W_k$ and $n \in \mathbb{N}$. We see that

$$\begin{aligned}\sum_{i\in S}\rho_i\mathfrak{m}_{\mathcal{E}}\langle f\circ F_i\rangle(\sigma_i^{-1}(\Sigma_w)) &= \rho_{w_1}\mathfrak{m}_{\mathcal{E}}\langle f\circ F_{w_1}\rangle(\sigma_{w_1}^{-1}(\Sigma_w)) = \rho_{w_1}\mathfrak{m}_{\mathcal{E}}\langle f\circ F_{w_1}\rangle(\Sigma_{w_2\dots w_k})\\ &= \rho_{w_1}\rho_{w_2\dots w_k}\mathcal{E}((f\circ F_{w_1})\circ F_{w_2\dots w_k}) = \mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma_w)\end{aligned}$$

Since $w \in W_*$ is arbitrary, by Dynkin's π-λ theorem, we deduce that

$$\mathfrak{m}_{\mathcal{E}}\langle f\rangle(A) = \sum_{i\in S}\rho_i(\mathfrak{m}_{\mathcal{E}}\langle f\circ F_i\rangle\circ\sigma_i^{-1})(A), \quad A\in\mathcal{B}(\Sigma).$$

We obtain (5.14) by $\chi\circ\sigma_i = F_i\circ\chi$.

(e): Assume that $(\mathcal{E},\mathcal{F})$ satisfies (Cla)$_p$. Then $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ satisfies (Cla)$_p$ by (5.16) (see also Proposition 2.3-(e),(f)). Now we obtain (5.15) by letting $t\downarrow 0$ in

$$\Gamma_{\mathcal{E}}\langle f+tg\rangle(A) = \sum_{i\in S}\rho_i\Gamma_{\mathcal{E}}\langle f\circ F_i + t(g\circ F_i)\rangle\bigl(F_i^{-1}(A)\bigr).$$

(f): This is immediate from (5.11), (4.5) and (5.9). □

We next prove the chain rules (CL1) and (CL2) for $\Gamma_{\mathcal{E}}\langle\,\cdot\,\rangle$. Such chain rules have been obtained also in [BV05], but we provide here self-contained proofs because our present framework is different from that of [BV05] and our version (CL2) is stronger than the chain rule proved in [BV05].

Theorem 5.11. *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$ and that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies (2.4). Then $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ satisfies (CL1), i.e., for any $u\in\mathcal{F}\cap C(K)$ and any $\Phi\in C^1(\mathbb{R})$, we have $\Phi(u)\in\mathcal{F}\cap C(K)$ and*

$$d\Gamma_{\mathcal{E}}\langle\Phi(u)\rangle = |\Phi'(u)|^p\, d\Gamma_{\mathcal{E}}\langle u\rangle. \tag{5.17}$$

Proof. First, let us observe a few consequences of (2.4). The proof of Corollary 2.5-(a) works even if we assume (2.4) instead of (GC)$_p$, so we have (2.11). We then obtain $\Phi(u)\in\mathcal{F}\cap C(K)$ by (2.11) and $\mathbb{R}\mathbb{1}_K\subseteq\mathcal{E}^{-1}(0)$. Also, by (2.11), the identity $ab = \frac{1}{4}\bigl[(a+b)^2-(a-b)^2\bigr]$ for $a,b\in\mathbb{R}$, and the triangle inequality for $\mathcal{E}^{1/p}$, there exists a constant $c_p\in(0,\infty)$ depending only on p such that for any $u,v\in\mathcal{F}\cap C(K)$,

$$uv\in\mathcal{F}\cap C(K) \quad\text{and}\quad \mathcal{E}(uv)\le c_p\bigl(\|v\|_{\sup}^p\,\mathcal{E}(u)+\|u\|_{\sup}^p\,\mathcal{E}(v)\bigr); \tag{5.18}$$

indeed, (5.18) for $u,v\in\mathcal{F}\cap C(K)$ with $\|u\|_{\sup}=\|v\|_{\sup}=1$ is easily verified, and this special case applied to $\|u\|_{\sup}^{-1}u, \|v\|_{\sup}^{-1}v$ yields (5.18) for general $u,v\in\mathcal{F}\cap C(K)$.

Next we will prove that

$$\lim_{l\to\infty}\Bigl|\rho_w\mathcal{E}\bigl(\Phi(u\circ F_w)\bigr)-\mathcal{S}_l^{(1)}(w)\Bigr| = 0 \quad\text{for any } w\in W_*, \tag{5.19}$$

where for $w \in W_*$ and $l \in \mathbb{N} \cup \{0\}$ we set, with an arbitrarily fixed $x_0 \in K$,

$$\mathcal{S}_l^{(1)}(w) := \sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}\bigl(\Phi'\bigl(u(F_{w\tau}(x_0))\bigr) \cdot (u \circ F_{w\tau})\bigr).$$

We need some preparations to prove (5.19). Note that, for any $z \in W_*$ and any $x \in K$,

$$\begin{aligned}
&\Phi\bigl(u(F_z(x))\bigr) - \Phi\bigl(u(F_z(x_0))\bigr) \\
&= \bigl[u(F_z(x)) - u(F_z(x_0))\bigr]\Biggl(\Phi'\bigl(u(F_z(x_0))\bigr) \\
&\qquad + \int_0^1 \Bigl[\Phi'\bigl(u(F_z(x_0)) + t\bigl(u(F_z(x)) - u(F_z(x_0))\bigr)\bigr) - \Phi'(u(F_z(x_0)))\Bigr]\, dt\Biggr).
\end{aligned}$$

In particular,

$$\Phi(u \circ F_z) - \widehat{u}_z = \Phi\bigl(u(F_z(x_0)\bigr) - \Phi'\bigl(u(F_z(x_0))\bigr)u(F_z(x_0)) + D_z I_z, \tag{5.20}$$

where $\widehat{u}_z, D_z, I_z \in C(K)$ are given by

$$\begin{aligned}
\widehat{u}_z(x) &:= \Phi'\bigl(u(F_z(x_0))\bigr) \cdot (u \circ F_z)(x), \\
D_z(x) &:= u(F_z(x)) - u(F_z(x_0)), \\
I_z(x) &:= \int_0^1 \Bigl[\Phi'\bigl(u(F_z(x_0)) + tD_z(x)\bigr) - \Phi'\bigl(u(F_z(x_0))\bigr)\Bigr]\, dt, \quad x \in K.
\end{aligned}$$

Note that $\widehat{u}_z \in \mathcal{F}$ by (5.5). By (2.11), we have that $I_z \in \mathcal{F}$ and that there exists a constant $C_{u,\Phi} \in (0,\infty)$ depending only on $p, \|u\|_{\sup}, \|\Phi'\|_{\sup,[-2\|u\|_{\sup},2\|u\|_{\sup}]}$ such that $\mathcal{E}(I_z) \le C_{u,\Phi}\mathcal{E}(u \circ F_z)$ and $\mathcal{E}\bigl(\Phi(u \circ F_z)\bigr) \le C_{u,\Phi}\mathcal{E}(u \circ F_z)$. Therefore, for any $l \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}\bigl(\Phi(u \circ F_{w\tau}) - \widehat{u}_{w\tau}\bigr) \\
&\overset{(5.20)}{=} \sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}(D_{w\tau} I_{w\tau}) \\
&\overset{(5.18)}{\le} c_p \sum_{\tau \in W_l} \rho_{w\tau} \bigl(\|I_{w\tau}\|_{\sup}^p \mathcal{E}(D_{w\tau}) + \|D_{w\tau}\|_{\sup}^p \mathcal{E}(I_{w\tau})\bigr) \\
&\le c_p \biggl(\max_{\tau' \in W_l} \|I_{w\tau'}\|_{\sup} + \max_{\tau' \in W_l} \|D_{w\tau'}\|_{\sup}\biggr)^p \sum_{\tau \in W_l} \rho_{w\tau} \Bigl(\mathcal{E}(D_{w\tau}) + C_{u,\Phi}\mathcal{E}(u \circ F_{w\tau})\Bigr) \\
&\le c_p(1 + C_{u,\Phi})\mathcal{E}(u) \biggl(\max_{\tau' \in W_l} \|I_{w\tau'}\|_{\sup} + \max_{\tau' \in W_l} \|D_{w\tau'}\|_{\sup}\biggr)^p.
\end{aligned} \tag{5.21}$$

Since u and Φ' are uniformly continuous on K, it follows from (5.3) that $\max_{\tau' \in W_l} \|I_{w\tau'}\|_{\sup}$ and $\max_{\tau' \in W_l} \|D_{w\tau'}\|_{\sup}$ converge to 0 as $l \to \infty$, and hence

$$\lim_{l \to \infty} \sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}\bigl(\Phi(u \circ F_{w\tau}) - \widehat{u}_{w\tau}\bigr) = 0, \tag{5.22}$$

which implies (5.19).

By the uniform continuity of Φ' and the fact that $\mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma_w) = \rho_w\mathcal{E}(f \circ F_w)$ for any $f \in \mathcal{F} \cap C(K)$ and any $w \in W_*$, we easily observe that

$$\lim_{l\to\infty}\left|\sum_{k=1}^{n}\int_{\Sigma_w}\left|\Phi'(u\circ\chi)\right|^p\,d\mathfrak{m}_{\mathcal{E}}\langle u\rangle - \mathcal{S}_l^{(1)}(w)\right| = 0.$$

Hence, by (5.19) and Dynkin's π-λ theorem,

$$d\mathfrak{m}_{\mathcal{E}}\langle\Phi(u)\rangle = \sum_{k=1}^{n}\left|\Phi'(u\circ\chi)\right|^p\,d\mathfrak{m}_{\mathcal{E}}\langle u\rangle. \tag{5.23}$$

Then we obtain the desired equality (5.17) by (5.23) and Proposition 5.10-(f). □

To prove (CL2), in addition to $(\mathrm{Cla})_p$, we need to assume the closedness of $(\mathcal{E}, \mathcal{F} \cap L^p(K,m))$ in $L^p(K,m)$. Recall the definition (3.24) of the norm $\|\cdot\|_{\mathcal{E},1}$, which we here define on $\mathcal{F} \cap L^p(K,m)$ without assuming that $\mathcal{F} \subseteq L^p(K,m)$.

Theorem 5.12 (Chain rule)**.** *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$, that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies* (2.4) *and* $(\mathrm{Cla})_p$*, and that $(\mathcal{F} \cap L^p(K,m), \|\cdot\|_{\mathcal{E},1})$ is a Banach space. Then $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ satisfies* (CL2)*, i.e., for any $n \in \mathbb{N}$, $u \in \mathcal{F} \cap C(K)$, $\boldsymbol{v} = (v_1,\dots,v_n) \in (\mathcal{F} \cap C(K))^n$, $\Phi \in C^1(\mathbb{R})$ and $\Psi \in C^1(\mathbb{R}^n)$, we have $\Phi(u), \Psi(\boldsymbol{v}) \in \mathcal{F} \cap C(K)$ and*

$$d\Gamma_{\mathcal{E}}\langle\Phi(u);\Psi(\boldsymbol{v})\rangle = \sum_{k=1}^{n}\operatorname{sgn}\bigl(\Phi'(u)\bigr)\left|\Phi'(u)\right|^{p-1}\partial_k\Psi(\boldsymbol{v})\,d\Gamma_{\mathcal{E}}\langle u; v_k\rangle. \tag{5.24}$$

Proof. Let $n \in \mathbb{N}$, $u \in \mathcal{F} \cap C(K)$, $\boldsymbol{v} = (v_1,\dots,v_n) \in (\mathcal{F} \cap C(K))^n$, $\Phi \in C^1(\mathbb{R})$ and $\Psi \in C^1(\mathbb{R}^n)$, so that $\Phi(u) \in \mathcal{F} \cap C(K)$ as observed at the beginning of the proof of Theorem 5.11. *We fix these $n, u, \boldsymbol{v} = (v_1,\dots,v_n), \Phi, \Psi$ throughout this proof*, and first, under the additional assumption that $\Psi(\boldsymbol{v}) \in \mathcal{F} \cap C(K)$, we will prove that

$$\lim_{l\to\infty}\left|\rho_w\mathcal{E}\bigl(\Phi(u\circ F_w);\Psi(\boldsymbol{v}\circ F_w)\bigr) - \mathcal{S}_l^{(2)}(w)\right| = 0 \quad \text{for any } w \in W_*, \tag{5.25}$$

where for $w \in W_*$ and $l \in \mathbb{N}\cup\{0\}$ we set, with an arbitrarily fixed $x_0 \in K$,

$$\mathcal{S}_l^{(2)}(w) \coloneqq \sum_{\tau\in W_l}\rho_{w\tau}\mathcal{E}\left(\Phi'(u\circ F_{w\tau}(x_0))\cdot(u\circ F_{w\tau}); \sum_{k=1}^{n}\partial_k\Psi(v\circ F_{w\tau}(x_0))\cdot(v_k\circ F_{w\tau})\right).$$

To prove (5.25), we observe that $\left|\rho_w\mathcal{E}\bigl(\Phi(u\circ F_w);\Psi(v\circ F_w)\bigr) - \mathcal{S}_l^{(2)}(w)\right| \leq A_{1,l} + A_{2,l}$, where

$$\begin{aligned}
\widehat{u}_z(x) &\coloneqq \Phi'\bigl(u(F_z(x_0))\bigr)\cdot(u\circ F_z)(x),\\
\widehat{v}_z(x) &\coloneqq \sum_{k=1}^{n}\partial_k\Psi\bigl(\boldsymbol{v}(F_z(x_0))\bigr)\cdot(v_k\circ F_z)(x) \quad \text{for } z \in W_*, x \in K,
\end{aligned}$$

$$A_{1,l} \coloneqq \sum_{\tau \in W_l} \rho_{w\tau} \left| \mathcal{E}\bigl(\Phi(u \circ F_{w\tau}); \Psi(\boldsymbol{v} \circ F_{w\tau})\bigr) - \mathcal{E}\bigl(\Phi(u \circ F_{w\tau}); \widehat{v}_{w\tau}\bigr) \right|,$$
$$A_{2,l} \coloneqq \sum_{\tau \in W_l} \rho_{w\tau} \left| \mathcal{E}\bigl(\Phi(u \circ F_{w\tau}); \widehat{v}_{w\tau}\bigr) - \mathcal{E}\bigl(\widehat{u}_{w\tau}; \widehat{v}_{w\tau}\bigr) \right|.$$

Similar to (5.22), we can show that

$$\lim_{l \to \infty} \sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}\bigl(\Psi(\boldsymbol{v} \circ F_{w\tau}) - \widehat{v}_{w\tau}\bigr) = 0. \tag{5.26}$$

By (3.10), (3.11) and Hölder's inequality, we have

$$A_{1,l} \lesssim \mathcal{E}(u \circ F_w)^{(p-1)/p} \left(\sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}(\Psi(\boldsymbol{v} \circ F_{w\tau}) - \widehat{v}_{w\tau}) \right)^{1/p},$$

and

$$\begin{aligned}
A_{2,l} &\lesssim \sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}(u \circ F_{w\tau})^{(p-1-\alpha_p)/p} \mathcal{E}\bigl(\Phi(u \circ F_{w\tau}) - \widehat{u}_{w\tau}\bigr)^{\alpha_p/p} \mathcal{E}\bigl(\widehat{v}_{w\tau}\bigr)^{1/p} \\
&\le \mathcal{E}(u \circ F_w)^{(p-1-\alpha_p)/p} \left(\sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}(\Phi(u \circ F_{w\tau}) - \widehat{u}_{w\tau}) \right)^{\alpha_p/p} \left(\sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}\bigl(\widehat{v}_{w\tau}\bigr) \right)^{1/p} \\
&\lesssim \mathcal{E}(u \circ F_w)^{(p-1-\alpha_p)/p} \left(\sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}(\Phi(u \circ F_{w\tau}) - \widehat{u}_{w\tau}) \right)^{\alpha_p/p} \max_{k \in \{1,\dots,n\}} \mathcal{E}(v_k \circ F_w)^{1/p}.
\end{aligned}$$

Combining these estimates with (5.22) and (5.26), we obtain $\lim_{l \to \infty} A_{i,l} = 0$ and thus (5.25) holds.

Continuing to assume that $\Psi(\boldsymbol{v}) \in \mathcal{F} \cap C(K)$, by the uniform continuities of $\Phi', \partial\Psi_k$ and the fact that $\mathfrak{m}_{\mathcal{E}}\langle f; g\rangle(\Sigma_w) = \rho_w \mathcal{E}(f \circ F_w; g \circ F_w)$ for any $f, g \in \mathcal{F} \cap C(K)$ and any $w \in W_*$, we easily observe that

$$\lim_{l \to \infty} \left| \sum_{k=1}^{n} \int_{\Sigma_w} \operatorname{sgn}\bigl(\Phi'(u \circ \chi)\bigr) \left|\Phi'(u \circ \chi)\right|^{p-1} \partial_k \Psi(\boldsymbol{v} \circ \chi)\, d\mathfrak{m}_{\mathcal{E}}\langle u; v_k\rangle - \mathcal{S}_l^{(2)}(w) \right| = 0.$$

Hence, by (5.25) and Dynkin's π-λ theorem,

$$d\mathfrak{m}_{\mathcal{E}}\langle \Phi(u); \Psi(\boldsymbol{v})\rangle = \sum_{k=1}^{n} \operatorname{sgn}\bigl(\Phi'(u \circ \chi)\bigr) \left|\Phi'(u \circ \chi)\right|^{p-1} \partial_k \Psi(\boldsymbol{v} \circ \chi)\, d\mathfrak{m}_{\mathcal{E}}\langle u; v_k\rangle. \tag{5.27}$$

Then by (5.27) and Proposition 5.10-(f), we obtain the desired equality (5.24) under the additional assumption that $\Psi(\boldsymbol{v}) \in \mathcal{F} \cap C(K)$. We stress here that *the arguments in this and the last paragraphs do NOT require the assumption that* $(\mathcal{F} \cap L^p(K, m), \|\cdot\|_{\mathcal{E},1})$ *is a Banach space.* (Note also that $u \in \mathcal{F} \cap C(K)$ and $\Phi \in C^1(\mathbb{R})$ are arbitrary here, and hence can be chosen to be $u = \Psi(\boldsymbol{v})$ and $\Phi = \mathrm{id}_{\mathbb{R}}$ as long as $\Psi(\boldsymbol{v}) \in \mathcal{F} \cap C(K)$.)

Thus it remains to prove that $\Psi(\boldsymbol{v}) \in \mathcal{F} \cap C(K)$. We can assume that $\Psi(0) = 0$ since $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$. Define $Q(\boldsymbol{v}) \subseteq \mathbb{R}^n$ by

$$Q(\boldsymbol{v}) := \Bigl[- \|v_1\|_{\sup}, \|v_1\|_{\sup}\Bigr] \times \Bigl[- \|v_2\|_{\sup}, \|v_2\|_{\sup}\Bigr] \times \cdots \times \Bigl[- \|v_n\|_{\sup}, \|v_n\|_{\sup}\Bigr].$$

Then there exists a sequence $\{\Psi_l\}_{l\in\mathbb{N}}$ of polynomials in n variables with real coefficients such that $\Psi_l(0) = 0$, $\|\Psi - \Psi_l\|_{\sup,Q(\boldsymbol{v})} \to 0$ and $\|\partial_k\Psi - \partial_k\Psi_l\|_{\sup,Q(\boldsymbol{v})} \to 0$ for each $k \in \{1, \dots, n\}$ as $l \to \infty$ (see [CH, Chapter II.4.3]). Let $l \in \mathbb{N}$. By the mean value theorem, for any $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in Q(\boldsymbol{v})$,

$$|\Psi_l(x) - \Psi_l(y)| \le \sum_{k=1}^{n} \|\partial_k\Psi_l\|_{\sup,Q(\boldsymbol{v})} |x_k - y_k| . \tag{5.28}$$

Noting that $\Psi_l(\boldsymbol{v}) \in \mathcal{F} \cap C(K)$ by (5.18) and hence that (5.24) with Ψ_l in place of Ψ holds by the result of the previous paragraph, we see from Propositions 5.10-(a) and 4.8 that

$$\begin{aligned}
\mathcal{E}(\Psi_l(\boldsymbol{v})) &= \Gamma_{\mathcal{E}}\langle\Psi_l(\boldsymbol{v})\rangle(K) \qquad \text{(by Proposition 5.10-(a))}\\
&= \int_K \sum_{k=1}^{n} \partial_k\Psi_l(\boldsymbol{v}(x))\, \Gamma_{\mathcal{E}}\langle\Psi_l(\boldsymbol{v}); v_k\rangle(dx) \qquad \text{(by (5.24) with } \Psi_l \text{ in place of } \Psi)\\
&\le \sum_{k=1}^{n} \|\partial_k\Psi_l\|_{\sup,Q(\boldsymbol{v})} \Gamma_{\mathcal{E}}\langle\Psi_l(\boldsymbol{v})\rangle(K)^{\frac{p-1}{p}} \Gamma_{\mathcal{E}}\langle v_k\rangle(K)^{\frac{1}{p}} \qquad \text{(by Proposition 4.8)}\\
&= \mathcal{E}(\Psi_l(\boldsymbol{v}))^{\frac{p-1}{p}} \sum_{k=1}^{n} \|\partial_k\Psi_l\|_{\sup,Q(\boldsymbol{v})} \mathcal{E}(v_k)^{1/p} \qquad \text{(by Proposition 5.10-(a))},
\end{aligned}$$

which implies that $\sup_{l\in\mathbb{N}} \mathcal{E}(\Psi_l(\boldsymbol{v})) < \infty$. Also, by $\Psi_l(0) = 0$, (5.28) and the dominated convergence theorem, $\{\Psi_l(\boldsymbol{v})\}_{l\in\mathbb{N}}$ converges in $L^p(K, m)$ to $\Psi(\boldsymbol{v})$ as $l \to \infty$. Now we conclude from Lemma 3.17 applied to $\bigl(\mathcal{E}, \overline{\mathcal{F} \cap C(K)}^{\mathcal{F}\cap L^p(K,m)}\bigr)$, which clearly satisfies (Cla)$_p$, that $\Psi(\boldsymbol{v}) \in \overline{\mathcal{F} \cap C(K)}^{\mathcal{F}\cap L^p(K,m)} \cap C(K) = \mathcal{F} \cap C(K)$, completing the proof. □

In the following corollaries, we recall useful consequences of the chain rule in Theorem 5.12, which are immediate from Proposition 4.16 (or more precisely, (4.36)), Theorems 4.17 and 4.18.

Corollary 5.13. *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$ and that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies (2.4) and (Cla)$_p$. Then for any $u, \varphi \in \mathcal{F} \cap C(K)$,*

$$\int_K \varphi\, d\Gamma_{\mathcal{E}}\langle u\rangle = \mathcal{E}(u; u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \mathcal{E}\bigl(|u|^{\frac{p}{p-1}}; \varphi\bigr). \tag{5.29}$$

Proof. For any $u, \varphi \in \mathcal{F} \cap C(K)$, since $u\varphi \in \mathcal{F} \cap C(K)$ by (5.18), we have (5.24) with either of $(u, u\varphi)$ and $\bigl(|u|^{\frac{p}{p-1}}, \varphi\bigr)$ in place of $(\Phi(u), \Psi(\boldsymbol{v}))$ by the second paragraph of the above proof of Theorem 5.12, and therefore (5.29) follows from Proposition 5.10-(a) and the argument in (4.36). □

Corollary 5.14. *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$, that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies (2.4) and $(\mathrm{Cla})_p$, and that $(\mathcal{F} \cap L^p(K, m), \|\cdot\|_{\mathcal{E},1})$ is a Banach space. Then, for any $u \in \mathcal{F} \cap C(K)$, the Borel measure $\Gamma_{\mathcal{E}}\langle u\rangle \circ u^{-1}$ on $\mathbb{R}$ defined by $(\Gamma_{\mathcal{E}}\langle u\rangle \circ u^{-1})(A) \coloneqq \Gamma_{\mathcal{E}}\langle u\rangle(u^{-1}(A))$, $A \in \mathcal{B}(\mathbb{R})$, is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Corollary 5.15. *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$, that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies (2.4) and $(\mathrm{Cla})_p$, and that $(\mathcal{F} \cap L^p(K, m), \|\cdot\|_{\mathcal{E},1})$ is a Banach space. Let $u, u_1, u_2, v \in \mathcal{F} \cap C(K)$, $a, a_1, a_2, b \in \mathbb{R}$ and $A \in \mathcal{B}(K)$.*

(a) *If $A \subseteq u^{-1}(a)$, then $\Gamma_{\mathcal{E}}\langle u\rangle(A) = 0$.*

(b) *If $A \subseteq (u-v)^{-1}(a)$, then $\Gamma_{\mathcal{E}}\langle u\rangle(A) = \Gamma_{\mathcal{E}}\langle v\rangle(A)$.*

(c) *If $A \subseteq u_1^{-1}(a_1) \cup u_2^{-1}(a_2)$, then*

$$\Gamma_{\mathcal{E}}\langle u_1 + u_2 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A), \tag{5.30}$$

$$\Gamma_{\mathcal{E}}\langle u_1 + u_2; v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_1; v\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2; v\rangle(A). \tag{5.31}$$

(d) *If $A \subseteq (u_1 - u_2)^{-1}(a) \cup v^{-1}(b)$, then*

$$\Gamma_{\mathcal{E}}\langle u_1; v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_2; v\rangle(A) \quad \textit{and} \quad \Gamma_{\mathcal{E}}\langle v; u_1\rangle(A) = \Gamma_{\mathcal{E}}\langle v; u_2\rangle(A). \tag{5.32}$$

5.3 Extensions of self-similar p-energy measures

In this subsection, we fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ with K connected, a self-similar measure m on $\mathcal{L}$, $p \in (1, \infty)$, and a self-similar p-energy form $(\mathcal{E}, \mathcal{F})$ on $(\mathcal{L}, m)$ with weight $(\rho_i)_{i\in S} \in (0, \infty)^S$, and further assume that $\mathcal{F} \subseteq L^p(K, m)$. In this setting, we first discuss the extension of self-similar p-energy measures to $\overline{\mathcal{F} \cap C(K)}^{\mathcal{F}} =: \mathcal{F}^0$. Recall the feature noted in Remark 5.5 of m as a self-similar measure on $\mathcal{L}$.

Lemma 5.16. *Assume that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is a Banach space. Let $u \in \mathcal{F}$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \cap C(K)$. If $\{u_n\}_{n\in\mathbb{N}}$ converges in $\mathcal{F}$ to u, then $\{u_n \circ F_w\}_{n\in\mathbb{N}}$ converges in $\mathcal{F}$ to $u \circ F_w$ for any $w \in W_*$. In particular,*

$$u \circ F_w \in \mathcal{F}^0 \quad \textit{for any } u \in \mathcal{F}^0 \textit{ and any } w \in W_*. \tag{5.33}$$

$$\mathcal{E}(u) = \sum_{i\in S} \rho_i \mathcal{E}(u \circ F_i) \quad \textit{for any } u \in \mathcal{F}^0. \tag{5.34}$$

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ satisfy $\lim_{n\to\infty} \|u - u_n\|_{\mathcal{E},1} = 0$. Then we easily see from the self-similarity of m that $\{u_n \circ F_w\}_{n\in\mathbb{N}}$ converges in $L^p(K, m)$ to $u \circ F_w$ for any $w \in W_*$. Since $\mathcal{E}(u_n \circ F_w - u_k \circ F_w) \leq \rho_w^{-1}\mathcal{E}(u_n - u_k)$ for any $n, k \in \mathbb{N}$ by (5.6), $\{u_n \circ F_w\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{F}$. Therefore, it has to converge to $u \circ F_w$ in $\mathcal{F}$, which shows (5.33). By letting $n \to \infty$ in (5.6) for u_n, we obtain (5.34). $\square$

Now that we have obtained the identity (5.34), in a similar way using Kolmogorov's extension theorem as in the previous subsection, for each $u \in \mathcal{F}^0$ we get a unique Borel measure $\mathfrak{m}_{\mathcal{E}}\langle u\rangle$ on Σ such that $\mathfrak{m}_{\mathcal{E}}\langle u\rangle(\Sigma_w) = \rho_w \mathcal{E}(u \circ F_w)$ for any $w \in W_*$. The following lemma states the triangle inequality for $\mathfrak{m}_{\mathcal{E}}\langle\,\cdot\,\rangle(A)^{1/p}$ on $\mathcal{F}^0$.

Lemma 5.17. *Assume that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is a Banach space. Then for any $u, v \in \mathcal{F}^0$ and any $A \in \mathcal{B}(\Sigma)$,*

$$\mathfrak{m}_{\mathcal{E}}\langle u+v\rangle(A)^{1/p} \leq \mathfrak{m}_{\mathcal{E}}\langle u\rangle(A)^{1/p} + \mathfrak{m}_{\mathcal{E}}\langle v\rangle(A)^{1/p}.$$

Proof. This follows from the triangle inequality for $\mathcal{E}^{1/p}$ and the argument in the proof of Proposition 5.9-(a). □

Now we identify the p-energy measures $\{\Gamma_{\mathcal{E}}\langle u\rangle\}_{u \in \mathcal{F}^0}$, obtained by applying Proposition 4.11 to the measures defined in (5.11), as $\{\mathfrak{m}_{\mathcal{E}}\langle u\rangle \circ \chi^{-1}\}_{u \in \mathcal{F}^0}$.

Proposition 5.18. *Assume that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is a Banach space. Then for any $u \in \mathcal{F}^0$,*

$$\Gamma_{\mathcal{E}}\langle u\rangle = \mathfrak{m}_{\mathcal{E}}\langle u\rangle \circ \chi^{-1} \quad \textit{(as Borel measures on } K\textit{)}. \tag{5.35}$$

Proof. The equality (5.35) for $u \in \mathcal{F} \cap C(K)$ is obvious from the definition of $\Gamma_{\mathcal{E}}\langle u\rangle$ in (5.11). Then the desired assertion immediately follows from (4.19), Lemma 5.17 and $\sup_{A \in \mathcal{B}(\Sigma)} \mathfrak{m}_{\mathcal{E}}\langle u\rangle(A) \leq \mathcal{E}(u)$. □

We conclude this subsection by seeing that self-similar p-energy measures can be extended to *functions belonging locally to $\mathcal{F}^0$* in Definition 5.20 below. To this end, we need the following lemma.

Lemma 5.19 (Weak locality of self-similar p-energy measures; [MS23+, Lemma 9.6])**.** *Assume that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is a Banach space. Let U be an open subset of K. If $u, v \in \mathcal{F}^0$ satisfy $u = v$ m-a.e. on U, then $\Gamma_{\mathcal{E}}\langle u\rangle(U) = \Gamma_{\mathcal{E}}\langle v\rangle(U)$.*

Proof. The proof is exactly the same as [MS23+, Lemma 9.6], but we recall the details here for the reader's convenience. By the inner regularity of $\Gamma_{\mathcal{E}}\langle u\rangle$ and $\Gamma_{\mathcal{E}}\langle v\rangle$ (see, e.g., [Dud, Theorem 7.1.3]), it suffices to show $\Gamma_{\mathcal{E}}\langle u\rangle(A) = \Gamma_{\mathcal{E}}\langle v\rangle(A)$ for any closed subset A of U. Let d be a metric on K giving the original topology of K. By (5.3), we can choose $\delta \in (0, \mathrm{dist}_d(A, K \setminus U))$ and $N \in \mathbb{N}$ so that $\max_{w \in W_n} \mathrm{diam}(K_w, d) < \delta$ for any $n \geq N$. For $n \in \mathbb{N}$, define $C_n := \{w \in W_n \mid \Sigma_w \cap \chi^{-1}(A) \neq \emptyset\}$. Since $u \circ F_w = v \circ F_w$ (m-a.e. on K) for any $n \geq N$ and any $w \in C_n$, we have

$$\mathfrak{m}_{\mathcal{E}}\langle u\rangle(\Sigma_{C_n}) = \sum_{w \in C_n} \rho_w \mathcal{E}(u \circ F_w) = \sum_{w \in C_n} \rho_w \mathcal{E}(v \circ F_w) = \mathfrak{m}_{\mathcal{E}}\langle v\rangle(\Sigma_{C_n}).$$

Since $\{\Sigma_{C_n}\}_{n \in \mathbb{N}}$ is a decreasing sequence satisfying $\bigcap_{n \in \mathbb{N}} \Sigma_{C_n} = \chi^{-1}(A)$ (see [Hin05, Proof of Lemma 4.1] or [MS23+, Proof of Proposition 9.3]), we obtain $\Gamma_{\mathcal{E}}\langle u\rangle(A) = \Gamma_{\mathcal{E}}\langle v\rangle(A)$ by letting $n \to \infty$ in the equality above. □

Definition 5.20. Let U be a non-empty open subset of K.

(1) For each linear subspace $\mathcal{D}$ of $\mathcal{F}$, we define a linear subspace $\mathcal{D}_{\mathrm{loc}}(U)$ of $L^0(U, m|_U)$ by

$$\mathcal{D}_{\mathrm{loc}}(U) := \left\{ f \in L^0(U, m|_U) \,\middle|\, \begin{array}{l} f = f^{\#} \ m\text{-a.e. on } V \text{ for some } f^{\#} \in \mathcal{D} \text{ for} \\ \text{each relatively compact open subset } V \text{ of } U \end{array} \right\}. \tag{5.36}$$

(2) Assume that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is a Banach space. In this setting, for each $f \in (\mathcal{F}^0)_{\mathrm{loc}}(U) =: \mathcal{F}^0_{\mathrm{loc}}(U)$, we further define a Radon measure $\Gamma_{\mathcal{E}}\langle f\rangle$ on U as follows. We first define $\Gamma_{\mathcal{E}}\langle f\rangle(E) := \Gamma_{\mathcal{E}}\langle f^{\#}\rangle(E)$ for each relatively compact Borel subset E of U, with $A \subseteq U$ and $f^{\#} \in \mathcal{F}^0$ as in (5.36) chosen so that $E \subseteq A$; this definition of $\Gamma_{\mathcal{E}}\langle f\rangle(E)$ is independent of a particular choice of such A and $f^{\#}$ by Lemma 5.19. We then define $\Gamma_{\mathcal{E}}\langle f\rangle(E) := \lim_{n\to\infty} \Gamma_{\mathcal{E}}\langle f\rangle(E \cap A_n)$ for each $E \in \mathcal{B}|_U$, where $\{A_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence of relatively compact open subsets of U such that $\bigcup_{n\in\mathbb{N}} A_n = U$; it is clear that this definition of $\Gamma_{\mathcal{E}}\langle f\rangle(E)$ is independent of a particular choice of $\{A_n\}_{n\in\mathbb{N}}$, coincides with the previous one when E is relatively compact in U, and gives a Radon measure on U.

5.4 Self-similar p-energy form as a fixed point

This subsection is devoted to presenting a standard method to construct a self-similar p-energy form. The main result of this subsection (Theorem 5.22) is essentially the same as the fixed point theorem in [Kig00, Theorem 1.5], but we present the details to show a useful version of this fixed point theorem where a fixed point is explicitly given as a limit.

In this subsection, we fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ with K connected, a self-similar measure m on $\mathcal{L}$, $p \in (1,\infty)$, and a linear subspace $\mathcal{F}$ of $L^p(K,m)$ with the following property:

$$u \circ F_w \in \mathcal{F} \quad \text{for any } u \in \mathcal{F} \text{ and any } w \in W_* \tag{5.37}$$

(recall Remark 5.5). We define

$$\mathfrak{E}_p(\mathcal{F}) := \{\mathcal{E}\colon \mathcal{F} \to [0,\infty) \mid (\mathcal{E},\mathcal{F}) \text{ is a } p\text{-energy form on } (K,m)\}.$$

Definition 5.21. Let $\boldsymbol{\rho} = (\rho_i)_{i\in S}$. For $n \in \mathbb{N} \cup \{0\}$, we define $\mathcal{S}_{\boldsymbol{\rho},n}\colon \mathfrak{E}_p(\mathcal{F}) \to \mathfrak{E}_p(\mathcal{F})$ by

$$\mathcal{S}_{\boldsymbol{\rho},n}(E)(u) := \sum_{w\in W_n} \rho_w E(u \circ F_w) \quad \text{for } E \in \mathfrak{E}_p(\mathcal{F}) \text{ and } u \in \mathcal{F}. \tag{5.38}$$

(Note that the triangle inequality for $\mathcal{S}_{\boldsymbol{\rho},n}(E)^{1/p}$ can be shown easily.) Set $\mathcal{S}_{\boldsymbol{\rho}} := \mathcal{S}_{\boldsymbol{\rho},1}$ and $\mathcal{S}_{\boldsymbol{\rho},0} := \mathrm{id}_{\mathfrak{E}_p(\mathcal{F})}$ for simplicity. Clearly, $\mathcal{S}_{\boldsymbol{\rho},n} = \mathcal{S}_{\boldsymbol{\rho}}^n := \underbrace{\mathcal{S}_{\boldsymbol{\rho}} \circ \mathcal{S}_{\boldsymbol{\rho}} \circ \cdots \circ \mathcal{S}_{\boldsymbol{\rho}}}_{n}$.

The desired self-similar p-energy form with weight $\boldsymbol{\rho}$ will be constructed as a non-trivial fixed point of $\mathcal{S}_{\boldsymbol{\rho}}$. The following theorem, which can be regarded as a version of [Kig00, Theorem 1.5] in a specific situation, describes when we can find such a fixed point and how it is obtained.

Theorem 5.22. *Let $\boldsymbol{\rho} = (\rho_i)_{i\in S}$ and let $\mathcal{E}^0 \in \mathfrak{E}_p(\mathcal{F})$. Assume that the quotient normed space $\mathcal{F}/(\mathcal{E}^0)^{-1}(0)$ (equipped with the norm $\mathcal{E}^0(\,\cdot\,)^{1/p}$) is separable and that there exists a constant $C \in [1,\infty)$ such that*

$$C^{-1}\mathcal{E}^0(u) \le \mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u) \le C\mathcal{E}^0(u) \quad \text{for any } u \in \mathcal{F} \text{ and any } n \in \mathbb{N}. \tag{5.39}$$

Then there exists $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k < n_{k+1}$ for any $k \in \mathbb{N}$ such that the following limit exists in $[0,\infty)$ for any $u \in \mathcal{F}$:

$$\mathcal{E}(u) := \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \mathcal{S}_{\boldsymbol{\rho},j}(\mathcal{E}^0)(u). \tag{5.40}$$

Furthermore, $(\mathcal{E},\mathcal{F})$ is a p-energy form on (K,m) satisfying

$$C^{-1}\mathcal{E}^0(u) \leq \mathcal{E}(u) \leq C\mathcal{E}^0(u) \quad \textit{for any } u \in \mathcal{F} \textit{ and any } n \in \mathbb{N}\cup\{0\}, \tag{5.41}$$

where C is the constant in (5.39), and

$$\mathcal{E}(u) = \sum_{w\in W_n} \rho_w \mathcal{E}(u \circ F_w) \quad \textit{for any } u \in \mathcal{F} \textit{ and any } n \in \mathbb{N}\cup\{0\}. \tag{5.42}$$

Proof. Set $\mathcal{E}^n := \frac{1}{n}\sum_{j=0}^{n-1} \mathcal{S}_{\boldsymbol{\rho},j}(\mathcal{E}^0)$ for $n \in \mathbb{N}$ for ease of notation. Then it is clear that $\mathcal{E}^n \in \mathfrak{E}_p(\mathcal{F})$. Let $\mathscr{C}$ be a countable dense subset of $\mathcal{F}/(\mathcal{E}^0)^{-1}(0)$. Since $\{\mathcal{E}^n(u)\}_{n\in\mathbb{N}}$ is bounded in $[0,\infty)$ for any $u \in \mathcal{F}$ by (5.39), by a standard diagonal procedure, there exists $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k < n_{k+1}$ for any $k \in \mathbb{N}$ such that $\{\mathcal{E}^{n_k}(u')\}_{k\in\mathbb{N}}$ is convergent in $[0,\infty)$ for any $u' \in \mathscr{C}$. Let $u \in \mathcal{F}$, $\varepsilon > 0$ and $u_* \in \mathscr{C}$ satisfy $\mathcal{E}^0(u-u_*)^{1/p} < \varepsilon$. Then for any $k,l \in \mathbb{N}$, by the triangle inequality for $\mathcal{E}^n(\,\cdot\,)^{1/p}$ and (5.39),

$$\begin{aligned}
&\left|\mathcal{E}^{n_k}(u)^{1/p} - \mathcal{E}^{n_l}(u)^{1/p}\right| \\
&\leq \left|\mathcal{E}^{n_k}(u)^{1/p} - \mathcal{E}^{n_k}(u_*)^{1/p}\right| + \left|\mathcal{E}^{n_k}(u_*)^{1/p} - \mathcal{E}^{n_l}(u_*)^{1/p}\right| + \left|\mathcal{E}^{n_l}(u)^{1/p} - \mathcal{E}^{n_l}(u_*)^{1/p}\right| \\
&\leq 2C^{1/p}\varepsilon + \left|\mathcal{E}^{n_k}(u)^{1/p} - \mathcal{E}^{n_l}(u)^{1/p}\right|,
\end{aligned}$$

whence $\limsup_{k\wedge l\to\infty} \left|\mathcal{E}^{n_k}(u)^{1/p} - \mathcal{E}^{n_l}(u)^{1/p}\right| \leq 2C^{1/p}\varepsilon$. Therefore $\{\mathcal{E}^{n_k}(u)\}_{k\in\mathbb{N}}$ is convergent in $[0,\infty)$ for any $u \in \mathcal{F}$, so the limit in (5.40) exists. It is clear that $(\mathcal{E},\mathcal{F})$ is a p-energy form on (K,m) satisfying (5.41).

Let us show (5.42). For any $n \in \mathbb{N}$ and any $u \in \mathcal{F}$, we easily see that

$$\frac{1}{n}\mathcal{E}^0(u) + \mathcal{S}_{\boldsymbol{\rho}}(\mathcal{E}^n)(u) = \frac{1}{n}\mathcal{E}^0(u) + \frac{1}{n}\sum_{l=0}^{n-1} \mathcal{S}_{\boldsymbol{\rho},l+1}(\mathcal{E}^0)(u) = \mathcal{E}^n(u) + \frac{1}{n}\mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u). \tag{5.43}$$

Since $\lim_{k\to\infty} \mathcal{S}_{\rho}(\mathcal{E}^{n_k})(u) = \mathcal{S}_{\rho}(\mathcal{E})(u)$ and $\lim_{k\to\infty} n_k^{-1}\mathcal{S}_{\boldsymbol{\rho},n_k}(\mathcal{E}^0)(u) = 0$ by (5.39), we obtain $\mathcal{S}_{\boldsymbol{\rho}}(\mathcal{E}) = \mathcal{E}$ by letting $n \to \infty$ along $\{n_k\}_{k\in\mathbb{N}}$ in (5.43). Hence (5.42) holds. $\square$

By virtue of the explicit representation (5.42), the resulting p-energy form $(\mathcal{E},\mathcal{F})$ inherits some nice properties of $(\mathcal{E}^0,\mathcal{F})$. In the following proposition, we see that (GC)$_p$ and the invariance under good transformations are examples of such properties.

Proposition 5.23. *Assume the same conditions as in Theorem 5.22 and let $\mathcal{E}$ be given by (5.40).*

(a) *If $(\mathcal{E}^0,\mathcal{F})$ satisfies (GC)$_p$, then $(\mathcal{E},\mathcal{F})$ also satisfies (GC)$_p$.*

(b) *Let $\mathscr{T}$ be a family of Borel measurable maps from K to K. Assume that $u \circ T \in \mathcal{F}$ and $\mathcal{E}^0(u \circ T) = \mathcal{E}^0(u)$ for any $u \in \mathcal{F}$ and any $T \in \mathscr{T}$. Furthermore, we assume that for any $T \in \mathscr{T}$, there exists a bijection $\tau_T\colon W_* \to W_*$ such that*

$$\tau_T|_{W_n} \text{ is a bijection from } W_n \text{ to itself for each } n \in \mathbb{N} \cup \{0\}, \tag{5.44}$$

$$T(K_w) \subseteq K_{\tau_T(w)} \quad \text{and} \quad F_{\tau_T(w)}^{-1} \circ T \circ F_w \in \mathscr{T} \quad \text{for any } w \in W_*, \tag{5.45}$$

and

$$\rho_w = \rho_{\tau_T(w)} \quad \text{for any } w \in W_*. \tag{5.46}$$

Then $\mathcal{E}(u \circ T) = \mathcal{E}(u)$ for any $u \in \mathcal{F}$ and any $T \in \mathscr{T}$.

Proof. (a): Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$ and $T = (T_1, \dots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2). Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}$. Then $T_l(u_k \circ F_w) = T_l(u_k) \circ F_w \in \mathcal{F}$ for any $k \in \{1, \dots, n_1\}$ and any $w \in W_*$ by (GC)_p for $(\mathcal{E}^0, \mathcal{F})$ and Lemma 5.39. If $q_2 < \infty$, then by a similar estimate as (2.19),

$$\begin{aligned}
&\sum_{l=1}^{n_2} \mathcal{S}_\rho(\mathcal{E}^0)\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} = \sum_{l=1}^{n_2} \left[\sum_{i \in S} \rho_i \mathcal{E}^0(T_l(\boldsymbol{u}) \circ F_i)\right]^{q_2/p} \\
&\le \left(\sum_{i \in S} \rho_i \left[\sum_{l=1}^{n_2} \mathcal{E}^0\bigl(T_l(\boldsymbol{u}) \circ F_i\bigr)^{q_2/p}\right]^{p/q_2}\right)^{q_2/p} \qquad \text{(by the triangle ineq. for } \|\cdot\|_{\ell^{q_2/p}}\text{)} \\
&\overset{\text{(GC)}_p}{\le} \left(\sum_{i \in S} \rho_i \left[\sum_{k=1}^{n_1} \mathcal{E}^0(u_k \circ F_i)^{q_1/p}\right]^{p/q_1}\right)^{q_2/p} \\
&\overset{(2.18)}{\le} \left(\sum_{k=1}^{n_1} \left[\sum_{i \in S} \rho_i \mathcal{E}^0(u_k \circ F_i)\right]^{q_1/p}\right)^{\frac{p}{q_1} \cdot \frac{q_2}{p}} = \left(\sum_{k=1}^{n_1} \mathcal{S}_\rho(\mathcal{E}^0)(u_k)^{q_1/p}\right)^{q_2/q_1},
\end{aligned}$$

whence $\bigl\|\bigl(\mathcal{S}_\rho(\mathcal{E}^0)(T_l(\boldsymbol{u}))^{1/p}\bigr)_{l=1}^{n_2}\bigr\|_{\ell^{q_2}} \le \bigl\|\bigl(\mathcal{S}_\rho(\mathcal{E}^0)(u_k)^{1/p}\bigr)_{k=1}^{n_1}\bigr\|_{\ell^{q_1}}$. The case of $q_2 = \infty$ is similar, so $(\mathcal{S}_\rho(\mathcal{E}^0), \mathcal{F})$ satisfies (GC)_p. Similarly, one can easily show that $(\mathcal{S}_{\rho,n}(\mathcal{E}^0), \mathcal{F})$ satisfies (GC)_p for any $n \in \mathbb{N}$. Hence (GC)_p for $(\mathcal{E}, \mathcal{F})$ holds by (5.42) and Proposition 2.10-(b).

(b): By (5.42), it suffices to prove $\mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u \circ T) = \mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u)$ for any $u \in \mathcal{F}$, any $n \in \mathbb{N} \cup \{0\}$ and any $T \in \mathscr{T}$. We immediately see that

$$\begin{aligned}
\mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u \circ T) &= \sum_{w \in W_n} \rho_w \mathcal{E}^0((u \circ T) \circ F_w) \\
&= \sum_{w \in W_n} \rho_w \mathcal{E}^0\bigl((u \circ F_{\tau_T(w)}) \circ F_{\tau_T(w)}^{-1} \circ T \circ F_w\bigr) \\
&\overset{(5.45)}{=} \sum_{w \in W_n} \rho_w \mathcal{E}^0(u \circ F_{\tau_T(w)}) \overset{(5.44),(5.46)}{=} \mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u),
\end{aligned}$$

which completes the proof. $\square$

Also, $(\mathcal{E},\mathcal{F})$ in Theorem 5.22 turns out to be strongly local under a mild condition.

Proposition 5.24. *Assume the same conditions as in Theorem 5.22 and let $\mathcal{E}$ be given by (5.42). If $\{u \in \mathcal{F} \mid \mathcal{E}^0(u) = 0\} = \mathbb{R}\mathbb{1}_K$, then $\{u \in \mathcal{F} \mid \mathcal{E}(u) = 0\} = \mathbb{R}\mathbb{1}_K$ and $(\mathcal{E},\mathcal{F})$ satisfies the strong local property (SL1).*

Proof. It is immediate from (5.41) that $\{u \in \mathcal{F} \mid \mathcal{E}(u) = 0\} = \mathbb{R}\mathbb{1}_K$. We will show (SL1) for $(\mathcal{E},\mathcal{F})$. Let $u_1, u_2, v \in \mathcal{F}$ and $a_1, a_2 \in \mathbb{R}$. Set $A_i := \operatorname{supp}_m[u_i - a_i\mathbb{1}_K]$ for $i \in \{1,2\}$ and assume that $A_1 \cap A_2 = \emptyset$. By (5.3), there exists $n \in \mathbb{N}$ such that $(\bigcup_{w\in W_n[A_1]} K_w) \cap (\bigcup_{w\in W_n[A_2]} K_w) = \emptyset$, where $W_n[A_i] := \{w \in W_n \mid K_w \cap A_i \neq \emptyset\}$. Note that $u_i \circ F_w = a_i\mathbb{1}_K$ for $w \in W_n \setminus W_n[A_i]$. This together with $\mathcal{E}(\mathbb{1}_K) = 0$ and (5.42) yields that

$$
\begin{aligned}
&\mathcal{E}(u_1 + u_2 + v) \\
&= \sum_{w\in W_n[A_1]} \rho_w \mathcal{E}(u_1 \circ F_w + v \circ F_w) + \sum_{w\in W_n[A_2]} \rho_w \mathcal{E}(u_2 \circ F_w + v \circ F_w) \\
&\qquad\qquad\qquad\qquad + \sum_{w\in W_n\setminus(W_n[A_1]\cup W_n[A_2])} \rho_w \mathcal{E}(v \circ F_w) \\
&= \mathcal{E}(u_1 + v) + \mathcal{E}(u_2 + v) - \sum_{w\in W_n\setminus W_n[A_1]} \rho_w \mathcal{E}(v \circ F_w) - \sum_{w\in W_n\setminus W_n[A_2]} \rho_w \mathcal{E}(v \circ F_w) \\
&\qquad\qquad\qquad\qquad + \sum_{w\in W_n\setminus(W_n[A_1]\cup W_n[A_2])} \rho_w \mathcal{E}(v \circ F_w) \\
&= \mathcal{E}(u_1 + v) + \mathcal{E}(u_2 + v) - \mathcal{E}(v),
\end{aligned}
$$

which shows (SL1). □

6 p-Resistance forms and nonlinear potential theory

In this section, we will introduce the notion of p-resistance form as a special class of p-energy forms, and investigate harmonic functions with respect to a p-resistance form. In particular, we prove fundamental results on taking the operation of traces of p-resistance forms, weak comparison principle and Hölder continuity estimates for harmonic functions. We also show the elliptic Harnack inequality for non-negative harmonic functions under some assumptions, and introduce the notion of p-resistance metric with respect to a given p-resistance form.

Throughout this section, we fix $p \in (1,\infty)$, a non-empty set X, a linear subspace $\mathcal{F}$ of $\mathbb{R}^X$ and $\mathcal{E} \colon \mathcal{F} \to [0,\infty)$. (This setting corresponds to choosing as $(\mathcal{B}, m)$ the pair of 2^X and the counting measure on X in the previous sections; recall Remark 2.1.)

6.1 Basics of p-resistance forms

The next definition is an L^p-analogue of the notion of *resistance form* introduced by Kigami in [Kig95]; see [Kig01, Kig03, Kig12] for details on resistance forms.

Definition 6.1 (p-Resistance form)**.** The pair $(\mathcal{E}, \mathcal{F})$ of $\mathcal{F} \subseteq \mathbb{R}^X$ and $\mathcal{E} \colon \mathcal{F} \to [0, \infty)$ is said to be a *p-resistance form* on X if and only if it satisfies the following conditions $(\mathrm{RF1})_p$-$(\mathrm{RF5})_p$:

$(\mathrm{RF1})_p$ $\mathcal{F}$ is a linear subspace of $\mathbb{R}^X$ containing $\mathbb{R}\mathbb{1}_X$ and $\mathcal{E}(\,\cdot\,)^{1/p}$ is a seminorm on $\mathcal{F}$ satisfying $\{u \in \mathcal{F} \mid \mathcal{E}(u) = 0\} = \mathbb{R}\mathbb{1}_X$.

$(\mathrm{RF2})_p$ The quotient normed space $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$ is a Banach space.

$(\mathrm{RF3})_p$ If $x \neq y \in X$, then there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

$(\mathrm{RF4})_p$ For any $x, y \in X$,

$$R_{\mathcal{E}}(x, y) \coloneqq R_{(\mathcal{E},\mathcal{F})}(x, y) \coloneqq \sup\left\{ \frac{|u(x) - u(y)|^p}{\mathcal{E}(u)} \;\middle|\; u \in \mathcal{F} \setminus \mathbb{R}\mathbb{1}_X \right\} < \infty. \tag{6.1}$$

$(\mathrm{RF5})_p$ $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{GC})_p$.

Remark 6.2. (1) The notion of 2-resistance form coincides with the original notion of resistance form ([Kig01, Definition 2.3.1]) although the condition $(\mathrm{RF5})_2$ is stronger than (RF5) in [Kig01, Definition 2.3.1]. Indeed, we can obtain $(\mathrm{RF5})_2$ by [Kig12, Theorem 3.14] and the explicit definition of $\mathcal{E}_{L_m}$ in [Kig12, Proposition 3.8].

(2) Let $(\mathcal{E}, \mathcal{F})$ be a p-resistance form on a finite set V. Then $\mathcal{F} = \mathbb{R}^V$ by $(\mathrm{RF1})_p$, $(\mathrm{RF3})_p$ and $(\mathrm{RF5})_p$ (see also [Kig12, Proposition 3.2]), so we say simply that $\mathcal{E}$ is a p-resistance form on V if V is a finite set.

Example 6.3. (1) Consider the same setting as in Example 3.11-(1) and assume that Ω is a bounded domain satisfying the strong local Lipschitz condition (see [AF, Paragraph 4.9]). Then the p-energy form $(\int_\Omega |\nabla f|^p \, dx, W^{1,p}(\Omega))$ is a p-resistance form on Ω if and only if $p > D$. Indeed, $(\mathrm{RF1})_p$ and $(\mathrm{RF5})_p$ are clear from the definition (we used the boundedness of Ω to ensure $\mathbb{R}\mathbb{1}_\Omega \subseteq L^p(\Omega)$), $(\mathrm{RF2})_p$ and $(\mathrm{RF3})_p$ follow from [AF, Theorem 3.3 and Corollary 3.4] for any $p \in (1, \infty)$. If $p > D$, then we can use the Morrey-type inequality [AF, Lemma 4.28] to verify $(\mathrm{RF4})_p$. Conversely, the supremum in (6.1) is not finite when $p \leq D$. To see it, we can assume that $x = 0 \in \Omega$. Let $\delta \in (0, \infty)$ be small enough so that $\overline{B(0, \delta)} \subseteq \Omega$ and $y \notin \overline{B(0, \delta)}$. For all large $n \in \mathbb{N}$ so that $n^{-1} < \delta$, define $u_n \in C(\Omega)$ by

$$u_n(z) \coloneqq \left(\frac{\log |z|^{-1} - \log \delta^{-1}}{\log n - \log \delta^{-1}} \right)^+ \wedge 1, \quad z \in \Omega.$$

Then we easily see that $u_n(0) = 1$, $u_n(y) = 1$ and $u_n \in W^{1,p}(\Omega)$ with

$$\begin{aligned} \int_\Omega |\nabla u_n|^p \, dz &\leq \left| \frac{1}{\log (n\delta)} \right|^p \int_{B(0,\delta) \setminus B(0,n^{-1})} |z|^{-p} \, dz = |S_{D-1}| \left| \frac{1}{\log (n\delta)} \right|^p \int_{\frac{1}{n}}^{\delta} r^{-p+D-1} \, dr \\ &= \begin{cases} |S_{D-1}| \, |\log (n\delta)|^{-(p-1)} & \text{if } p = D, \\ \frac{|S_{D-1}|}{D-p} \, |\log (n\delta)|^{-p} \left(\delta^{D-p} - n^{-(D-p)} \right) & \text{if } p < D, \end{cases} \end{aligned}$$

where $|S_{D-1}|$ is the volume of the $(D-1)$-dimensional unit sphere. In particular, $\frac{|u_n(x) - u_n(y)|^p}{\| |\nabla u_n| \|_{L^p(\Omega)}^p} \to \infty$ as $n \to \infty$, so $(\mathrm{RF4})_p$ does not hold.

(2) The construction of a regular p-energy form on a p-conductively homogeneous compact metric space (K, d) in [Kig23, Theorem 3.21] requires the assumption $p > \dim_{\mathrm{ARC}}(K, d)$, where $\dim_{\mathrm{ARC}}(K, d)$ is the Ahlfors regular conformal dimension of (K, d). (See Definition 8.5-(4) for the definition of $\dim_{\mathrm{ARC}}(K, d)$. The same condition $p > \dim_{\mathrm{ARC}}(K, d)$ is also assumed in [Shi24].) This condition $p > \dim_{\mathrm{ARC}}(K, d)$ plays the same role as $p > D$ in (1) above (see also [CCK24, Theorem 1.1]). In Theorem 8.19, we will see that p-energy forms constructed in [Kig23, Theorem 3.21] are indeed p-resistance forms. We also show that p-energy forms on p.-c.f. self-similar sets in [CGQ22, Theorem 5.1] under the condition (**R**) in [CGQ22, p. 18] are p-resistance forms in Theorem 8.43.

(3) Here we recall typical p-resistance forms on finite sets given in [KS23+, Example 2.2-(1)] because these examples are important to construct self-similar p-resistance forms on p.-c.f. self-similar structures in Subsection 8.3. Let V be a non-empty finite set. Note that in this case $\mathcal{E}$ is a p-resistance form on V if and only if $\mathcal{E}\colon \mathbb{R}^V \to [0, \infty)$ satisfies (RF1)_p and (RF5)_p; indeed, (RF3)_p is obvious for $\mathcal{F} = \mathbb{R}^V$, (RF2)_p and (RF4)_p are easily implied by (RF1)_p and $\dim(\mathcal{F}/\mathbb{R}\mathbb{1}_V) < \infty$. Now, consider any functional $\mathcal{E}\colon \mathbb{R}^V \to [0, \infty)$ of the form

$$\mathcal{E}(u) = \frac{1}{2} \sum_{x,y \in V} L_{xy} \lvert u(x) - u(y) \rvert^p \tag{6.2}$$

for some $L = (L_{xy})_{x,y\in V} \in [0, \infty)^{V\times V}$ such that $L_{xy} = L_{yx}$ for any $x, y \in V$. It is obvious that $\mathcal{E}$ satisfies (RF1)_p if and only if the graph (V, E_L) is connected, where $E_L \coloneqq \{\{x, y\} \mid x, y \in V,\, x \neq y,\, L_{xy} > 0\}$. It is also easy to see that $\mathcal{E}$ satisfies (RF5)_p. It thus follows that $\mathcal{E}$ is a p-resistance form on V if and only if (V, E_L) is connected. Note that, while any 2-resistance form on V is of the form (6.2) with $p = 2$, *the counterpart of this fact for $p \neq 2$ is NOT true* unless $\#V \leq 2$.

In the rest of this section, we assume that $(\mathcal{E}, \mathcal{F})$ is a p-resistance form on X. Then the following proposition is immediate from the definition (6.1) of $R_{\mathcal{E}}$ and Theorem 3.24.

Proposition 6.4. (1) *For any $u \in \mathcal{F}$ and any $x, y \in X$,*

$$\lvert u(x) - u(y) \rvert^p \leq R_{\mathcal{E}}(x, y)\mathcal{E}(u). \tag{6.3}$$

(2) *$R_{\mathcal{E}}^{1/p}$ is a metric on X.*

(3) *$(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$ is a uniformly convex Banach space, and thus it is reflexive.*

In particular, the metric $R_{\mathcal{E}}^{1/p}$ induces a topology on X. Throughout the rest of this section, we consider X as a topological space with the topology induced by $R_{\mathcal{E}}^{1/p}$. Note that then $\mathcal{F} \subseteq C(X)$ by (6.3).

We introduce the regularity of p-resistance forms as follows.

Definition 6.5 (Regularity)**.** Assume that X is locally compact. We say that $(\mathcal{E}, \mathcal{F})$ is *regular* if and only if $\mathcal{F} \cap C_c(X)$ is dense in $(C_c(X), \|\cdot\|_{\mathrm{sup}})$.

The regularity ensures the existence of cutoff functions.

Proposition 6.6. *Assume that X is locally compact and that $(\mathcal{E},\mathcal{F})$ is regular. Then for any open subset U of X and any compact subset K of U, there exists $\psi \in \mathcal{F} \cap C_c(X)$ such that $0 \leq \psi \leq 1$, $\psi = 1$ on an open neighborhood of K and $\operatorname{supp}_X[\psi] \subseteq U$. In particular, $\mathcal{F} \cap C_c(X)$ is a special core.*

Proof. Since X is locally compact, we can pick an open subset Ω of X such that $K \subseteq \Omega$, $\overline{\Omega}^X \subseteq U$ and $\overline{\Omega}^X$ is compact. By Urysohn's lemma, there exists $\psi_0 \in C_c(X)$ such that $0 \leq \psi_0 \leq 1$, $\psi_0 = 1$ on Ω and $\operatorname{supp}_X[\psi_0] \subseteq U$. Since $(\mathcal{E},\mathcal{F})$ is regular, for any $\varepsilon \in (0,1/2)$ there exists $\psi_\varepsilon \in \mathcal{F} \cap C_c(X)$ such that $\|\psi_0 - \psi_\varepsilon\|_{\sup} < \varepsilon$, and then the function $\psi := \bigl[(1-2\varepsilon)^{-1}(\psi_\varepsilon - \varepsilon)^+\bigr] \wedge 1$ belongs to $\mathcal{F} \cap C_c(X)$ by (RF1)$_p$ and Proposition 2.3-(b) and has the desired properties. □

We need the following notation to define traces of a p-resistance form later.

Definition 6.7. Let B be a non-empty subset of X. Define a linear subspace $\mathcal{F}|_B$ of $\mathcal{F}$ by $\mathcal{F}|_B := \bigl\{u|_B \bigm| u \in \mathcal{F}\bigr\}$.

The following proposition is useful to discuss boundary conditions on finite sets.

Proposition 6.8. *Let B be a non-empty finite subset of X. Then $\mathcal{F}|_B = \mathbb{R}^B$.*

Proof. By virtue of (RF1)$_p$, it suffices to show that $\mathbb{1}_x^B \in \mathcal{F}|_B$ for any $x \in B$ under the assumption that $\#B \geq 2$. Let $x \in B$. For each $y \in B \setminus \{x\}$, by (RF1)$_p$ and (RF2)$_p$, there exists $u_y \in \mathcal{F}$ satisfying $u_y(x) = 1$ and $u_y(y) = 0$. Let $f := \sum_{y \in B\setminus\{x\}}(u_y^+ \wedge 1)$ and $g := \sum_{y \in B\setminus\{x\}}\bigl((1-u_y)^+ \wedge 1\bigr)$. Then $f, g \in \mathcal{F}$ by (RF1)$_p$ and (RF5)$_p$. Since $f(x) = \#B - 1$, $f|_{B\setminus\{x\}} \leq \#B - 2$, $g(x) = 0$ and $g|_{B\setminus\{x\}} \geq 1$, the function $h \in \mathcal{F}$ given by

$$h := \bigl(f - (\#B-2)(g^+ \wedge 1)\bigr)^+ \wedge 1$$

satisfies $h|_B = \mathbb{1}_x^B$ and hence $\mathbb{1}_x^B \in \mathcal{F}|_B$. □

The next definition is introduced to deal with Dirichlet-type boundary conditions.

Definition 6.9. For a non-empty subset $B \subseteq X$, define

$$\mathcal{F}^0(B) := \{u \in \mathcal{F} \mid u(x) = 0 \text{ for any } x \in X \setminus B\}, \quad B^{\mathcal{F}} := \bigcap_{u \in \mathcal{F}^0(X\setminus B)} u^{-1}(0).$$

For basic properties of $B^{\mathcal{F}}$, see [Kig12, Chapters 2, 5 and 6]. Here we only recall the following results, which will be used later.

Proposition 6.10 ([Kig12, Theorems 2.5 and 6.3])**.** *Let B be a non-empty subset of X.*

(a) $\mathbb{C}_{\mathcal{F}} := \bigl\{B \bigm| B \subseteq X, B = B^{\mathcal{F}}\bigr\}$ *satisfies the axiom of closed sets and it defines a topology on X. Moreover, $\{x\} \in \mathbb{C}_{\mathcal{F}}$ for any $x \in X$.*

(b) *For any* $B \subseteq X$ *and* $x \notin B^{\mathcal{F}}$, *there exists* $u \in \mathcal{F}$ *such that* $u \in \mathcal{F}^0(X \setminus B)$, $u(x) = 1$ *and* $0 \leq u \leq 1$.

(c) *Assume that* X *is locally compact and that* $(\mathcal{E}, \mathcal{F})$ *is regular. Then* $B = B^{\mathcal{F}}$ *for any closed set* B *of* X.

Proof. The statements (a) and (b) follow from [Kig12, Theorem 2.4 and Lemma 2.5]. The argument showing (R1) ⇒ (R2) in [Kig12, Proof of Theorem 6.3] proves (c). □

For $B \subseteq X$ and $x \notin B^{\mathcal{F}}$, we define

$$R_{\mathcal{E}}(x, B) := R_{(\mathcal{E},\mathcal{F})}(x, B) := \sup\left\{ \frac{|u(x)|^p}{\mathcal{E}(u)} \,\middle|\, u \in \mathcal{F}^0(X \setminus B),\, u(x) \neq 0 \right\} < \infty. \tag{6.4}$$

Note that $R_{\mathcal{E}}(x, \{y\}) = R_{\mathcal{E}}(x, y)$ for $y \in X \setminus \{x\}$ by Proposition 6.10-(a).

6.2 Harmonic functions and traces of p-resistance forms

In this subsection, we consider harmonic functions with respect to p-resistance forms and traces of p-resistance forms to subsets of the original domains.

The following proposition states that the variational and distributional formulations of harmonic functions coincide for p-resistance forms.

Proposition 6.11. *Let* $h \in \mathcal{F}$ *and* $B \subseteq X$. *Then the following conditions are equivalent:*

(1) $\mathcal{E}(h) = \inf\{\mathcal{E}(u) \mid u \in \mathcal{F},\, u|_B = h|_B\}$.

(2) $\mathcal{E}(h; \varphi) = 0$ *for any* $\varphi \in \mathcal{F}^0(X \setminus B)$.

Proof. For $\varphi \in \mathcal{F}$, define $E_\varphi \colon \mathbb{R} \to \mathbb{R}$ by $E_\varphi(t) := \mathcal{E}(h + t\varphi)$, so that E_φ is differentiable by Proposition 3.6. If $\mathcal{E}(h) = \inf\{\mathcal{E}(u) \mid u \in \mathcal{F},\, u|_B = h|_B\}$ and $\varphi \in \mathcal{F}^0(X \setminus B)$, then E_φ takes its minimum at $t = 0$, and thus $\mathcal{E}(h; \varphi) = \frac{1}{p}\frac{d}{dt}E_\varphi(t)\big|_{t=0} = 0$, proving (1) ⇒ (2).

Conversely, assume that $\mathcal{E}(h; \varphi) = 0$ for any $\varphi \in \mathcal{F}^0(X \setminus B)$, and let $u \in \mathcal{F}$ satisfy $u|_B = h|_B$. Then by $u - h \in \mathcal{F}^0(X \setminus B)$ we have $\frac{d}{dt}\big|_{t=0}E_{u-h}(t) = p\mathcal{E}(h; u - h) = 0$, which together with the convexity of E_{u-h} implies that $\mathcal{E}(u) = E_{u-h}(1) \geq E_{u-h}(0) = \mathcal{E}(h)$, proving (2) ⇒ (1). □

Definition 6.12 ($\mathcal{E}$-(sub,super)harmonic function)**.** Let $B \subseteq X$ and $h \in \mathcal{F}$. We say that h is $\mathcal{E}$-*subharmonic* on $X \setminus B$ if and only if

$$\mathcal{E}(h; \varphi) \leq 0 \quad \text{for any } \varphi \in \mathcal{F}^0(X \setminus B) \text{ with } \varphi \geq 0. \tag{6.5}$$

We say that h is $\mathcal{E}$-*superharmonic* on $X \setminus B$ if and only if $-h$ is $\mathcal{E}$-subharmonic on $X \setminus B$, and say that h is $\mathcal{E}$-*harmonic* on $X \setminus B$ if and only if h is both $\mathcal{E}$-subharmonic and $\mathcal{E}$-superharmonic on $X \setminus B$, i.e., h satisfies either (and hence both) of (1) and (2) in Proposition 6.11. We set $\mathcal{H}_{\mathcal{E},B} := \{h \in \mathcal{F} \mid h \text{ is } \mathcal{E}\text{-harmonic on } X \setminus B\}$.

$\mathcal{E}$-harmonic functions with given boundary values uniquely exist, and their energies under $\mathcal{E}$ define a new p-resistance form on the boundary set, as follows. This new p-resistance form is called the *trace* of $(\mathcal{E}, \mathcal{F})$ on the boundary set.

Theorem 6.13. *Let $B \subseteq X$ be non-empty, and define $\mathcal{E}|_B \colon \mathcal{F}|_B \to [0,\infty)$ by*

$$\mathcal{E}|_B(u) := \inf\{\mathcal{E}(v) \mid v \in \mathcal{F}, v|_B = u\}, \quad u \in \mathcal{F}|_B. \tag{6.6}$$

Then $(\mathcal{E}|_B, \mathcal{F}|_B)$ is a p-resistance form on B and $R_{\mathcal{E}|_B} = R_{\mathcal{E}}|_{B\times B}$. Moreover, for any $u \in \mathcal{F}|_B$ there exists a unique $h^{\mathcal{E}}_B[u] \in \mathcal{F}$ such that $h^{\mathcal{E}}_B[u]\big|_B = u$ and $\mathcal{E}\bigl(h^{\mathcal{E}}_B[u]\bigr) = \mathcal{E}|_B(u)$, so that $h^{\mathcal{E}}_B(\mathcal{F}|_B) = \mathcal{H}_{\mathcal{E},B}$, and

$$h^{\mathcal{E}}_B[au + b\mathbb{1}_B] = ah^{\mathcal{E}}_B[u] + b\mathbb{1}_X \quad \text{for any } u \in \mathcal{F}|_B \text{ and any } a, b \in \mathbb{R}, \tag{6.7}$$

$$\mathcal{E}|_B(u; v) = \mathcal{E}\bigl(h^{\mathcal{E}}_B[u]; h^{\mathcal{E}}_B[v]\bigr) \quad \text{for any } u, v \in \mathcal{F}|_B, \tag{6.8}$$

$$\mathcal{E}|_B(f|_B; g|_B) = \mathcal{E}(f; g) \quad \text{for any } f \in \mathcal{H}_{\mathcal{E},B} \text{ and any } g \in \mathcal{F}, \tag{6.9}$$

where $\mathcal{E}|_B(u; v) := \frac{1}{p}\frac{d}{dt}\mathcal{E}|_B(u + tv)\big|_{t=0}$ for $u, v \in \mathcal{F}|_B$ (recall (3.7)).

Remark 6.14. The map $h^{\mathcal{E}}_B[\,\cdot\,]$ does *not* satisfy either $h^{\mathcal{E}}_B[u+v] \leq h^{\mathcal{E}}_B[u] + h^{\mathcal{E}}_B[u]$ for any $u, v \in \mathcal{F}|_B$ or $h^{\mathcal{E}}_B[u+v] \geq h^{\mathcal{E}}_B[u] + h^{\mathcal{E}}_B[u]$ for any $u, v \in \mathcal{F}|_B$ in general, unless $p = 2$ or $\#B \leq 2$.

Proof of Theorem 6.13. We first show the desired existence of $h^{\mathcal{E}}_B[u]$ for any $u \in \mathcal{F}|_B$. Let us fix $y_* \in B$ and let $\alpha := \inf\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F} \text{ with } v|_B = u\bigr\} \in [0,\infty)$. Then there exists $\{v_n\}_{n\in\mathbb{N}}$ such that $v_n \in \mathcal{F}$, $v_n|_B = u$ and $\mathcal{E}(v_n) \leq \alpha + n^{-1}$ for any $n \in \mathbb{N}$. Note that $\frac{v_k+v_l}{2} \in \mathcal{F}$ also satisfies $\bigl(\frac{v_k+v_l}{2}\bigr)\big|_B = u$ for any $k, l \in \mathbb{N}$. In the case of $p \in (1, 2]$, we see that

$$\begin{aligned}\mathcal{E}(v_k - v_l)^{1/(p-1)} &\overset{(2.8)}{\leq} 2\bigl(\mathcal{E}(v_k) + \mathcal{E}(v_l)\bigr)^{1/(p-1)} - \mathcal{E}(v_k + v_l)^{1/(p-1)} \\ &\leq 2\bigl(2\alpha + k^{-1} + l^{-1}\bigr)^{1/(p-1)} - 2^{p/(p-1)}\alpha^{1/(p-1)} \\ &\xrightarrow[k\wedge l\to\infty]{} 2(2\alpha)^{1/(p-1)} - 2^{p/(p-1)}\alpha^{1/(p-1)} = 0.\end{aligned} \tag{6.10}$$

Similarly, in the case of $p \in [2,\infty)$, we have

$$\begin{aligned}\mathcal{E}(v_k - v_l) &\overset{(2.9)}{\leq} 2\bigl(\mathcal{E}(v_k)^{1/(p-1)} + \mathcal{E}(v_l)^{1/(p-1)}\bigr)^{p-1} - \mathcal{E}(v_k + v_l) \\ &\leq 2\bigl((\alpha + k^{-1})^{1/(p-1)} + (\alpha + l^{-1})^{1/(p-1)}\bigr)^{p-1} - 2^p\alpha \\ &\xrightarrow[k\wedge l\to\infty]{} 2\bigl(2\alpha^{1/(p-1)}\bigr)^{p-1} - 2^p\alpha = 0.\end{aligned} \tag{6.11}$$

Consequently, $\{v_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$. By (RF2)$_p$, there exists $h \in \mathcal{F}$ such that $h(y_*) = u(y_*)$ and $\lim_{n\to\infty}\mathcal{E}(h - v_n) = 0$. For any $y \in B$, by (RF4)$_p$,

$$|h(y) - u(y)|^p = |h(y) - v_n(y)|^p = |(h - v_n)(y) - (h - v_n)(y_*)|^p \leq R_{\mathcal{E}}(y, y_*)\mathcal{E}(h - v_n) \to 0$$

as $n \to \infty$, and hence $h|_B = u$. In particular, h is a minimizer of α. Assume that $g \in \mathcal{F}$ also satisfies $g|_B = u$ and $\mathcal{E}(g) = \alpha$. Then a similar estimate to (6.10) or to (6.11) imply that $\mathcal{E}(h-g) = 0$. Since $h - g \in \mathcal{F}^0(X \setminus B)$ and $B \neq \emptyset$, we have $h = g =: h^{\mathcal{E}}_B[u]$ by (RF1)_p. The property (6.7) immediately follows from (RF1)_p for $(\mathcal{E}, \mathcal{F})$.

Next we prove that $(\mathcal{E}|_B, \mathcal{F}|_B)$ is a p-resistance form on B. It is clear that $\mathcal{E}|_B(au) = |a|^p\,\mathcal{E}|_B(u)$ for any $u \in \mathcal{F}|_B$. Let us show the triangle inequality for $\mathcal{E}|_B(\,\cdot\,)^{1/p}$, Since $\bigl(h^{\mathcal{E}}_B[u] + h^{\mathcal{E}}_B[v]\bigr)\big|_B = u + v$ for any $u, v \in \mathcal{F}|_B$, we see that

$$\begin{aligned}\mathcal{E}|_B(u+v)^{1/p} &= \mathcal{E}\bigl(h^{\mathcal{E}}_B[u+v]\bigr)^{1/p} \leq \mathcal{E}\bigl(h^{\mathcal{E}}_B[u] + h^{\mathcal{E}}_B[v]\bigr)^{1/p} \\ &\leq \mathcal{E}\bigl(h^{\mathcal{E}}_B[u]\bigr)^{1/p} + \mathcal{E}\bigl(h^{\mathcal{E}}_B[v]\bigr)^{1/p} = \mathcal{E}|_B(u)^{1/p} + \mathcal{E}|_B(v)^{1/p}.\end{aligned}$$

By (6.7), we easily see that $\mathcal{F}|_B$ contains $\mathbb{R}\mathbb{1}_B$. If $u \in \mathcal{F}|_B$ satisfies $\mathcal{E}|_B(u) = 0$, then $h^{\mathcal{E}}_B[u] \in \mathbb{R}\mathbb{1}_X$ and hence $h^{\mathcal{E}}_B[u]\big|_B = u \in \mathbb{R}\mathbb{1}_B$. Thus (RF1)_p for $(\mathcal{E}|_B, \mathcal{F}|_B)$ holds. To prove (RF2)_p for $(\mathcal{E}|_B, \mathcal{F}|_B)$, let $\{u_n\} \subseteq \mathcal{F}|_B$ satisfy $\lim_{n\wedge m\to\infty} \mathcal{E}|_B(u_n - u_m) = 0$. Then, by the triangle inequality for $\mathcal{E}|_B(\,\cdot\,)^{1/p}$, we easily see that $\{\mathcal{E}|_B(u_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $[0,\infty)$. By (Cla)_p for $(\mathcal{E}, \mathcal{F})$ and a similar argument to (6.10) (or to (6.11)), we have $\lim_{n\wedge m\to\infty} \mathcal{E}\bigl(h^{\mathcal{E}}_B[u_n] - h^{\mathcal{E}}_B[u_m]\bigr) = 0$. Hence there exists $h \in \mathcal{F}$ such that $\lim_{n\to\infty} \mathcal{E}(h - h^{\mathcal{E}}_B[u_n]) = 0$ by (RF2)_p for $(\mathcal{E}, \mathcal{F})$. Then $\mathcal{E}|_B(h|_B - u_n) \leq \mathcal{E}(h - h^{\mathcal{E}}_B[u_n]) \to 0$, which proves the completeness of $\bigl(\mathcal{F}|_B/\mathbb{R}\mathbb{1}_B, \mathcal{E}|_B(\,\cdot\,)^{1/p}\bigr)$. The condition (RF3)_p for $\mathcal{F}|_B$ is clear from that of $\mathcal{F}$. The inequality $R_{\mathcal{E}|_B} \leq R_{\mathcal{E}}|_{B\times B}$ (and hence (RF4)_p for $(\mathcal{E}|_B, \mathcal{F}|_B)$) follows from the following estimate:

$$\frac{|u(x)-u(y)|^p}{\mathcal{E}|_B(u)} = \frac{\bigl|h^{\mathcal{E}}_B[u](x) - h^{\mathcal{E}}_B[u](y)\bigr|^p}{\mathcal{E}(h^{\mathcal{E}}_B[u])} \leq R_{\mathcal{E}}(x,y) \quad \text{for any } x, y \in B,\, u \in \mathcal{F}|_B.$$

To show the converse inequality $R_{\mathcal{E}|_B} \geq R_{\mathcal{E}}|_{B\times B}$, let $x, y \in B$ and let $u \in \mathcal{F} \setminus \mathbb{R}\mathbb{1}_X$ be such that $u(x) \neq u(y)$. Then $u|_B \in \mathcal{F}|_B \setminus \mathbb{R}\mathbb{1}_B$ and $\mathcal{E}(u) \geq \mathcal{E}|_B(u|_B) > 0$. Therefore,

$$\frac{|u(x)-u(y)|^p}{\mathcal{E}(u)} \leq \frac{|u|_B(x) - u|_B(y)|^p}{\mathcal{E}|_B(u|_B)} \leq R_{\mathcal{E}|_B}(x,y).$$

The same estimate is clear if $u(x) = u(y)$, so taking the supremum over $u \in \mathcal{F} \setminus \mathbb{R}\mathbb{1}_X$ yields $R_{\mathcal{E}}(x,y) \leq R_{\mathcal{E}|_B}(x,y)$. Lastly, we prove (RF5)_p for $(\mathcal{E}|_B, \mathcal{F}|_B)$. Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$, and $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2), and let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \bigl(\mathcal{F}|_B\bigr)^{n_1}$. Note that $T_l(\boldsymbol{u}) = T_l\bigl(h^{\mathcal{E}}_B[u_1], \dots, h^{\mathcal{E}}_B[u_{n_1}]\bigr)\big|_B \in \mathcal{F}|_B$. Therefore, if $q_2 < \infty$, then

$$\begin{aligned}\left(\sum_{l=1}^{n_2} \mathcal{E}|_B\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p}\right)^{1/q_2} &\leq \left(\sum_{l=1}^{n_2} \mathcal{E}\Bigl(T_l\bigl(h^{\mathcal{E}}_B[u_1], \dots, h^{\mathcal{E}}_B[u_{n_1}]\bigr)\Bigr)^{q_2/p}\right)^{1/q_2} \\ &\leq \left(\sum_{k=1}^{n_1} \mathcal{E}\bigl(h^{\mathcal{E}}_B[u_k]\bigr)^{q_1/p}\right)^{1/q_1} = \left(\sum_{k=1}^{n_1} \mathcal{E}|_B(u_k)^{q_1/p}\right)^{1/q_1}.\end{aligned}$$

The case $q_2 = \infty$ is similar, so $(\mathcal{E}|_B, \mathcal{F}|_B)$ satisfies (GC)_p.

We conclude the proof by showing (6.8) and (6.9). By Proposition 3.6, we know that

$$\lim_{t\downarrow 0}\frac{\mathcal{E}|_B(u\pm tv)-\mathcal{E}|_B(u)}{\pm t}=\frac{d}{dt}\mathcal{E}|_B(u+tv)\Big|_{t=0},$$

and

$$\lim_{t\downarrow 0}\frac{\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\pm th_B^{\mathcal{E}}[v]\bigr)-\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr)}{\pm t}=p\mathcal{E}\bigl(h_B^{\mathcal{E}}[u];h_B^{\mathcal{E}}[v]\bigr).$$

For any $t>0$, we have

$$\begin{aligned}\frac{\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]-th_B^{\mathcal{E}}[v]\bigr)-\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr)}{-t}&\leq\frac{\mathcal{E}|_B(u-tv)-\mathcal{E}|_B(u)}{-t}\\&\leq\frac{\mathcal{E}|_B(u+tv)-\mathcal{E}|_B(u)}{t}\leq\frac{\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]+th_B^{\mathcal{E}}[v]\bigr)-\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr)}{t},\end{aligned}$$

and hence we obtain (6.8) by letting $t\downarrow 0$. If $f\in\mathcal{H}_{\mathcal{E},B}$, i.e., $h_B^{\mathcal{E}}[f|_B]=f$, then $\mathcal{E}(f;g)=\mathcal{E}(f;h_B^{\mathcal{E}}[g])=\mathcal{E}|_B(f|_B;g|_B)$ since $g-h_B^{\mathcal{E}}[g|_B]\in\mathcal{F}^0(X\setminus B)$ for any $g\in\mathcal{F}$. This proves (6.9). □

The following proposition states a compatibility of the operation taking traces.

Proposition 6.15. *Let A,B be subsets of X such that $\emptyset\neq A\subseteq B$. Then $(\mathcal{E}|_B|_A,\mathcal{F}|_B|_A)=(\mathcal{E}|_A,\mathcal{F}|_A)$ and $h_B^{\mathcal{E}}\circ h_A^{\mathcal{E}|_B}=h_A^{\mathcal{E}}$. In particular, $h_A^{\mathcal{E}|_B}[u]=h_A^{\mathcal{E}}[u]\big|_B$ for any $u\in\mathcal{F}|_A$.*

Proof. Clearly, we have $\mathcal{F}|_B|_A=\mathcal{F}|_A$. For any $u\in\mathcal{F}|_A$, we see that

$$\begin{aligned}\mathcal{E}|_A(u)=\mathcal{E}\bigl(h_A^{\mathcal{E}}[u]\bigr)&\geq\min\bigl\{\mathcal{E}(v)\bigm| v\in\mathcal{F}\text{ such that }v|_B=h_A^{\mathcal{E}}[u]\big|_B\bigr\}\\&=\mathcal{E}|_B\bigl(h_A^{\mathcal{E}}[u]\big|_B\bigr)\\&\geq\min\bigl\{\mathcal{E}|_B(w)\bigm| w\in\mathcal{F}|_B\text{ such that }w|_A=h_A^{\mathcal{E}}[u]\big|_A=u\bigr\}\\&=\mathcal{E}|_B|_A(u)=\mathcal{E}|_B\bigl(h_A^{\mathcal{E}|_B}[u]\bigr)=\mathcal{E}\Bigl(h_B^{\mathcal{E}}\bigl[h_A^{\mathcal{E}|_B}[u]\bigr]\Bigr)\\&\geq\min\Bigl\{\mathcal{E}(v)\Bigm| v\in\mathcal{F}\text{ such that }v|_A=\bigl(h_B^{\mathcal{E}}\circ h_A^{\mathcal{E}|_B}\bigr)[u]\big|_A=u\Bigr\}=\mathcal{E}|_A(u),\end{aligned}$$

which implies $\mathcal{E}|_A(u)=\mathcal{E}|_B|_A(u)$ and $\mathcal{E}\bigl(h_A^{\mathcal{E}}[u]\bigr)=\mathcal{E}\bigl((h_B^{\mathcal{E}}\circ h_A^{\mathcal{E}|_B})[u]\bigr)$. Since the restrictions to A of both functions $h_A^{\mathcal{E}}[u]$ and $(h_B^{\mathcal{E}}\circ h_A^{\mathcal{E}|_B})[u]$ are u, the uniqueness in Theorem 6.13 implies $h_A^{\mathcal{E}}[u]=\bigl(h_B^{\mathcal{E}}\circ h_A^{\mathcal{E}|_B}\bigr)[u]$. Taking their restrictions to B yields $h_A^{\mathcal{E}|_B}[u]=h_A^{\mathcal{E}}[u]\big|_B$. □

The following theorem presents an expression of $(\mathcal{E},\mathcal{F})$ as the "inductive limit" of its traces $\{\mathcal{E}|_V\}_{V\subseteq X,1\leq\#V<\infty}$ to finite subsets, which is a straightforward generalization of the counterpart for resistance forms given in [Kaj, Corollary 2.37]. This expression can be applied to get a few useful results on convergences of the seminorm $\mathcal{E}^{1/p}$.

Theorem 6.16. *It holds that*

$$\mathcal{F}=\biggl\{u\in\mathbb{R}^X\biggm|\sup_{V\subseteq X;1\leq\#V<\infty}\mathcal{E}|_V(u|_V)<\infty\biggr\},\tag{6.12}$$

$$\mathcal{E}(u)=\sup_{V\subseteq X;1\leq\#V<\infty}\mathcal{E}|_V(u|_V)\quad\text{for any }u\in\mathcal{F}.\tag{6.13}$$

Proof. Define $(\mathcal{E}_*, \mathcal{F}_*)$ by

$$\mathcal{E}_*(u) := \sup_{V \subseteq X; 1 \leq \#V < \infty} \mathcal{E}|_V(u|_V), \quad u \in \mathbb{R}^X,$$

and $\mathcal{F}_* := \{u \in \mathbb{R}^X \mid \mathcal{E}_*(u) < \infty\}$. Then $\mathcal{E}_*^{1/p}$ is clearly a seminorm on $\mathcal{F}_*$ and $\{u \in \mathcal{F}_* \mid \mathcal{E}_*(u) = 0\} = \mathbb{R}\mathbb{1}_X$. We first show that, for any $V \subseteq X$ with $1 \leq \#V < \infty$ and any $u \in \mathbb{R}^V$,

$$h_V^{\mathcal{E}}[u] \in \mathcal{F}_* \quad \text{and} \quad \mathcal{E}|_V(u) = \min\{\mathcal{E}_*(v) \mid v \in \mathcal{F}, v|_V = u\} = \mathcal{E}_*\bigl(h_V^{\mathcal{E}}[u]\bigr), \tag{6.14}$$

both of which are obtained by seeing that, for any $U \subseteq X$ with $1 \leq \#U < \infty$,

$$\mathcal{E}|_U\bigl(h_V^{\mathcal{E}}[u]\bigr|_U\bigr) \leq \mathcal{E}\bigl(h_V^{\mathcal{E}}[u]\bigr) = \mathcal{E}|_V(u).$$

Indeed, taking the supremum over U, we get $\mathcal{E}_*\bigl(h_V^{\mathcal{E}}[u]\bigr) \leq \mathcal{E}|_V(u)$ and hence (6.14) holds. (The converse $\mathcal{E}|_V(u) \leq \mathcal{E}_*\bigl(h_V^{\mathcal{E}}[u]\bigr)$ is clear from the definition.) We also note that $\mathcal{E}_*$ satisfies (Cla)_p since $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is a p-resistance form for each $Y \subseteq X$ and $\mathcal{E}|_V(u|_V) \leq \mathcal{E}|_U(u|_U)$ for any $U, V \subseteq X$ with $\emptyset \neq V \subseteq U$ and $u \in \mathbb{R}^U$.

The inclusion $\mathcal{F} \subseteq \mathcal{F}_*$ and the estimate $\mathcal{E}_* \leq \mathcal{E}$ (on $\mathcal{F}$) easily follow from the following estimate:

$$\mathcal{E}|_V(u|_V) = \mathcal{E}\bigl(h_V^{\mathcal{E}}[u|_V]\bigr) \leq \mathcal{E}(u) \quad \text{for any } u \in \mathcal{F} \text{ and any } V \subseteq X \text{ with } 1 \leq \#V < \infty.$$

To show $\mathcal{F}_* \subseteq \mathcal{F}$ and $\mathcal{E} \leq \mathcal{E}_*$, let $u \in \mathcal{F}_*$, let us choose a subset $V_n \subseteq X$ for each $n \in \mathbb{N}$ such that $1 \leq \#V_n < \infty$ and $\mathcal{E}|_{V_n}(u|_{V_n}) \geq \mathcal{E}_*(u) - n^{-1}$, and set $u_n := h_{V_n}^{\mathcal{E}}[u|_{V_n}]$. Then

$$\mathcal{E}_*(u) - n^{-1} \leq \mathcal{E}|_{V_n}(u|_{V_n}) \overset{(6.14)}{=} \mathcal{E}_*(u_n) \overset{(6.14)}{\leq} \mathcal{E}_*(u),$$

which implies that $\lim_{n\to\infty} \mathcal{E}_*(u_n) = \lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}_*(u)$. Using (Cla)_p for $\mathcal{E}_*$ and $\mathcal{E}_*\bigl(\frac{u+u_n}{2}\bigr) \geq \mathcal{E}_*(u_n)$, we easily obtain $\lim_{n\to\infty} \mathcal{E}_*(u - u_n) = 0$ similarly as (6.10) or (6.11). We next show that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$. From (Cla)_p for $\mathcal{E}$, $\lim_{n\to\infty} \mathcal{E}(u_n) = \lim_{n\to\infty} \mathcal{E}_*(u_n) = \mathcal{E}_*(u)$ and

$$\mathcal{E}(u_k + u_l) \geq \mathcal{E}\bigl(h_{V_k \cup V_l}^{\mathcal{E}}[(u_k + u_l)|_{V_k \cup V_l}]\bigr) \geq 2^p \mathcal{E}|_{V_k \cup V_l}(u|_{V_k \cup V_l}) \overset{(6.14)}{=} 2^p \mathcal{E}_*(u_{k+l}),$$

we can obtain $\lim_{k\wedge l\to\infty} \mathcal{E}(u_k - u_l) = 0$ similarly as (6.10) or (6.11). Hence, by (RF1)_p for $(\mathcal{E}, \mathcal{F})$, there exists $v \in \mathcal{F}$ such that $\lim_{n\to\infty} \mathcal{E}(v - u_n) = 0$. By $\mathcal{E}_* \leq \mathcal{E}$ on $\mathcal{F}$, we conclude that $\lim_{n\to\infty} \mathcal{E}_*(v - u_n) = 0$, which together with the triangle inequality for $\mathcal{E}_*^{1/p}$ and $\lim_{n\to\infty} \mathcal{E}_*(u - u_n) = 0$ implies that $\mathcal{E}_*(u - v) = 0$ and thus $u - v \in \mathbb{R}\mathbb{1}_X$. In particular, $u = (u - v) + v \in \mathcal{F}_*$ and $\mathcal{E}(u) = \lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}_*(u)$, completing the proof. □

Corollary 6.17. *Let $u \in \mathcal{F}$ and let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$.*

(a) *Assume that $\lim_{n\to\infty}(u_n(x) - u_n(y)) = u(x) - u(y)$ for any $x, y \in X$. Then $\mathcal{E}(u) \leq \liminf_{n\to\infty} \mathcal{E}(u_n)$.*

(b) $\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$ *if and only if* $\limsup_{n\to\infty}\mathcal{E}(u_n)\leq\mathcal{E}(u)$ *and* $\lim_{n\to\infty}(u_n(x)-u_n(y))=u(x)-u(y)$ *for any* $x,y\in X$.

Proof. Assume that $u,u_n\in\mathcal{F}$, $n\in\mathbb{N}$, satisfy $\lim_{n\to\infty}(u_n(x)-u_n(y))=u(x)-u(y)$ for any $x,y\in X$. For any $\varepsilon>0$, by Theorem 6.16, there exists $V\subseteq X$ with $1\leq\#V<\infty$ such that $\mathcal{E}|_V(u|_V)>\mathcal{E}(u)-\varepsilon$. Then we have

$$\lim_{n\to\infty}\mathcal{E}|_V(u_n|_V)=\mathcal{E}|_V(u|_V)>\mathcal{E}(u)-\varepsilon,$$

since $\mathbb{R}^V$ is a finite-dimensional vector space, $\mathcal{E}|_V(\,\cdot\,)^{1/p}$ is a seminorm on $\mathbb{R}^V$ and $\lim_{n\to\infty}\max_{x,y\in V}|(u_n(x)-u_n(y))-(u(x)-u(y))|=0$. In particular, there exists $N_1\in\mathbb{N}$ (depending on ε) such that $\mathcal{E}(u_n)\geq\mathcal{E}|_V(u_n|_V)>\mathcal{E}(u)-\varepsilon$ for any $n\geq N_1$ and hence $\liminf_{n\to\infty}\mathcal{E}(u_n)\geq\mathcal{E}(u)$, proving (a). Next, in addition, we assume that $\limsup_{n\to\infty}\mathcal{E}(u_n)\leq\mathcal{E}(u)$. Then $\lim_{n\to\infty}\mathcal{E}(u_n)=\mathcal{E}(u)$. Since $\{\frac{u+u_n}{2}\}_{n\in\mathbb{N}}$ satisfies the same conditions as $\{u_n\}_{n\in\mathbb{N}}$, we obtain $\lim_{n\to\infty}\mathcal{E}\bigl(\frac{u+u_n}{2}\bigr)=\mathcal{E}(u)$. Similar to (6.10) or (6.11), we have from $(\mathrm{Cla})_p$ for $\mathcal{E}$ that $\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$. The converse part of (b) is clear from (6.3). □

Corollary 6.18. (a) *Let* $\{\varphi_n\}_{n\in\mathbb{N}}\subseteq C(\mathbb{R})$ *satisfy* $\lim_{n\to\infty}\varphi_n(t)=t$ *for any* $t\in\mathbb{R}$ *and* $|\varphi_n(t)-\varphi_n(s)|\leq|t-s|$ *for any* $n\in\mathbb{N}$ *and any* $s,t\in\mathbb{R}$. *Then* $\{\varphi_n(u)\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$ *and* $\lim_{n\to\infty}\mathcal{E}(u-\varphi_n(u))=0$ *for any* $u\in\mathcal{F}$.

(b) *Let* $u\in\mathcal{F}$, $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$ *and* $\varphi\in C(\mathbb{R})$ *satisfy* $\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$, $\lim_{n\to\infty}u_n(x)=u(x)$ *for some* $x\in X$, $|\varphi(t)-\varphi(s)|\leq|t-s|$ *for any* $s,t\in\mathbb{R}$ *and* $\varphi(u)=u$. *Then* $\{\varphi(u_n)\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$ *and* $\lim_{n\to\infty}\mathcal{E}(u-\varphi(u_n))=0$.

Proof. (a): This is immediate from Corollary 6.17 and $(\mathrm{RF5})_p$.

(b): For any $y\in X$, we have

$$|u(y)-u_n(y)|\leq R_{\mathcal{E}}(x,y)^{1/p}\mathcal{E}(u-u_n)^{1/p}+|u(x)-u_n(x)|\xrightarrow{n\to\infty}0,$$

and hence $\lim_{n\to\infty}\varphi(u_n(y))=\varphi(u(y))=u(y)$. By $(\mathrm{RF5})_p$ we also have $\{\varphi(u_n)\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$ and $\limsup_{n\to\infty}\mathcal{E}(\varphi(u_n))\leq\lim_{n\to\infty}\mathcal{E}(u_n)=\mathcal{E}(u)$. Thus $\lim_{n\to\infty}\mathcal{E}(u-\varphi(u_n))=0$ by Corollary 6.17-(b). □

In the following proposition, we record a useful variant of Theorem 6.16.

Proposition 6.19. *Let* $\{\mathcal{V}_n\}_{n\in\mathbb{N}\cup\{0\}}$ *be a non-decreasing sequence of non-empty finite subsets of* X, *and set* $\mathcal{V}_*:=\bigcup_{n\in\mathbb{N}\cup\{0\}}\mathcal{V}_n$. *If*

$$\text{the map } \mathcal{F}\ni u\mapsto u|_{\mathcal{V}_*}\in\mathcal{F}|_{\mathcal{V}_*} \text{ is injective (and hence a linear isomorphism),} \tag{6.15}$$

then (note that $\{\mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n})\}_{n\in\mathbb{N}\cup\{0\}}$ is non-decreasing since $\{\mathcal{V}_n\}_{n\in\mathbb{N}\cup\{0\}}$ is non-decreasing),

$$\mathcal{F}|_{\mathcal{V}_*}=\Bigl\{u\in\mathbb{R}^{\mathcal{V}_*}\Bigm|\lim_{n\to\infty}\mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n})<\infty\Bigr\}, \tag{6.16}$$

$$\mathcal{E}(u;v)=\lim_{n\to\infty}\mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n};v|_{\mathcal{V}_n})\quad\text{for any } u,v\in\mathcal{F}, \tag{6.17}$$

$$\lim_{n\to\infty} \mathcal{E}\bigl(u - h^{\mathcal{E}}_{\mathcal{V}_n}[u|_{\mathcal{V}_n}]\bigr) = 0 \quad \textit{for any } u \in \mathcal{F}. \tag{6.18}$$

In particular, if $\overline{\mathcal{V}_*}^X = X$*, then* (6.16)*,* (6.17) *and* (6.18) *hold, and*

$$\mathcal{F} = \Bigl\{ u \in C(X) \Bigm| \lim_{n\to\infty} \mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n}) < \infty \Bigr\}. \tag{6.19}$$

Proof. Assume (6.15). By Theorem 6.16, we have $\mathcal{F}|_{\mathcal{V}_*} \subseteq \bigl\{ u \in \mathbb{R}^{\mathcal{V}_*} \bigm| \lim_{n\to\infty} \mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n}) < \infty \bigr\}$ and $\mathcal{E}(u) \geq \lim_{n\to\infty} \mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n})$ for any $u \in \mathcal{F}$. To show the converse, let $u \in \mathbb{R}^{\mathcal{V}_*}$ satisfy $\lim_{n\to\infty} \mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n}) < \infty$, set $u_n := h^{\mathcal{E}}_{\mathcal{V}_n}(u|_{\mathcal{V}_n}) \in \mathcal{F}$ for each $n \in \mathbb{N} \cup \{0\}$ and fix $x_0 \in \mathcal{V}_0$. We can assume that $u(x_0) = 0$ by considering $u - u(x_0)$ instead of u. A similar estimate to (6.10) or (6.11) for $\mathcal{E}$ and (RF2)$_p$ together imply that $\lim_{n\to\infty} \mathcal{E}(v - u_n) = 0$ for some $v \in \mathcal{F}$ with $v(x_0) = 0$. Since $|v(x) - u(x)|^p \leq R_{\mathcal{E}}(x, x_0)\mathcal{E}(v - u_n)$ for any $x \in \mathcal{V}_*$ and any $n \in \mathbb{N}$ with $x \in \mathcal{V}_n$ by (6.3), we get $u = v|_{\mathcal{V}_*} \in \mathcal{F}|_{\mathcal{V}_*}$, proving $\mathcal{F}|_{\mathcal{V}_*} \supseteq \bigl\{ u \in \mathbb{R}^{\mathcal{V}_*} \bigm| \lim_{n\to\infty} \mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n}) < \infty \bigr\}$ and thereby (6.16). We then have (6.18) by (6.15) and $\lim_{n\to\infty} \mathcal{E}(v - u_n) = 0$, and obtain (6.17) from (6.18), (3.10), (3.11) and (6.8).

Lastly, if $\overline{\mathcal{V}_*}^X = X$, then since $\mathcal{F} \subseteq C(X)$ by (6.3) we have (6.15), hence (6.16), (6.17) and (6.18) hold, and (6.19) follows from (6.16) and $\mathcal{F} \subseteq C(X)$. □

Based on Proposition 6.19, standard machinery for constructing the "inductive limit" of p-energy forms on p.-c.f. self-similar structures can be stated in Theorems 6.21 and 6.22 below, which are extensions of the counterpart for resistance forms given in [Kaj, Lemma 2.24, Theorem 2.25 and Corollary 2.43] to p-resistance forms. This approach will be used in Subsection 8.3, where the construction of p-energy forms due to [CGQ22] is reviewed. See also [Kig01, Sections 2.2, 2.3 and 3.3] for the details in the case of $p = 2$.

Definition 6.20 (Compatible sequence of p-resistance forms on finite sets)**.** Let $\mathcal{V}_n$ be a non-empty finite set and let $\mathcal{E}^{(n)}$ be a p-resistance form on $\mathcal{V}_n$ for each $n \in \mathbb{N} \cup \{0\}$. We say that the sequence $\mathcal{S} := \{(\mathcal{V}_n, \mathcal{E}^{(n)})\}_{n \in \mathbb{N}\cup\{0\}}$ is a *compatible sequence of p-resistance forms* if and only if $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$ and $\mathcal{E}^{(n+1)}|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$ for any $n \in \mathbb{N} \cup \{0\}$.

Theorem 6.21. *Let $\mathcal{S} = \{(\mathcal{V}_n, \mathcal{E}^{(n)})\}_{n\in\mathbb{N}\cup\{0\}}$ be a compatible sequence of p-resistance forms. We define $\mathcal{V}_* := \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{V}_n$,*

$$\mathcal{F}_{\mathcal{S}} := \Bigl\{ u \in \mathbb{R}^{\mathcal{V}_*} \Bigm| \lim_{n\to\infty} \mathcal{E}^{(n)}(u|_{\mathcal{V}_n}) < \infty \Bigr\}, \quad \textit{and} \tag{6.20}$$

$$\mathcal{E}_{\mathcal{S}}(u) := \lim_{n\to\infty} \mathcal{E}^{(n)}(u|_{\mathcal{V}_n}), \quad u \in \mathcal{F}_{\mathcal{S}}. \tag{6.21}$$

Then $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ is a p-resistance form on $\mathcal{V}_*$ and $\mathcal{E}_{\mathcal{S}}|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$ for any $n \in \mathbb{N} \cup \{0\}$.

Proof. Noting that $\{\mathcal{E}^{(n)}(u|_{\mathcal{V}_n})\}_{n\in\mathbb{N}\cup\{0\}}$ is non-decreasing for any $u \in \mathbb{R}^{\mathcal{V}_*}$, we easily obtain (RF1)$_p$ for $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$. To see (RF5)$_p$ for $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$, let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$ and $T = (T_1, \dots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2), and let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}_{\mathcal{S}}^{n_1}$. Then, for any $l \in \{1, \dots, n_2\}$, (GC)$_p$ for $\mathcal{E}^{(n)}$ implies that

$$\mathcal{E}^{(n)}(T_l(\boldsymbol{u})|_{\mathcal{V}_n})^{1/p} \leq \Bigl\| \bigl(\mathcal{E}^{(n)}(T_l(\boldsymbol{u}|_{\mathcal{V}_n}))^{1/p} \bigr)_{l=1}^{n_2} \Bigr\|_{\ell^{q_2}}$$

$$\leq \left\| \left(\mathcal{E}^{(n)}(u_k|_{\mathcal{V}_n})^{1/p}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} \leq \left\| \left(\mathcal{E}_{\mathcal{S},*}(u_k)^{1/p}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} < \infty.$$

By letting $n \to \infty$, we obtain (GC)$_p$ for $(\mathcal{E}_\mathcal{S}, \mathcal{F}_\mathcal{S})$, i.e., (RF5)$_p$ for $(\mathcal{E}_\mathcal{S}, \mathcal{F}_\mathcal{S})$ holds. Before proving (RF2)$_p$–(RF4)$_p$ for $(\mathcal{E}_\mathcal{S}, \mathcal{F}_\mathcal{S})$, we shall show the following claim:

$$\begin{aligned}&\text{For any } n \in \mathbb{N} \cup \{0\} \text{ and any } u \in \mathbb{R}^{\mathcal{V}_n}, \text{ there exists a unique } h^{\mathcal{S}}_{\mathcal{V}_n}[u] \in \mathcal{F}_\mathcal{S} \text{ such}\\ &\text{that } h^{\mathcal{S}}_{\mathcal{V}_n}[u]\big|_{\mathcal{V}_n} = u \text{ and } \mathcal{E}_\mathcal{S}\bigl(h^{\mathcal{S}}_{\mathcal{V}_n}[u]\bigr) = \min\{\mathcal{E}_\mathcal{S}(v) \mid v \in \mathcal{F}_\mathcal{S}, v|_{\mathcal{V}_n} = u\} = \mathcal{E}^{(n)}(u).\end{aligned} \tag{6.22}$$

To prove (6.22), by (RF1)$_p$ and (RF5)$_p$ for $(\mathcal{E}_\mathcal{S}, \mathcal{F}_\mathcal{S})$, we first note that $\#\{v \in \mathcal{F}_\mathcal{S} \mid \mathcal{E}_\mathcal{S}(v) = \alpha\} \leq 1$, where $\alpha := \min\{\mathcal{E}_\mathcal{S}(v) \mid v \in \mathcal{F}_\mathcal{S}, v|_{\mathcal{V}_n} = u\}$. (Recall the arguments in (6.10) and (6.11).) Hence it suffices to show the existence of the minimizer realizing α. For any $k_2 \geq k_1 \geq n$, we have $h^{\mathcal{E}^{(k_2)}}_{\mathcal{V}_n}[u]\big|_{\mathcal{V}_{k_1}} = h^{\mathcal{E}^{(k_1)}}_{\mathcal{V}_n}[u]$ by $\mathcal{E}^{(k_2)}|_{\mathcal{V}_{k_1}} = \mathcal{E}^{(k_1)}$ and Proposition 6.15, which implies that $u_*(x) := h^{\mathcal{E}^{(k)}}_{\mathcal{V}_n}[u](x)$ for $x \in \mathcal{V}_k$ with $k \geq n$ is well-defined. Clearly, $u_*|_{\mathcal{V}_n} = u$. For any $k \geq n$, we have $\mathcal{E}^{(k)}(u_*|_{\mathcal{V}_k}) = \mathcal{E}^{(k+1)}(u_*|_{\mathcal{V}_{k+1}})$ by Proposition 6.15 again, whence $u_* \in \mathcal{F}_\mathcal{S}$ and $\mathcal{E}_\mathcal{S}(u_*) = \mathcal{E}^{(n)}(u)$. Since $\mathcal{E}^{(n)}(u) \leq \mathcal{E}_\mathcal{S}(v)$ for any $v \in \mathcal{F}_\mathcal{S}$ with $v|_{\mathcal{V}_n} = u$, we also get $\mathcal{E}_\mathcal{S}(u_*) = \alpha$, so $h^{\mathcal{S}}_{\mathcal{V}_n}[u] := u_*$ is the desired function.

Now let us go back to the proof of (RF2)$_p$–(RF4)$_p$.

(RF3)$_p$: This is immediate since $\mathcal{F}_\mathcal{S}|_{\mathcal{V}_n} = \mathbb{R}^{V_n}$ for any $n \in \mathbb{N} \cup \{0\}$ by (6.22).

(RF4)$_p$: Let $x, y \in \mathcal{V}_*$ with $x \neq y$ and let $n \in \mathbb{N} \cup \{0\}$ satisfy $x, y \in \mathcal{V}_n$. Let $u := h^{\mathcal{E}^n}_{\{x,y\}}\bigl[\mathbb{1}^{\{x,y\}}_x\bigr] \in \mathbb{R}^{\mathcal{V}_n}$. Then for any $v \in \mathcal{F}_\mathcal{S}$ with $v|_{\{x,y\}} = \mathbb{1}^{\{x,y\}}_x$,

$$\mathcal{E}_\mathcal{S}(v) \overset{(6.22)}{\geq} \mathcal{E}^{(n)}(v|_{\mathcal{V}_n}) \geq R_{\mathcal{E}^n}(x,y)^{-1} = \mathcal{E}^{(n)}(u) \overset{(6.22)}{=} \mathcal{E}_\mathcal{S}\bigl(h^{\mathcal{S}}_{\mathcal{V}_n}[u]\bigr).$$

Therefore, we have

$$R_{\mathcal{E}_\mathcal{S}}(x,y) = \mathcal{E}_\mathcal{S}\bigl(h^{\mathcal{S}}_{\mathcal{V}_n}[u]\bigr)^{-1} = R_{\mathcal{E}^{(n)}}(x,y) < \infty. \tag{6.23}$$

(RF2)$_p$: Fix $x_* \in \mathcal{V}_*$, and let $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathcal{F}_\mathcal{S}$ satisfy $u_k(x_*) = 0$ for any $k \in \mathbb{N}$ and $\lim_{k\wedge l\to\infty} \mathcal{E}_\mathcal{S}(u_k - u_l) = 0$. From (RF4)$_p$, $\{u_k(x)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for any $x \in \mathcal{V}_*$, so we can define $u \in \mathbb{R}^{V_*}$ by $u(x) := \lim_{k\to\infty} u_k(x)$. Let $\varepsilon \in (0,\infty)$. Then there exists $N_0 \in \mathbb{N}$ such that $\sup_{k,l\geq N_0} \mathcal{E}_\mathcal{S}(u_k - u_l) \leq \varepsilon$. Since $\mathcal{E}^{(n)}(\,\cdot\,)^{1/p}$ is a norm on the finite-dimensional vector space $\mathbb{R}^{\mathcal{V}_n}/\mathbb{R}\mathbb{1}_{\mathcal{V}_n}$, we obtain

$$\mathcal{E}^{(n)}(u|_{\mathcal{V}_n} - u_l|_{\mathcal{V}_n}) \leq \liminf_{k\to\infty} \mathcal{E}_\mathcal{S}(u_k - u_l) \leq \varepsilon \quad \text{for any } l \geq N_0 \text{ and any } n \in \mathbb{N} \cup \{0\}.$$

Letting $n \to \infty$ here, for any $l \geq N_0$ we obtain $u - u_l \in \mathcal{F}_\mathcal{S}$, therefore $u = (u - u_l) + u_l \in \mathcal{F}_\mathcal{S}$, also $\mathcal{E}_\mathcal{S}(u - u_l) \leq \varepsilon$, and thus $\lim_{l\to\infty} \mathcal{E}_\mathcal{S}(u - u_l) = 0$, which proves that $(\mathcal{F}_\mathcal{S}/\mathbb{R}\mathbb{1}_{\mathcal{V}_*}, \mathcal{E}_\mathcal{S}^{1/p})$ is a Banach space.

Now we know that $(\mathcal{E}_\mathcal{S}, \mathcal{F}_\mathcal{S})$ is a p-resistance form on $\mathcal{V}_*$. Then (6.22) means that $h^{\mathcal{S}}_{\mathcal{V}_n} = h^{\mathcal{E}_\mathcal{S}}_{\mathcal{V}_n}[u]$ for any $u \in \mathbb{R}^{\mathcal{V}_n}$, whence $\mathcal{E}_\mathcal{S}|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$ by (6.22) again. □

The following theorem yields a p-resistance form on the completion of $(X, R_\mathcal{E}^{1/p})$.

Theorem 6.22. *Let $(\widehat{X}, \widehat{d})$ be the completion of the metric space $(X, R_{\mathcal{E}}^{1/p})$. Define $\widehat{\mathcal{F}} \subseteq \mathbb{R}^{\widehat{X}}$ and $\widehat{\mathcal{E}} \colon \widehat{\mathcal{F}} \to [0, \infty)$ by*

$$\widehat{\mathcal{F}} \coloneqq \bigl\{u \in C(\widehat{X}) \bigm| u|_X \in \mathcal{F}\bigr\}, \tag{6.24}$$

$$\widehat{\mathcal{E}}(u) \coloneqq \mathcal{E}(u|_X), \quad u \in \widehat{\mathcal{F}}. \tag{6.25}$$

Then $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ is a p-resistance form on $\widehat{X}$, $R_{\widehat{\mathcal{E}}}^{1/p} = \widehat{d}$, and the map $\widehat{\mathcal{F}} \ni u \mapsto u|_X \in \mathcal{F}$ is a linear isomorphism.

Proof. Set $\widehat{R}(x, y) \coloneqq \widehat{d}(x, y)^p$ for ease of notation, then $\widehat{R}\,\bigr|_{X \times X} = R_{\mathcal{E}}$. For any $u \in \mathcal{F}$, we know that u is uniformly continuous with respect to $\widehat{d}$ by (6.3) for $(\mathcal{E}, \mathcal{F})$, so there exists a unique $\widehat{u} \in C(\widehat{X})$ satisfying $\widehat{u}|_X = u$ and then $\widehat{u} \in \widehat{\mathcal{F}}$. This implies that the map $\widehat{\mathcal{F}} \ni u \mapsto u|_X \in \mathcal{F}$ is a bijection and thus it is a linear isomorphism. Also, for $u \in \widehat{\mathcal{F}}$, we define the continuous function $\eta_u \colon \widehat{X} \times \widehat{X} \to \mathbb{R}$ by $\eta_u(x, y) \coloneqq |u(x) - u(y)|^p - \widehat{R}(x, y)\widehat{\mathcal{E}}(u)$. Since $\eta_u|_{X \times X} \leq 0$ by (6.3) for $R_{\mathcal{E}}$, the continuity of η_u yields

$$|u(x) - u(y)|^p \leq \widehat{R}(x, y)\widehat{\mathcal{E}}(u), \quad x, y \in \widehat{X}. \tag{6.26}$$

Now we show (RF1)$_p$-(RF5)$_p$ for $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$.

(RF1)$_p$: Clearly, $\widehat{\mathcal{F}}$ is a linear subspace of $\mathbb{R}^{\widehat{X}}$ containing $\mathbb{R}\mathbb{1}_{\widehat{X}}$ and $\widehat{\mathcal{E}}(\,\cdot\,)^{1/p}$ is a seminorm on $\widehat{\mathcal{F}}$. By $\mathbb{1}_{\widehat{X}}|_X = \mathbb{1}_X$ and (RF1)$_p$ for $(\mathcal{E}, \mathcal{F})$, it holds that $\{u \in \widehat{\mathcal{F}} \mid \widehat{\mathcal{E}}(u) = 0\} = \mathbb{R}\mathbb{1}_{\widehat{X}}$.

(RF2)$_p$: This is immediate from (RF2)$_p$ for $(\mathcal{E}, \mathcal{F})$ since $\widehat{\mathcal{F}} \ni u \mapsto u|_X \in \mathcal{F}$ is a linear isomorphism.

(RF5)$_p$: This is immediate from (RF5)$_p$ for $(\mathcal{E}, \mathcal{F})$.

(RF3)$_p$ and (RF4)$_p$: Let $x, y \in \widehat{X}$ with $x \neq y$ and let $\{x_n\}_{n \geq 0}, \{y_n\}_{n \geq 0} \subseteq X$ satisfy $\lim_{n \to \infty} \widehat{R}(x, x_n) = \lim_{n \to \infty} \widehat{R}(y, y_n) = 0$. We can assume that $x_n \neq y_n$ for any $n \geq 0$. Let $u_n \in \widehat{\mathcal{F}}$ be the unique function satisfying $u_n|_X = h^{\mathcal{E}}_{\{x_n, y_n\}}\bigl[\mathbb{1}_{x_n}^{\{x_n, y_n\}}\bigr]$. Then $\{\widehat{\mathcal{E}}(u_n)\}_{n \geq 0}$ is bounded in $[0, \infty)$ since $\widehat{\mathcal{E}}(u_n) = R_{\mathcal{E}}(x_n, y_n)^{-1} = \widehat{R}(x_n, y_n)^{-1} \to \widehat{R}(x, y)^{-1}$ as $n \to \infty$. Also, it is easy to see that $0 \leq u_n \leq 1$. From (6.26) and the Arzelá–Ascoli theorem, there exist a subsequence $\{u_{n_k}\}_k$ and $u_* \in C(\widehat{X})$ such that $\lim_{k \to \infty} \|u_* - u_{n_k}\|_{\sup, \widehat{X}} = 0$. A similar argument as in the proof of (RF2)$_p$ for $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ in Theorem 6.21 implies that $u_* \in \widehat{\mathcal{F}}$ and $\lim_{k \to \infty} \widehat{\mathcal{E}}(u_* - u_{n_k}) = 0$. Now we define $u \in \widehat{\mathcal{F}}$ by $u \coloneqq u_* - u_*(y)$ so that $u(y) = 0$. Then we have from (6.26) that

$$|u(x_{n_k}) - u(y_{n_k}) - 1|^p \leq \widehat{R}(x_{n_k}, y_{n_k})\widehat{\mathcal{E}}(u - u_{n_k}) \xrightarrow[k \to \infty]{} 0,$$

whence $u(x) = 1$, in particular, (RF3)$_p$ holds. By (6.26) again, we obtain $R_{\widehat{\mathcal{E}}}(x, y) \leq \widehat{R}(x, y) < \infty$, so (RF4)$_p$ holds. Moreover, this also shows $R_{\widehat{\mathcal{E}}}(x, y) = \widehat{R}(x, y) = \widehat{\mathcal{E}}(u)^{-1}$. $\square$

Corollary 6.23. *Let $\mathcal{S} = \{(\mathcal{V}_n, \mathcal{E}^{(n)})\}_{n \in \mathbb{N} \cup \{0\}}$ be a compatible sequence of p-resistance forms and let (K, d) be the completion of $(\mathcal{V}_*, R_{\mathcal{E}_{\mathcal{S}}}^{1/p})$, where $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ is the p-resistance* *form on* $\mathcal{V}_* = \bigcup_{n \in \mathbb{N}\cup\{0\}} \mathcal{V}_n$ *given in Theorem 6.21. Define* $\widehat{\mathcal{F}}_{\mathcal{S}} \subseteq \mathbb{R}^K$ *and* $\widehat{\mathcal{E}}_{\mathcal{S}} \colon \widehat{\mathcal{F}}_{\mathcal{S}} \to [0,\infty)$ *by*

$$\widehat{\mathcal{F}}_{\mathcal{S}} := \bigl\{ u \in C(K) \bigm| u|_{\mathcal{V}_*} \in \mathcal{F}_{\mathcal{S}} \bigr\} = \Bigl\{ u \in C(K) \Bigm| \lim_{n\to\infty} \mathcal{E}^{(n)}(u|_{\mathcal{V}_n}) < \infty \Bigr\}, \tag{6.27}$$

$$\widehat{\mathcal{E}}_{\mathcal{S}}(u) := \mathcal{E}_{\mathcal{S}}(u|_{\mathcal{V}_*}) = \lim_{n\to\infty} \mathcal{E}^{(n)}(u|_{\mathcal{V}_n}), \quad u \in \widehat{\mathcal{F}}_{\mathcal{S}}. \tag{6.28}$$

Then $(\widehat{\mathcal{E}}_{\mathcal{S}}, \widehat{\mathcal{F}}_{\mathcal{S}})$ *is a p-resistance form on K,* $R_{\widehat{\mathcal{E}}_{\mathcal{S}}}^{1/p} = d$*, and the map* $\widehat{\mathcal{F}}_{\mathcal{S}} \ni u \mapsto u|_{\mathcal{V}_*} \in \mathcal{F}_{\mathcal{S}}$ *is a linear isomorphism. In particular,* $\widehat{\mathcal{E}}_{\mathcal{S}}\big|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$ *for any* $n \in \mathbb{N}\cup\{0\}$*.*

Proof. We obtain the desired assertions by applying Theorem 6.22 with $\mathcal{V}_*$, $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ in place of X, $(\mathcal{E}, \mathcal{F})$. Also, by $\mathcal{E}_{\mathcal{S}}|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$ (see Theorem 6.21) and the fact that $\widehat{\mathcal{F}}_{\mathcal{S}} \ni u \mapsto u|_{\mathcal{V}_*} \in \mathcal{F}_{\mathcal{S}}$ is a linear isomorphism, we have $\widehat{\mathcal{E}}_{\mathcal{S}}\big|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$. □

We conclude this subsection with a discussion of strong locality of p-resistance forms.

Definition 6.24 (Strong locality of p-resistance form)**.** (1) We say that $(\mathcal{E}, \mathcal{F})$ has the strong local property $(\mathrm{SL1})_{\mathrm{s}}$ if and only if

$$\mathcal{E}(u_1 + u_2 + v) + \mathcal{E}(v) = \mathcal{E}(u_1 + v) + \mathcal{E}(u_2 + v). \tag{6.29}$$

for any $u_1, u_2, v \in \mathcal{F}$ with either $\mathrm{supp}_X[u_1 - a_1 \mathbb{1}_X]$ or $\mathrm{supp}_X[u_2 - a_2 \mathbb{1}_X]$ compact and $(u_1(x) - a_1)(u_2(x) - a_2) = 0$ for any $x \in X$ for some $a_1, a_2 \in \mathbb{R}$.

(2) We say that $(\mathcal{E}, \mathcal{F})$ has the strong local property $(\mathrm{SL2})_{\mathrm{s}}$, or $(\mathcal{E}, \mathcal{F})$ is *strongly local*, if and only if

$$\mathcal{E}(u_1; v) = \mathcal{E}(u_2; v) \tag{6.30}$$

for any $u_1, u_2, v \in \mathcal{F}$ with either $\mathrm{supp}_X[u_1 - u_2 - a\mathbb{1}_X]$ or $\mathrm{supp}_X[v - b\mathbb{1}_X]$ compact and $(u_1(x) - u_2(x) - a)(v(x) - b) = 0$ for any $x \in X$ for some $a, b \in \mathbb{R}$.

(3) We say that $(\mathcal{E}, \mathcal{F})$ has the strong local property $(\mathrm{SL1})_{\mathrm{w}}$ if and only if $(\mathrm{SL1})_{\mathrm{s}}$ with "$(u_1(x) - a_1)(u_2(x) - a_2) = 0$ for any $x \in X$" replaced by "$\mathrm{supp}_X[u_1 - a_1\mathbb{1}_X] \cap \mathrm{supp}_X[u_2 - a_2\mathbb{1}_X] = \emptyset$" holds.

(4) We say that $(\mathcal{E}, \mathcal{F})$ has the strong local property $(\mathrm{SL2})_{\mathrm{w}}$ if and only if $(\mathrm{SL2})_{\mathrm{s}}$ with "$(u_1(x) - u_2(x) - a)(v(x) - b) = 0$ for any $x \in X$" replaced by "$\mathrm{supp}_X[u_1 - u_2 - a\mathbb{1}_X] \cap \mathrm{supp}_X[v - b\mathbb{1}_X] = \emptyset$" holds.

Note that $(\mathrm{SL1})_{\mathrm{w}}$ and $(\mathrm{SL2})_{\mathrm{w}}$ are exactly (SL1) and (SL2), respectively, in Definition 3.30 with "supp_m" replaced by "supp_X". In the following proposition, we discuss relations among the strong local properties $(\mathrm{SL1})_{\mathrm{s}}$, $(\mathrm{SL2})_{\mathrm{s}}$, $(\mathrm{SL1})_{\mathrm{w}}$ and $(\mathrm{SL2})_{\mathrm{w}}$ introduced in Definition 6.24.

Proposition 6.25. (a) *If X is locally compact and $(\mathcal{E}, \mathcal{F})$ is regular and satisfies $(\mathrm{SL2})_{\mathrm{w}}$, then $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{SL1})_{\mathrm{w}}$.*

(b) *If $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{SL1})_{\mathrm{w}}$, then $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{SL2})_{\mathrm{w}}$.*

(c) *$(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{SL1})_{\mathrm{s}}$ if and only if $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{SL1})_{\mathrm{w}}$.*

(d) *$(\mathcal{E}, \mathcal{F})$ satisfies* $(\mathrm{SL2})_{\mathrm{s}}$ *if and only if* $(\mathcal{E}, \mathcal{F})$ *satisfies* $(\mathrm{SL2})_{\mathrm{w}}$.

In particular, if X is locally compact and $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathrm{SL1})_{\mathrm{s}}$, $(\mathrm{SL2})_{\mathrm{s}}$, $(\mathrm{SL1})_{\mathrm{w}}$ *and* $(\mathrm{SL2})_{\mathrm{w}}$ *are equivalent to each other.*

Proof. (a): Since $(\mathcal{E}, \mathcal{F})$ satisfies (3.34), (3.35) and (3.36) with m the counting measure on X by Proposition 2.3-(d), Corollary 6.18-(a) and Proposition 6.6, the implication from $(\mathrm{SL2})_{\mathrm{w}}$ to $(\mathrm{SL1})_{\mathrm{w}}$ is proved in exactly the same way as the proof of Proposition 3.32-(b) (note that the separability of X is used there only to define $\mathrm{supp}_m[\,\cdot\,]$).

(b): This is proved in exactly the same way as the proof of Proposition 3.32-(a).

(c): The implication from $(\mathrm{SL1})_{\mathrm{s}}$ to $(\mathrm{SL1})_{\mathrm{w}}$ is obvious. Conversely, assume $(\mathrm{SL1})_{\mathrm{w}}$, let $u_1, u_2, v \in \mathcal{F}$, $a_1, a_2 \in \mathbb{R}$ and assume that either $\mathrm{supp}_X[u_1 - a_1 \mathbb{1}_X]$ or $\mathrm{supp}_X[u_2 - a_2 \mathbb{1}_X]$ is compact and $(u_1(x) - a_1)(u_2(x) - a_2) = 0$ for any $x \in X$. For $n \in \mathbb{N}$, let $\varphi_n \in C(\mathbb{R})$ be given by $\varphi_n(t) \coloneqq t - (-\frac{1}{n}) \vee (t \wedge \frac{1}{n})$ and set $u_{1,n} \coloneqq \varphi_n(u_1 - a_1 \mathbb{1}_X)$ and $u_{2,n} \coloneqq \varphi_n(u_2 - a_2 \mathbb{1}_X)$, so that $u_{i,n} \in \mathcal{F}$ and $\lim_{n\to\infty} \mathcal{E}(u_i - u_{i,n}) = 0$ for $i \in \{1, 2\}$ by Corollary 6.18-(a) and $(\mathrm{RF1})_p$. Then for each $n \in \mathbb{N}$, since $\mathrm{supp}_X[u_{1,n}] \cap \mathrm{supp}_X[u_{2,n}] = \emptyset$ and either $\mathrm{supp}_X[u_{1,n}]$ or $\mathrm{supp}_X[u_{2,n}]$ is compact by the assumptions on u_1, u_2, it follows from $(\mathrm{SL1})_{\mathrm{w}}$ that $\mathcal{E}(u_{1,n} + u_{2,n} + v) + \mathcal{E}(v) = \mathcal{E}(u_{1,n} + v) + \mathcal{E}(u_{2,n} + v)$, and we obtain $\mathcal{E}(u_1 + u_2 + v) + \mathcal{E}(v) = \mathcal{E}(u_1 + v) + \mathcal{E}(u_2 + v)$ by letting $n \to \infty$, proving $(\mathrm{SL1})_{\mathrm{s}}$.

(d): The implication from $(\mathrm{SL2})_{\mathrm{s}}$ to $(\mathrm{SL2})_{\mathrm{w}}$ is obvious. Conversely, assume $(\mathrm{SL2})_{\mathrm{w}}$, let $u_1, u_2, v \in \mathcal{F}$, $a, b \in \mathbb{R}$ and assume that either $\mathrm{supp}_X[u_1 - u_2 - a\mathbb{1}_X]$ or $\mathrm{supp}_X[v - b\mathbb{1}_X]$ is compact and $(u_1(x) - u_2(x) - a)(v(x) - b) = 0$ for any $x \in X$. For $n \in \mathbb{N}$, set $v_n \coloneqq \varphi_n(v - b\mathbb{1}_X)$, where φ_n is the same as in the proof of (c), so that $v_n \in \mathcal{F}$ and $\lim_{n\to\infty} \mathcal{E}(v - v_n) = 0$ by Corollary 6.18-(a) and $(\mathrm{RF1})_p$. Then for each $n \in \mathbb{N}$, since $\mathrm{supp}_X[u_1 - u_2 - a\mathbb{1}_X] \cap \mathrm{supp}_X[v_n] = \emptyset$ and either $\mathrm{supp}_X[u_1 - u_2 - a\mathbb{1}_X]$ or $\mathrm{supp}_X[v_n]$ is compact by the assumptions on u_1, u_2, v, it follows from $(\mathrm{SL2})_{\mathrm{w}}$ that $\mathcal{E}(u_1; v_n) = \mathcal{E}(u_2; v_n)$, and we obtain $\mathcal{E}(u_1; v) = \mathcal{E}(u_2; v)$ by letting $n \to \infty$, proving $(\mathrm{SL2})_{\mathrm{s}}$. □

6.3 Weak comparison principles

In this subsection, we show some weak comparison principles in this context. The first one is obtained as an application of the strong subadditivity.

Proposition 6.26 (Weak comparison principle I)**.** *Let B be a non-empty subset of X. Then, for any $u, v \in \mathcal{F}|_B$ satisfying $u(y) \leq v(y)$ for any $y \in B$, it holds that*

$$h_B^{\mathcal{E}}[u](x) \leq h_B^{\mathcal{E}}[v](x) \quad \text{for any } x \in X. \tag{6.31}$$

In particular,

$$\inf_B u \leq h_B^{\mathcal{E}}[u](x) \leq \sup_B u \quad \text{for any } x \in X. \tag{6.32}$$

Proof. Let $f \coloneqq h_B^{\mathcal{E}}[u]$ and $g \coloneqq h_B^{\mathcal{E}}[v]$. We will prove $f \wedge g = f$, which immediately implies (6.31). Since $(f \wedge g)|_B = u$ and $(f \vee g)|_B = v$, we have

$$\mathcal{E}(f) \leq \mathcal{E}(f \wedge g) \quad \text{and} \quad \mathcal{E}(g) \leq \mathcal{E}(f \vee g).$$

By the strong subadditivity in (2.6), we obtain $\mathcal{E}(f \wedge g) = \mathcal{E}(f)$ (and $\mathcal{E}(f \vee g) = \mathcal{E}(g)$), which together with the uniqueness in Theorem 6.13, we have $f \wedge g = f$. $\square$

We can extend the weak comparison principle above to arbitrary open subsets if X is locally compact and $(\mathcal{E}, \mathcal{F})$ is regular and strongly local. See Proposition 6.30 below. This version of weak comparison principle will be used to prove the *strong comparison principle* on p.-c.f. self-similar structures in a forthcoming paper [KS+a]. We begin with some preparations.

Definition 6.27. Let U be a non-empty open subset of X.

(1) We define

$$\mathcal{F}_{\mathrm{loc}}(U) := \left\{ f \in \mathbb{R}^U \;\middle|\; \begin{array}{l} f\mathbb{1}_V = f^{\#}\mathbb{1}_V \text{ for some } f^{\#} \in \mathcal{F} \text{ for each} \\ \text{relatively compact open subset } V \text{ of } U \end{array} \right\}.$$

(2) Assume that $(\mathcal{E}, \mathcal{F})$ is strongly local. Let $V \subseteq U$ be an open subset. A function $h \in \mathcal{F}_{\mathrm{loc}}(U)$ is said to be $\mathcal{E}$-harmonic on V if $\mathcal{E}(h^{\#}; \varphi) = 0$ for any $\varphi \in \mathcal{F}^0(V)$ with $\mathrm{supp}[\varphi]$ compact (with respect to the metric topology of $R_{\mathcal{E}}^{1/p}$), where $h^{\#} \in \mathcal{F}$ satisfies $h\mathbb{1}_{\mathrm{supp}[\varphi]} = h^{\#}\mathbb{1}_{\mathrm{supp}[\varphi]}$.

Remark 6.28. (1) If $X =: K$ comes from a self-similar structure and the topology induced by $R_{\mathcal{E}}^{1/p}$ coincides with the original topology of K, then the definition of $\mathcal{F}_{\mathrm{loc}}(U)$ above is the same as (5.36) by virtue of $\mathcal{F} \subseteq C(K)$.

(2) By the strong locality of $(\mathcal{E}, \mathcal{F})$, the value $\mathcal{E}(h^{\#}; \varphi)$ is independent of a particular choice of $h^{\#}$.

We need the following proposition to achieve the desired weak comparison principle.

Proposition 6.29. *Assume that X is locally compact and that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local. Let U be a non-empty open subset of X and let $u \in \mathcal{F}$ satisfy $u(x) = 0$ for any $x \in \partial_X U = \overline{U}^X \setminus U$. Then $u\mathbb{1}_U \in \mathcal{F}$.*

Proof. Define $\varphi_n \in C(\mathbb{R})$ by $\varphi_n(t) := t - \left(\frac{1}{n}\right) \vee \left(t \wedge \frac{1}{n}\right)$ and set $A_n := U \cap \mathrm{supp}_X[\varphi_n(u)]$ for each $n \in \mathbb{N}$. Since $u|_{\partial U} = 0$, $A_n = \overline{U}^X \cap \mathrm{supp}_X[\varphi_n(u)]$ and thus A_n is a compact subset of U. By Proposition 6.6, there exists $v_n \in \mathcal{F}$ such that $\mathbb{1}_{A_n} \leq v_n \leq \mathbb{1}_U$. Then we easily obtain $\varphi_n(u)\mathbb{1}_U = \varphi_n(u)v_n$, hence by Corollary 6.18-(a) and Proposition 2.3-(d) we have $\varphi_n(u)\mathbb{1}_U \in \mathcal{F}$. By the strong locality and Corollary 6.18-(a), $\{\varphi_n(u)\mathbb{1}_U\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$. Thus, by (RF2)$_p$ and (6.3), $\{\varphi_n(u)\mathbb{1}_U\}_{n \in \mathbb{N}}$ converges in norm in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$ to its pointwise limit $u\mathbb{1}_U$, whence $u\mathbb{1}_U \in \mathcal{F}$. $\square$

Now we can state the desired version of the weak comparison principle.

Proposition 6.30 (Weak comparison principle II)**.** *Assume that X is locally compact and that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local. Let U be non-empty open subset of X such that $\overline{U}^X$ is compact and $U \neq X$. If $u, v \in C(\overline{U}^X) \cap \mathcal{F}_{\mathrm{loc}}(U)$ are $\mathcal{E}$-harmonic on U and $u(x) \leq v(x)$ for any $x \in \partial_X U = \overline{U}^X \setminus U$, then $u(x) \leq v(x)$ for any $x \in \overline{U}^X$.*

Proof. We first observe that $\partial_X O \neq \emptyset$ for any non-empty open subset of X such that $\overline{O}^X$ is compact and $O \neq X$. To this end, suppose that $\partial_X O = \emptyset$ and then show $O = X$. We see from Proposition 3.28 that there exists $\varphi \in \mathcal{F} \cap C_c(X)$ such that $\varphi|_O = 1$ and $\varphi|_{X \setminus O} = 0$ since $O = \overline{O}^X$ is compact. By the strong locality of $(\mathcal{E}, \mathcal{F})$ and (RF1)$_p$, we have $\mathcal{E}(\varphi) = 0$ and hence $\varphi \in \mathbb{R}\mathbb{1}_X$. Therefore, $X \setminus O = \emptyset$ since O is non-empty.

Let us go back to the proof. Since u and v are uniformly continuous on $\overline{U}^X$ and $\partial_X U \neq \emptyset$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$V := \left\{ x \in U \;\middle|\; \mathrm{dist}_{R_{\mathcal{E}}^{1/p}}(x, \partial_X U) > \delta \right\} \neq \emptyset,$$

and $u(x) \leq v(x) + \varepsilon$ for any $x \in \overline{U}^X \setminus V$. Then V is a relatively compact open subset of U and hence there exist $u^{\#}, v^{\#} \in \mathcal{F}$ such that $u\mathbb{1}_V = u^{\#}\mathbb{1}_V$ and $v\mathbb{1}_V = v^{\#}\mathbb{1}_V$. Define $f := u^{\#} - (u^{\#} - v^{\#})^{+}\mathbb{1}_{X \setminus V}$, $g := v^{\#} + (u^{\#} - v^{\#})^{+}\mathbb{1}_{X \setminus V}$. Then $f, g \in \mathcal{F}$ by $u^{\#}(x) \leq v^{\#}(x)$ for any $x \in \partial_X V \neq \emptyset$, Propositions 2.3-(b) and 6.29. We also have $f, g \in \mathcal{H}_{\mathcal{E}, X \setminus V}$ by the strong locality of $(\mathcal{E}, \mathcal{F})$. Since $f(x) = (u^{\#} \wedge v^{\#})(x) \leq (u^{\#} \vee v^{\#})(x) = g(x)$ for any $x \in X \setminus V$, Proposition 6.26 implies that $u(x) = u^{\#}(x) = f(x) \leq g(x) = v^{\#}(x) = v(x)$ for any $x \in V$. Therefore, we conclude that $u(x) \leq v(x) + \varepsilon$ for any $x \in \overline{U}^X$. Since $\varepsilon > 0$ is arbitrary, we complete the proof. $\square$

6.4 Sharp Hölder regularity of harmonic functions

In this subsection, we present a sharp Hölder regularity estimate on $\mathcal{E}$-harmonic functions and prove that $R_{\mathcal{E}}^{1/(p-1)}$ is a metric on X.

As an application of Proposition 3.10, we can show the following Hölder continuity estimate for $\mathcal{E}$-harmonic functions.

Theorem 6.31. *Let B be a non-empty subset of X, $x \in X \setminus B^{\mathcal{F}}$ and $y \in X$. Then*

$$h_{B \cup \{x\}}^{\mathcal{E}}\bigl[\mathbb{1}_B^{B \cup \{x\}}\bigr](y) \leq \frac{R_{\mathcal{E}}(x, y)^{1/(p-1)}}{R_{\mathcal{E}}(x, B)^{1/(p-1)}}. \tag{6.33}$$

Moreover, for any $h \in \mathcal{H}_{\mathcal{E}, B}$ with $\sup_B |h| < \infty$,

$$|h(x) - h(y)| \leq \frac{R_{\mathcal{E}}(x, y)^{1/(p-1)}}{R_{\mathcal{E}}(x, B)^{1/(p-1)}} \operatorname*{osc}_B[h]. \tag{6.34}$$

Proof. Since (6.33) and (6.34) are obvious if $x = y$, we may and do assume that $x \neq y$.

To show (6.33), on one hand, we see that

$$\begin{aligned} -\mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_x) &= \mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_{B \cup \{x\}}) - \mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_x) \\ &= \mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_B) = R_{\mathcal{E}}(x, B)^{-1}. \end{aligned} \tag{6.35}$$

On the other hand,

$$- \mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_x)$$

$$
\begin{aligned}
&= -\mathcal{E}\Bigl(h^{\mathcal{E}}_{B\cup\{x\}}[\mathbb{1}_B]; h^{\mathcal{E}}_{B\cup\{x,y\}}[\mathbb{1}_x]\Bigr) \quad \text{(by (6.9))} \\
&= -\mathcal{E}|_{B\cup\{x,y\}}\Bigl(h^{\mathcal{E}}_{B\cup\{x\}}[\mathbb{1}_B]\big|_{B\cup\{x,y\}}; \mathbb{1}_x\Bigr) \quad \text{(by Proposition 6.15 and (6.9))} \\
&\geq -\mathcal{E}|_{B\cup\{x,y\}}\Bigl(\bigl(h^{\mathcal{E}}_{B\cup\{x\}}[\mathbb{1}_B](y)\cdot h^{\mathcal{E}}_{\{x,y\}}[\mathbb{1}_y]\bigr)\big|_{B\cup\{x,y\}}; \mathbb{1}_x\Bigr) \quad \text{(by Proposition 3.10)} \\
&= -h^{\mathcal{E}}_{B\cup\{x\}}[\mathbb{1}_B](y)^{p-1}\mathcal{E}|_{B\cup\{x,y\}}\Bigl(h^{\mathcal{E}}_{\{x,y\}}[\mathbb{1}_y]\big|_{B\cup\{x,y\}}; \mathbb{1}_x\Bigr) \\
&= -h^{\mathcal{E}}_{B\cup\{x\}}[\mathbb{1}_B](y)^{p-1}\mathcal{E}|_{\{x,y\}}(\mathbb{1}_y; \mathbb{1}_{\{x,y\}} - \mathbb{1}_y) \quad \text{(by Proposition 6.15 and (6.9))} \\
&= h^{\mathcal{E}}_{B\cup\{x\}}[\mathbb{1}_B](y)^{p-1}R_{\mathcal{E}}(x,y)^{-1}.
\end{aligned} \tag{6.36}
$$

We obtain (6.33) by combining (6.35) and (6.36).

Next we prove (6.34). Let $h \in \mathcal{H}_{\mathcal{E},B}$ satisfy $\sup_B |h| < \infty$. Then we see that

$$
\begin{aligned}
h - h(x) &\leq h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[(h-h(x))^+\big|_{B\cup\{x\}}\Bigr] \quad \text{(by Propositions 6.26 and 6.15)} \\
&\leq h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[\operatorname*{osc}_B[h]\cdot \mathbb{1}_B^{B\cup\{x\}}\Bigr] \quad \text{(by Proposition 6.26 and } (h-h(x))^+(x) = 0) \\
&= \operatorname*{osc}_B[h]\cdot h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[\mathbb{1}_B^{B\cup\{x\}}\Bigr].
\end{aligned}
$$

Similarly, we have

$$
h - h(x) \geq -h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[(h-h(x))^-\big|_{B\cup\{x\}}\Bigr] \geq -\operatorname*{osc}_B[h]\cdot h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[\mathbb{1}_B^{B\cup\{x\}}\Bigr].
$$

Hence, by combining these estimates with (6.33), we get (6.34). □

Using Theorem 6.31, we can show the triangle inequality for $R_{\mathcal{E}}^{1/(p-1)}$.

Corollary 6.32. *$R_{\mathcal{E}}^{1/(p-1)}\colon X \times X \to [0,\infty)$ is a metric on X.*

Definition 6.33 (p-Resistance metric)**.** We define $\widehat{R}_{p,\mathcal{E}} \coloneqq R_{\mathcal{E}}^{1/(p-1)}$ and call $\widehat{R}_{p,\mathcal{E}}$ the *p-resistance metric* of $(\mathcal{E},\mathcal{F})$.

Proof of Corollary 6.32. It suffices to prove the triangle inequality $R_{\mathcal{E}}(x,z)^{1/(p-1)} \leq R_{\mathcal{E}}(x,y)^{1/(p-1)} + R_{\mathcal{E}}(y,z)^{1/(p-1)}$ for any $x,y,z \in X$ with $\#\{x,y,z\} = 3$. By (6.33) with $B = \{z\}$, we have $h^{\mathcal{E}}_{\{x,z\}}[\mathbb{1}_x^{\{x,z\}}](y) \leq \frac{R_{\mathcal{E}}(x,y)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}}$. By exchanging the roles of x and z, we get $h^{\mathcal{E}}_{\{x,z\}}[\mathbb{1}_z^{\{x,z\}}](y) \leq \frac{R_{\mathcal{E}}(y,z)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}}$. Since $\mathbb{1}_X = h^{\mathcal{E}}_{\{x,z\}}[\mathbb{1}_x^{\{x,z\}}] + h^{\mathcal{E}}_{\{x,z\}}[\mathbb{1}_z^{\{x,z\}}]$, we have

$$
1 \leq \frac{R_{\mathcal{E}}(x,y)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}} + \frac{R_{\mathcal{E}}(y,z)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}},
$$

which proves the desired triangle inequality for $R_{\mathcal{E}}^{1/(p-1)}$. □

Example 6.34. Let $p \in (1,\infty)$ and $(\mathcal{E},\mathcal{F})$ be a p-resistance form on the unit open interval $(0,1)$ given by

$$
\mathcal{F} \coloneqq W^{1,p}(0,1) \quad \text{and} \quad \mathcal{E}(u) \coloneqq \int_0^1 |\nabla u|^p \, dx.
$$

(Recall Example 6.3-(1).) For any $x, y \in (0,1)$ with $0 < x < y < 1$, we easily see that $u \in W^{1,p}(0,1)$ defined by $u(t) := (y-x)^{-1}(t-x)\mathbb{1}_{[x,y]}(t)$, $t \in (0,1)$, is $\mathcal{E}$-harmonic on $(0,1) \setminus \{x,y\}$. Therefore we have $R_{\mathcal{E}}(x,y) = (y-x)^{p-1}$ and the p-resistance metric $\widehat{R}_{p,\mathcal{E}}$ coincides with the Euclidean metric on $(0,1)$. In particular, the Hölder regularity estimate (6.34) is sharp. This example also shows that exponent $1/(p-1)$ in the p-resistance metric is sharp, that is, $R_{\mathcal{E}}^{\alpha}$ is not a metric for $\alpha > 1/(p-1)$ in general.

6.5 Elliptic Harnack inequality for non-negative harmonic functions

Throughout this subsection, we assume that $\{\Gamma\langle u\rangle\}_{u\in\mathcal{F}}$ is a family of p-energy measures on $(X, \mathcal{B}(X))$ dominated by $(\mathcal{E}, \mathcal{F})$ and satisfies (Cla)$_p$. For ease of the notation, we set $\widehat{R}_p := \widehat{R}_{p,\mathcal{E}} = R_{\mathcal{E}}^{1/(p-1)}$.

In this subsection, we establish the elliptic Harnack inequality for non-negative $\mathcal{E}$-superharmonic functions under some extra analytic conditions (Theorem 6.36). We mainly follow the argument in [Cap07], but we assume the two-point estimate (6.39) instead of the Poincaré inequality [Cap07, (2.4)] (see also Remark 7.13). Let us start with the following log-Caccioppoli inequality under the assumption of the chain rule (CL2).

Lemma 6.35 (Log-Caccioppoli type inequality)**.** *Assume that $\{\Gamma\langle u\rangle\}_{u\in\mathcal{F}}$ satisfies the chain rule (CL2). Then there exists $C \in (0,\infty)$ (depending only on p) such that for any $A, \varepsilon \in (0,\infty)$ with $A > 1$, any $(x,s) \in X \times (0,\infty)$ and any $u \in \mathcal{F}$ such that $u \geq 0$ on X, u is $\mathcal{E}$-superharmonic on $B_{\widehat{R}_p}(x, As)$ and $\Gamma\langle u\rangle(X) = \mathcal{E}(u)$, it holds that*

$$\int_{B_{\widehat{R}_p}(x,s)} d\Gamma\langle\Phi_\varepsilon(u)\rangle \leq C \inf\bigl\{\mathcal{E}(\varphi) \bigm| \varphi \in \mathcal{F}, \varphi|_{B_{\widehat{R}_p}(x,s)} = 1, \operatorname{supp}_X[\varphi] \subseteq B_{\widehat{R}_p}(x, As)\bigr\}, \tag{6.37}$$

where $\Phi_\varepsilon \in C^1(\mathbb{R})$ is any function satisfying $\Phi_\varepsilon(x) = \log(x+\varepsilon) - \log\varepsilon$ for any $x \in [0,\infty)$.

Proof. Let $\varphi \in \mathcal{F}$ satisfy $\varphi|_{B_{\widehat{R}_p}(x,s)} = 1$, $\operatorname{supp}_X[\varphi] \subseteq B_{\widehat{R}_p}(x, As)$ and

$$\mathcal{E}(\varphi) = \inf\bigl\{\mathcal{E}(\varphi) \bigm| \varphi \in \mathcal{F}, \varphi|_{B_{\widehat{R}_p}(x,s)} = 1, \operatorname{supp}_X[\varphi] \subseteq B_{\widehat{R}_p}(x, As)\bigr\},$$

which exists by Theorem 6.13. Let $\varepsilon > 0$ and set $u_\varepsilon := u + \varepsilon$. Note that $\varphi^p u_\varepsilon^{1-p} \in \mathcal{F}$ by Proposition 2.3-(d) and Corollary 2.5-(a). We see that

$$\begin{aligned}
\int_{B_{\widehat{R}_p}(x,s)} d\Gamma\langle\Phi_\varepsilon(u)\rangle &\leq \int_{B_{\widehat{R}_p}(x,As)} \varphi^p \, d\Gamma\langle\Phi_\varepsilon(u)\rangle \\
&\overset{\text{(CL2)}}{=} \frac{1}{p-1} \int_{B_{\widehat{R}_p}(x,As)} \varphi^p \, d\Gamma\langle u_\varepsilon; u_\varepsilon^{1-p}\rangle \\
&\overset{\text{(CL2)}}{=} \frac{1}{1-p} \left(\int_{B_{\widehat{R}_p}(x,As)} d\Gamma\langle u_\varepsilon; \varphi^p u_\varepsilon^{1-p}\rangle - \int_{B_{\widehat{R}_p}(x,As)} u_\varepsilon^{1-p} \, d\Gamma\langle u_\varepsilon; \varphi^p\rangle \right)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(*)}{\leq} \frac{1}{1-p}\biggl(\mathcal{E}(u_{\varepsilon};\varphi^{p}u_{\varepsilon}^{1-p}) - \int_{B_{\widehat{R}_p}(x,As)} u_{\varepsilon}^{1-p}\,d\Gamma\langle u_{\varepsilon};\varphi^{p}\rangle\biggr) \\
&\overset{(**)}{\leq} \frac{-1}{1-p}\int_{B_{\widehat{R}_p}(x,As)} u_{\varepsilon}^{1-p}\,d\Gamma\langle u_{\varepsilon};\varphi^{p}\rangle \\
&\overset{\text{(CL2)}}{=} \frac{p}{p-1}\int_{B_{\widehat{R}_p}(x,As)} \varphi^{p-1}\,d\Gamma\langle \Phi_{\varepsilon}(u);\varphi\rangle \\
&\overset{(4.13)}{\leq} \frac{p}{p-1}\biggl(\frac{1}{2}\int_{B_{\widehat{R}_p}(x,As)} \varphi^{p}\,d\Gamma\langle \Phi_{\varepsilon}(u)\rangle\biggr)^{\frac{p-1}{p}}\biggl(2^{p-1}\int_{B_{\widehat{R}_p}(x,As)} d\Gamma\langle\varphi\rangle\biggr)^{\frac{1}{p}} \\
&\leq \frac{p}{p-1}\biggl(\frac{p-1}{2p}\int_{B_{\widehat{R}_p}(x,As)} \varphi^{p}\,d\Gamma\langle \Phi_{\varepsilon}(u)\rangle + \frac{2^{p-1}}{p}\int_{B_{\widehat{R}_p}(x,As)} d\Gamma\langle\varphi\rangle\biggr),
\end{aligned}
$$

where we used Theorem 4.18 and $\Gamma\langle u_{\varepsilon}\rangle(X) = \mathcal{E}(u_{\varepsilon})$ in $(*)$, the fact that u_{ε} is $\mathcal{E}$-superharmonic on $B_{\widehat{R}_p}(x,As)$ in $(**)$, and Young's inequality in the last inequality. Hence we obtain $\int_{B_{\widehat{R}_p}(x,s)} d\Gamma\langle\Phi_{\varepsilon}(u)\rangle \leq p^{-1}2^{p}\mathcal{E}(\varphi)$. □

Now we can prove the desired elliptic Harnack inequality as in the following theorem. We will see later in Theorem 7.15 that Theorem 6.36 is applicable to p.-c.f. self-similar structures equipped with good self-similar p-resistance forms (see Subsection 7.2 for the precise setting).

Theorem 6.36 (Elliptic Harnack inequality)**.** *Assume that there exist $\Upsilon\colon X\times(0,\infty)\to(0,\infty)$ and $A_1, A_2, C \in (0,\infty)$ with $A_1 \geq 1$ and $A_2 > 1$ such that the following hold:*

(i) *For any $(x,s)\in X\times(0,\infty)$,*

$$\Upsilon(x,2s) \leq C\Upsilon(x,s). \tag{6.38}$$

(ii) *For any $(x,s)\in X\times(0,\infty)$ and any $u\in\mathcal{F}$,*

$$\sup_{y,z\in B_{\widehat{R}_p}(x,s)} |u(y)-u(z)|^{p} \leq C\Upsilon(x,s)^{-1}\Gamma\langle u\rangle\bigl(B_{\widehat{R}_p}(x,A_1 s)\bigr). \tag{6.39}$$

(iii) *For any $(x,s)\in X\times(0,\infty)$ with $B_{\widehat{R}_p}(x,A_2 s)\neq X$,*

$$\inf\bigl\{\mathcal{E}(\varphi)\bigm|\varphi\in\mathcal{F},\, \varphi|_{B_{\widehat{R}_p}(x,s)} = 1,\, \operatorname{supp}_X[\varphi]\subseteq B_{\widehat{R}_p}(x,A_2 s)\bigr\} \leq C\Upsilon(x,s). \tag{6.40}$$

(iv) *$\{\Gamma\langle u\rangle\}_{u\in\mathcal{F}}$ satisfies the chain rule (CL2).*

Then there exist $C_{\mathrm{H}}\in(0,\infty)$ and $\delta_{\mathrm{H}}\in(0,1)$ such that for any $(x,s)\in X\times(0,\infty)$ with $B_{\widehat{R}_p}(x,\delta_{\mathrm{H}}^{-1}s)\neq X$ and any $u\in\mathcal{F}$ such that $u\geq 0$ on X, u is $\mathcal{E}$-superharmonic on $B_{\widehat{R}_p}(x,\delta_{\mathrm{H}}^{-1}s)$ and $\Gamma\langle u\rangle(X)=\mathcal{E}(u)$, it holds that

$$\sup_{B_{\widehat{R}_p}(x,s)} u \leq C_{\mathrm{H}} \inf_{B_{\widehat{R}_p}(x,s)} u. \tag{6.41}$$

Proof. Let $\varepsilon \in (0,\infty)$ and set $\delta_{\mathrm{H}} := (A_1A_2)^{-1}$. Let $(x,s) \in X \times (0,\infty)$ satisfy $B_{\widehat{R}_p}(x,\delta_{\mathrm{H}}^{-1}s) \neq X$, and let $u \in \mathcal{F}$ be such that $u \geq 0$ on X, u is $\mathcal{E}$-superharmonic on $B_{\widehat{R}_p}(x,\delta_{\mathrm{H}}^{-1}s)$ and $\Gamma\langle u\rangle(X) = \mathcal{E}(u)$. Set $u_\varepsilon := u + \varepsilon$, $M_\varepsilon := \sup_{B_{\widehat{R}_p}(x,s)} u_\varepsilon$ and $m_\varepsilon := \inf_{B_{\widehat{R}_p}(x,s)} u_\varepsilon$. From (RF1)$_p$, (3.9), (EM1)$_p$ and (EM2)$_p$, u_ε is $\mathcal{E}$-superharmonic on $B_{\widehat{R}_p}(x,\delta_{\mathrm{H}}^{-1}s)$ and $\Gamma\langle u_\varepsilon\rangle(X) = \mathcal{E}(u_\varepsilon)$. By (6.38), (6.39), (6.37) and (6.40), there exists $C_0 \in (0,\infty)$ independent of x,s,u,ε such that

$$\sup_{B_{\widehat{R}_p}(x,s)} \log u_\varepsilon - \inf_{B_{\widehat{R}_p}(x,s)} \log u_\varepsilon \leq C_0,$$

whence $\log\left(\frac{M_\varepsilon}{m_\varepsilon}\right) \leq C_0$. In particular, $M_\varepsilon/m_\varepsilon \leq e^{C_0}$. We obtain (6.41) by letting $\varepsilon \downarrow 0$. $\square$

7 Self-similar p-resistance forms and p-energy measures

In this section, we investigate p-resistance forms by focusing on the self-similar case as in Section 5. Throughout this section, we fix $p \in (1,\infty)$ and a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ with $\#S \geq 2$ and K connected.

7.1 Self-similar p-resistance forms

We first introduce the notion of *self-similar p-resistance form*.

Definition 7.1 (Self-similar p-resistance form)**.** Let $\rho = (\rho_i)_{i\in S} \in (0,\infty)^S$ and let $(\mathcal{E},\mathcal{F})$ be a p-resistance form on K. We say that $(\mathcal{E},\mathcal{F})$ is a *self-similar p-resistance form on* $\mathcal{L}$ *with weight* ρ if and only if $\mathcal{F} \subseteq C(K)$ and $(\mathcal{E},\mathcal{F})$ satisfies (5.5) and (5.6) (under the original topology of K implicit in $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ being a self-similar structure).

Throughout the rest of this section except Proposition 7.8 and Theorem 7.9, we fix a self-similar p-resistance form $(\mathcal{E},\mathcal{F})$ on $\mathcal{L}$ with weight $\rho = (\rho_i)_{i\in S} \in (0,\infty)^S$. Note that the topology induced by the p-resistance metric $\widehat{R}_{p,\mathcal{E}}$ of $(\mathcal{E},\mathcal{F})$ may be different from the original topology of K implicit in $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ being a self-similar structure. Under the present setting, in referring to a topology of K we always consider its original topology. Note also that then $\mathcal{F}$ is dense in $(C(K), \|\cdot\|_{\mathrm{sup}})$ by the compactness of K, (2.7), (RF1)$_p$, (RF3)$_p$ and the Stone–Weierstrass theorem (see, e.g., [Dud, Theorem 2.4.11]).

The following properties of the p-resistance metric are elementary.

Proposition 7.2. (1) *For any $x, y \in K$,*

$$R_{\mathcal{E}}(F_w(x), F_w(y)) \leq \rho_w^{-1} R_{\mathcal{E}}(x,y). \tag{7.1}$$

(2) *If $\min_{i\in S} \rho_i > 1$ and if either $\operatorname{diam}(K, \widehat{R}_{p,\mathcal{E}}) < \infty$ or $\mathcal{L}$ is a p.-c.f. self-similar structure, then $\widehat{R}_{p,\mathcal{E}}$ is compatible with the original topology of K, and in particular, V_* is dense in $(K, \widehat{R}_{p,\mathcal{E}})$.*

Remark 7.3. It is known that, if $p = 2$, $\min_{i \in S} \rho_i > 1$ and $\mathcal{L}$ is a p.-c.f. self-similar structure, then there exists $c \in (0, \infty)$ such that for any $x, y \in K$ and any $w \in W_*$,

$$R_{\mathcal{E}}(F_w(x), F_w(y)) \geq c\rho_w^{-1} R_{\mathcal{E}}(x, y); \tag{7.2}$$

see [Kig03, Theorem A.1]. We extend this result to the case of $p \in (1, \infty) \setminus \{2\}$ in Subsection B.3; see Theorem B.9.

Proof of Proposition 7.2. (1): This is immediate from (5.6). (See [Kig01, Lemma 3.3.5] for the case of $p = 2$.)

(2): We can show that $\widehat{R}_{p,\mathcal{E}}$ is compatible with the original topology of K, by following [Kig09, Proof of Proposition B.1] if $\operatorname{diam}(K, \widehat{R}_{p,\mathcal{E}}) < \infty$, and by following [Kig01, Proof of Theorem 3.3.4] if $\mathcal{L}$ is a p.-c.f. self-similar structure (see also Lemma 8.42 below). Then V_* is dense in $(K, \widehat{R}_{p,\mathcal{E}})$ since $\overline{V_*}^K = K$ by [Kig01, Lemma 1.3.11]. $\square$

The following proposition presents compatible sequences of p-resistance forms having a self-similarity.

Proposition 7.4. *Let $n \in \mathbb{N} \cup \{0\}$, let Λ be a partition of Σ and set $V_{n,\Lambda} \coloneqq \bigcup_{w \in \Lambda} F_w(V_n)$. Then for any $u \in \mathcal{F}|_{V_{n,\Lambda}}$,*

$$\mathcal{E}|_{V_{n,\Lambda}}(u) = \sum_{w \in \Lambda} \rho_w \mathcal{E}|_{V_n}(u \circ F_w), \tag{7.3}$$

$$h^{\mathcal{E}}_{V_{n,\Lambda}}(u) \circ F_w = h^{\mathcal{E}}_{V_n}(u \circ F_w) \quad \textit{for any } w \in \Lambda. \tag{7.4}$$

In particular, for any $m \in \mathbb{N} \cup \{0\}$ and any $u \in \mathcal{F}|_{V_{n+m}}$,

$$\mathcal{E}|_{V_{n+m}}(u) = \sum_{w \in W_m} \rho_w \mathcal{E}|_{V_n}(u \circ F_w). \tag{7.5}$$

Proof. Note that (7.5) follows from (7.3) by choosing $\Lambda = W_m$ and that the sequence $\mathcal{S} \coloneqq \bigl\{(V_{n,\Lambda}, \mathcal{E}_{V_{n,\Lambda}})\bigr\}_{n \in \mathbb{N} \cup \{0\}}$ is a compatible sequence of p-resistance forms by Proposition 6.15. Let $u \in \mathcal{F}|_{V_{n,\Lambda}}$. Then we see that

$$\begin{aligned}
\mathcal{E}|_{V_{n,\Lambda}}(u) &= \min\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F} \text{ with } v|_{V_{n,\Lambda}} = u\bigr\} \\
&\overset{(5.7)}{=} \min\left\{\sum_{w \in \Lambda} \rho_w \mathcal{E}(v \circ F_w) \,\middle|\, v \in \mathcal{F} \text{ with } v|_{V_{n,\Lambda}} = u\right\} \\
&\geq \sum_{w \in \Lambda} \rho_w \min\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F} \text{ with } v|_{V_n} = u \circ F_w\bigr\} = \sum_{w \in \Lambda} \rho_w \mathcal{E}|_{V_n}(u \circ F_w).
\end{aligned}$$

To prove the converse, define $v \in C(K)$ so that $v \circ F_w = h^{\mathcal{E}}_{V_n}[u \circ F_w]$ for any $w \in \Lambda$; note that such v is well-defined by (5.2). Then $v|_{V_{n,\Lambda}} = u$ and $v \in \mathcal{F}_{\mathcal{S}}$ by (5.5). Since

$$\mathcal{E}|_{V_{n,\Lambda}}(u) \leq \mathcal{E}(v) \overset{(5.7)}{=} \sum_{w \in \Lambda} \rho_w \mathcal{E}(v \circ F_w) = \sum_{w \in \Lambda} \rho_w \mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u \circ F_w]\bigr) = \sum_{w \in \Lambda} \rho_w \mathcal{E}|_{V_n}(u \circ F_w),$$

we have (7.3). Next we prove (7.4). We have $\mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u]\circ F_w\bigr)\geq\mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u\circ F_w]\bigr)$ for any $w\in\Lambda$. Since

$$\begin{aligned}\mathcal{E}|_{V_{n,\Lambda}}(u)=\mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u]\bigr)&=\sum_{w\in\Lambda}\rho_w\mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u]\circ F_w\bigr)\\&\geq\sum_{w\in\Lambda}\rho_w\mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u\circ F_w]\bigr)=\sum_{w\in\Lambda}\rho_w\mathcal{E}|_{V_n}(u\circ F_w)=\mathcal{E}|_{V_{n,\Lambda}}(u),\end{aligned}$$

we obtain $\mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u]\circ F_w\bigr)=\mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u\circ F_w]\bigr)$ for any $w\in\Lambda$. The uniqueness in Theorem 6.13 implies $h^{\mathcal{E}}_{V_{n,\Lambda}}[u]\circ F_w=h^{\mathcal{E}}_{V_n}[u\circ F_w]$. □

The following corollary is an immediate consequence of Proposition 6.19.

Corollary 7.5. *Assume that $\mathcal{L}=(K,S,\{F_i\}_{i\in S})$ is a p.-c.f. self-similar structure. Then*

$$\mathcal{F}=\Bigl\{u\in C(K)\Bigm|\lim_{n\to\infty}\mathcal{E}|_{V_n}(u|_{V_n})<\infty\Bigr\},\tag{7.6}$$

$$\mathcal{E}(u;v)=\lim_{n\to\infty}\mathcal{E}|_{V_n}(u|_{V_n};v|_{V_n})\quad\text{for any }u,v\in\mathcal{F},\tag{7.7}$$

$$\lim_{n\to\infty}\mathcal{E}\bigl(u-h^{\mathcal{E}}_{V_n}[u|_{V_n}]\bigr)=0\quad\text{for any }u\in\mathcal{F}.\tag{7.8}$$

Proof. By $\mathcal{F}\subseteq C(K)$ and $\overline{V_*}^K=K$ we have (6.15), and therefore (7.6) follows from (6.16), $\mathcal{F}\subseteq C(K)$ and $\overline{V_*}^K=K$, (7.7) from (6.17), and (7.8) from (6.18). (Note that (6.19) may not be applicable to the present situation because the topology considered in (6.19) is that induced by $R_{\mathcal{E}}^{1/p}$ and may be different from the original topology of K.) □

The following proposition gives characterizations of $\mathcal{E}$-harmonic functions on $K\setminus V_n$.

Proposition 7.6. *Let $n\in\mathbb{N}\cup\{0\}$ and $h\in C(K)$. Then the following two conditions are equivalent to each other:*

(1) $h\in\mathcal{H}_{\mathcal{E},V_n}$.

(2) $h\circ F_w\in\mathcal{H}_{\mathcal{E},V_0}$ *for any* $w\in W_n$.

If in addition $\mathcal{L}$ is a p.-c.f. self-similar structure, then each of (1) and (2) above is equivalent also to the following condition:

(3) *For any $m\in\mathbb{N}$ with $m>n$ and any $x\in V_m\setminus V_n$,*

$$\sum_{w\in W_m;\,x\in F_w(V_0)}\rho_w\mathcal{E}|_{V_0}\Bigl(h\circ F_w|_{V_0};\mathbb{1}^{V_0}_{F_w^{-1}(x)}\Bigr)=0.\tag{7.9}$$

Proof. To see (1) ⇒ (2), let $w\in W_n$, $\varphi\in\mathcal{F}^0(K\setminus V_0)$ and define $(F_w)_*\varphi\colon K\to\mathbb{R}$ by

$$(F_w)_*\varphi:=\begin{cases}\varphi\circ F_w^{-1}&\text{on }K_w,\\0&\text{on }K\setminus K_w.\end{cases}$$

Then since $(F_w)_*\varphi \in C(K)$ by $\varphi|_{V_0} = 0$ and (5.2), it follows from (5.5) that $(F_w)_*\varphi \in \mathcal{F}^0(K \setminus V_n)$, and then from (1) and (5.6) that $0 = \mathcal{E}(h; (F_w)_*\varphi) = \rho_w \mathcal{E}(h \circ F_w; \varphi)$, proving $h \circ F_w \in \mathcal{H}_{\mathcal{E},V_0}$, namely (2). The converse implication (2) $\Rightarrow$ (1) is obvious from (5.6).

Next we prove the equivalence between (1) and (3) for a p.-c.f. self-similar structure $\mathcal{L}$. We first show (1) $\Rightarrow$ (3). For any $m > n$ and any $\varphi \in \mathcal{F}^0(K \setminus V_n)$, we note that $h^{\mathcal{E}}_{V_m}[\varphi|_{V_m}]\big|_{V_n} = 0$. Then, for any $h \in \mathcal{H}_{\mathcal{E},V_n}$, we have from (7.5) that

$$0 = \mathcal{E}|_{V_m}(h|_{V_m}; \varphi|_{V_m}) = \sum_{w \in W_m} \rho_w \mathcal{E}|_{V_0}\bigl(h \circ F_w|_{V_0}; \varphi \circ F_w|_{V_0}\bigr) \quad \text{for any } \varphi \in \mathcal{F}^0(K \setminus V_0).$$

By choosing $\varphi \in \mathcal{F}^0(K \setminus V_n)$ so that $\varphi|_{V_m} = \mathbb{1}^{V_m}_x$ for $x \in V_m \setminus V_n$, we obtain (3). We next assume that $h \in C(K)$ satisfies (7.9) and fix $\varphi \in \mathcal{F}^0(K \setminus V_n)$ in order to show the converse implication (3) $\Rightarrow$ (1). For $m > n$, we see from (7.5), $\varphi|_{V_n} = 0$ and (7.9) that

$$\begin{aligned}
\mathcal{E}|_{V_m}(h|_{V_m}; \varphi|_{V_m}) &= \sum_{w \in W_m} \rho_w \mathcal{E}|_{V_0}\bigl(h \circ F_w|_{V_0}; \varphi \circ F_w|_{V_0}\bigr) \\
&= \sum_{w \in W_m} \sum_{y \in V_0} \varphi(F_w(y)) \rho_w \mathcal{E}|_{V_0}\bigl(h \circ F_w|_{V_0}; \mathbb{1}^{V_0}_y\bigr) \\
&= \sum_{x \in V_m \setminus V_n} \varphi(x) \sum_{w \in W_m; x \in F_w(V_0)} \rho_w \mathcal{E}|_{V_0}\Bigl(h \circ F_w|_{V_0}; \mathbb{1}^{V_0}_{F_w^{-1}(x)}\Bigr) = 0.
\end{aligned}$$

Letting $m \to \infty$ here on the basis of (7.7), we obtain $\mathcal{E}(h; \varphi) = 0$, and hence $h \in \mathcal{H}_{\mathcal{E},V_n}$. $\square$

Thanks to the self-similarity, we can get the following localized version of the weak comparison principle (recall Proposition 6.26).

Proposition 7.7 (A localized weak comparison principle)**.** *Let $n \in \mathbb{N} \cup \{0\}$, $w \in W_n$, and let $u, v \in \mathcal{H}_{\mathcal{E},V_n}$ satisfy $u(x) \le v(x)$ for any $x \in F_w(V_0)$. Then $u(x) \le v(x)$ for any $x \in K_w$.*

Proof. Since $h \circ F_w \in \mathcal{H}_{\mathcal{E},V_0}$ by the implication from (1) to (2) in Proposition 7.6, the assertion follows by applying Proposition 6.26 to $h \circ F_w$. $\square$

Next we show the monotonicity in p of the $\frac{1}{p-1}$-th power of the weight of a self-similar p-resistance form with constant weight on a p.-c.f. self-similar structure (Theorem 7.9 below); see also Theorem 8.32 for a similar result in another framework including the generalized Sierpiński carpets. The proof of Theorem 7.9 requires the following basic result, which is immediate from (5.2) and Proposition 2.10-(a).

Proposition 7.8. *Assume that $\mathcal{L}$ is a p.-c.f. self-similar structure. Let $k, n \in \mathbb{N} \cup \{0\}$ and let E be a p-resistance form on V_k. Let $\rho = (\rho_i)_{i \in S} \in (0,\infty)^S$ and define $\Lambda^n_\rho(E) \colon \mathbb{R}^{V_{k+n}} \to [0,\infty)$ by*

$$\Lambda^n_\rho(E)(u) := \sum_{w \in W_n} \rho_w E(u \circ F_w|_{V_k}), \quad u \in \mathbb{R}^{V_{k+n}}. \tag{7.10}$$

Then $\Lambda^n_\rho(E)$ is a p-resistance form on V_{k+n}.

Theorem 7.9. *Assume that $\mathcal{L}$ is a p.-c.f. self-similar structure. Let $p_1, p_2 \in (1,\infty)$ satisfy $p_1 \le p_2$, and for each $s \in \{1,2\}$, let $\rho_s \in (1,\infty)$ and let $(\mathcal{E}_s, \mathcal{F}_s)$ be a self-similar p_s-resistance form on $\mathcal{L}$ with weight $(\rho_s)_{i\in S}$. Then*

$$\rho_1^{1/(p_1-1)} \le \rho_2^{1/(p_2-1)}. \tag{7.11}$$

Proof. Let $s \in \{1,2\}$, $n \in \mathbb{N}\cup\{0\}$, and let $E_{s,n}$ be the p_s-resistance form on V_n given by

$$E_{s,n}(u) \coloneqq \rho_s^n \sum_{v\in W_n} \sum_{x,y\in V_0} |u(F_v(x)) - u(F_v(y))|^{p_s}, \quad u \in \mathbb{R}^{V_n},$$

so that $\Lambda^n_{(\rho_s)_{i\in S}}(E_{s,0}) = E_{s,n}$. Since both $E_{s,0}(\,\cdot\,)^{1/p_s}$ and $\mathcal{E}_s|_{V_0}(\,\cdot\,)^{1/p_s}$ are norms on the finite-dimensional vector space $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$, there exists $C_s \in [1,\infty)$ such that

$$C_s^{-1}E_{s,0}(u) \le \mathcal{E}_s|_{V_0}(u) \le C_s E_{s,0}(u) \quad \text{for any } u \in \mathbb{R}^{V_0}. \tag{7.12}$$

Since $\Lambda^n_{(\rho_s)_{i\in S}}(\mathcal{E}_s|_{V_0}) = \mathcal{E}_s|_{V_n}$ by (7.5), we see from (7.12) that

$$C_s^{-1}E_{s,n}(u) \le \mathcal{E}_s|_{V_n}(u) \le C_s E_{s,n}(u) \quad \text{for any } n \in \mathbb{N}\cup\{0\} \text{ and any } u \in \mathbb{R}^{V_n}. \tag{7.13}$$

Now we move to the proof of (7.11). Let us fix $x_0, y_0 \in V_0$ with $x_0 \ne y_0$ and set $B \coloneqq \{x_0, y_0\}$. Then we can find $w \in W_*$ so that $B \cap K_w = \emptyset$ and $h_{1,w} \coloneqq h_1 \circ F_w \notin \mathbb{R}\mathbb{1}_K$, where $h_1 \coloneqq h_B^{\mathcal{E}_1}[\mathbb{1}_{x_0}]$. (Supposing that $h_1 \circ F_w \in \mathbb{R}\mathbb{1}_K$ for any $w \in W_*$ with $B \cap K_w = \emptyset$, we would easily get a contradiction by using the connectedness of K, [Kig01, Theorem 1.6.2], (6.3) and $h_1(x_0) \ne h_1(y_0)$.) Noting that $c \coloneqq \inf_{x\in K_w} R_{\mathcal{E}_1}(x,B) \ge \mathcal{E}_1\bigl(h^{\mathcal{E}_1}_{V_{|w|}}[\mathbb{1}_B]\bigr)^{-1} > 0$ by Proposition 7.7 and (6.4) and that $0 \le h_1 \le 1$ by (6.32), for any $n \in \mathbb{N}\cup\{0\}$ we obtain

$$\begin{aligned}
&\mathcal{E}_2|_{V_0}(h_{1,w}|_{V_0}) \le \mathcal{E}_2|_{V_n}(h_{1,w}|_{V_n}) \quad \text{(by Proposition 6.15 and (6.6))}\\
&\overset{(7.13)}{\le} C_2 E_{2,n}(h_{1,w}|_{V_n})\\
&= C_2\rho_2^n \sum_{v\in W_n}\sum_{x,y\in V_0} |h_1(F_{wv}(x)) - h_1(F_{wv}(y))|^{p_2-p_1} \cdot |h_{1,w}(F_v(x)) - h_{1,w}(F_v(y))|^{p_1}\\
&\overset{(6.34)}{\le} C_2\rho_2^n \sum_{v\in W_n}\sum_{x,y\in V_0} \left(\frac{R_{\mathcal{E}_1}(F_{wv}(x), F_{wv}(y))}{R_{\mathcal{E}_1}(F_{wv}(x), B)}\right)^{\frac{p_2-p_1}{p_1-1}} \cdot |h_{1,w}(F_v(x)) - h_{1,w}(F_v(y))|^{p_1}\\
&\overset{(7.1)}{\le} C_2\Bigl(c^{-1}\sup_{x,y\in K} R_{\mathcal{E}_1}(x,y)\Bigr)^{(p_2-p_1)/(p_1-1)} \bigl(\rho_2\rho_1^{-(p_2-1)/(p_1-1)}\bigr)^n E_{1,n}(h_{1,w}|_{V_n})\\
&\overset{(7.13)}{\le} C_1C_2\Bigl(c^{-1}\sup_{x,y\in K} R_{\mathcal{E}_1}(x,y)\Bigr)^{(p_2-p_1)/(p_1-1)} \bigl(\rho_2\rho_1^{-(p_2-1)/(p_1-1)}\bigr)^n \mathcal{E}_1(h_{1,w}).
\end{aligned} \tag{7.14}$$

Since $\sup_{x,y\in K} R_{\mathcal{E}_1}(x,y) < \infty$ by Proposition 7.2-(2) and $\mathcal{E}_2|_{V_0}(h_{1,w}|_{V_0}), \mathcal{E}_1(h_{1,w}) \in (0,\infty)$, we conclude by letting $n \to \infty$ in (7.14) that $\rho_2\rho_1^{-(p_2-1)/(p_1-1)} \ge 1$, proving (7.11). $\square$

7.2 Two-point estimate and capacity upper estimate

This subsection is devoted to proving the two-point estimate and the (p,p)-Poincaré inequality in terms of self-similar p-energy measures, and showing also the capacity upper estimate under the additional assumption that $\mathcal{L}$ is a p.-c.f. self-similar structure.

Recall that we fix a self-similar p-resistance form $(\mathcal{E},\mathcal{F})$ on $\mathcal{L}$ with weight $\boldsymbol{\rho}=(\rho_i)_{i\in S}\in(0,\infty)^S$. In this subsection, we further assume that $\min_{i\in S}\rho_i>1$ and that the p-resistance metric $\widehat{R}_p:=\widehat{R}_{p,\mathcal{E}}$ of $(\mathcal{E},\mathcal{F})$ is compatible with the original topology of K, which is, in view of Propositions 6.4-(1) and 7.2-(2), equivalent to assuming $\min_{i\in S}\rho_i>1$ and $\operatorname{diam}(K,\widehat{R}_p)<\infty$. (Also by Proposition 7.2-(2), the assumption of $\operatorname{diam}(K,\widehat{R}_p)<\infty$ can be dropped when $\mathcal{L}$ is a p.-c.f. self-similar structure.) We also let $\{\Gamma_{\mathcal{E}}\langle u\rangle\}_{u\in\mathcal{F}}$ be the associated p-energy measures defined in (5.11). In the following definition, we introduce natural *scales* $\{\Lambda_s\}_{s\in(0,1]}$ with respect to $\widehat{R}_p$. See [Kig09, Kig20] for further details on scales.

Definition 7.10. (1) We define $\Lambda_1^{\widehat{R}_p}:=\{\emptyset\}$,

$$\Lambda_s^{\widehat{R}_p}:=\left\{w\;\middle|\;w=w_1\dots w_n\in W_*\setminus\{\emptyset\},(\rho_{w_1\dots w_{n-1}})^{-1/(p-1)}>s\geq\rho_w^{-1/(p-1)}\right\}$$

for each $s\in(0,1)$. (Note that $\{\Lambda_s^{\widehat{R}_p}\}_{s\in(0,1]}$ is the scale associated with the weight function $g(w):=\rho_w^{-1/(p-1)}$; see [Kig20, Definition 2.3.1].)

(2) For each $(s,x)\in(0,1]\times K$, we define $\Lambda_{s,0}^{\widehat{R}_p}(x):=\{w\in\Lambda_s^{\widehat{R}_p}\mid x\in K_w\}$ and $U_0^{\widehat{R}_p}(x,s):=\bigcup_{w\in\Lambda_{s,0}^{\widehat{R}_p}(x)}K_w$. Inductively, for $M\in\mathbb{N}$, define $\Lambda_{s,M}^{\widehat{R}_p}(x):=\{w\in\Lambda_s^{\widehat{R}_p}\mid K_w\cap U_{M-1}^{\widehat{R}_p}(x,s)\neq\emptyset\}$ and $U_M^{\widehat{R}_p}(x,s):=\bigcup_{w\in\Lambda_{s,M}^{\widehat{R}_p}(x)}K_w$.

It is easy to see that $\lim_{s\downarrow0}\min\{|w|\mid w\in\Lambda_s^{\widehat{R}_p}\}=\infty$, that $\Lambda_s^{\widehat{R}_p}$ is a partition of Σ for any $s\in(0,1]$, and that $\Lambda_{s_1}^{\widehat{R}_p}\leq\Lambda_{s_2}^{\widehat{R}_p}$ for any $s_1,s_2\in(0,1]$ with $s_1\leq s_2$. By [Kig20, Proposition 2.3.7], for any $x\in K$ and any $M\in\mathbb{N}\cup\{0\}$, $\bigl\{U_M^{\widehat{R}_p}(x,s)\bigr\}_{s\in(0,1]}$ is non-decreasing in s and forms a fundamental system of neighborhoods of x in K.

If $\bigl\{U_{M_*}^{\widehat{R}_p}(x,s)\bigr\}_{(x,s)\in K\times(0,1]}$ is comparable to the metric balls with respect to $\widehat{R}_p$ (in the sense of (7.15) below) for some $M_*\in\mathbb{N}$, then we have the following two-point estimate.

Proposition 7.11 (Two-point estimate)**.** *Assume that there exist $\alpha_1,\alpha_2\in(0,\infty)$ such that for any $(x,s)\in K\times(0,1]$,*

$$B_{\widehat{R}_p}(x,\alpha_1 s)\subseteq U_{M_*}^{\widehat{R}_p}(x,s)\subseteq B_{\widehat{R}_p}(x,\alpha_2 s).\tag{7.15}$$

Then there exist $C,A\in(0,\infty)$ with $A\geq1$ such that for any $(x,s)\in K\times(0,\infty)$ and any $u\in\mathcal{F}_{\mathrm{loc}}\bigl(B_{\widehat{R}_p}(x,As)\bigr)$,

$$\sup_{y,z\in B_{\widehat{R}_p}(x,s)}|u(y)-u(z)|^p\leq Cs^{p-1}\Gamma_{\mathcal{E}}\langle u\rangle\bigl(B_{\widehat{R}_p}(x,As)\bigr).\tag{7.16}$$

Proof. We can assume that $\alpha_1 \le \alpha_2$ and $\alpha_1 \le 1$ without loss of generality. Throughout this proof, we fix $x \in K$ and set $A \coloneqq \alpha_1^{-1}(\alpha_2 \vee \operatorname{diam}(K, \widehat{R}_p))$. We first consider the case of $s \in (\alpha_1, \infty)$. Note that $B_{\widehat{R}_p}(x, As) = K$. By (6.3) and Proposition 5.10-(a), for any $y, z \in B_{\widehat{R}_p}(x, s)$ and any $u \in \mathcal{F}$,

$$|u(y) - u(z)|^p \le \operatorname{diam}(K, \widehat{R}_p)^{p-1}\mathcal{E}(u) = C_1\alpha_1^{p-1}\Gamma_{\mathcal{E}}\langle u\rangle(K),$$

where $C_1 \coloneqq \alpha_1^{-(p-1)} \operatorname{diam}(K, \widehat{R}_p)^{p-1}$. This shows (7.16) in the case of $s \in (\alpha_1, \infty)$.

Next let $s \in (0, \alpha_1]$. Let U be a relatively compact open subset of K such that $U \supseteq U_{M_*}^{\widehat{R}_p}(x, \alpha_1^{-1}s)$ and let $u^\# \in \mathcal{F}$ satisfy $u = u^\#$ on U. For any $y, z \in B_{\widehat{R}_p}(x, s)$, there exists $\{v(i)\}_{i=1}^{2M_*+1} \subseteq \Lambda^{\widehat{R}_p}_{\alpha_1^{-1}s, M_*}(x)$ such that $y \in K_{v(1)}$, $z \in K_{v(2M_*+1)}$ and $K_{v(i)} \cap K_{v(i+1)} \neq \emptyset$ for each $i \in \{1, 2, \dots, 2M_*\}$. Let us fix $x_i \in K_{v(i)} \cap K_{v(i+1)}$ and $q_i \in V_0$ that satisfy $x_i = F_{v(i)}(q_i)$. We note that, for any $y', z' \in K_{v(i)}$,

$$\begin{aligned}
|u(y') - u(z')|^p &= \left|u(F_{v(i)}(F_{v(i)}^{-1}(y'))) - u(F_{v(i)}(F_{v(i)}^{-1}(z')))\right|^p \\
&\le R_{\mathcal{E}}(F_{v(i)}^{-1}(y'), F_{v(i)}^{-1}(z'))\mathcal{E}(u^\# \circ F_{v(i)}) \\
&\overset{(5.12)}{\le} \operatorname{diam}(K, \widehat{R}_p)^{p-1}\rho_{v(i)}^{-1}\Gamma_{\mathcal{E}}\langle u^\#\rangle(K_{v(i)}) = \operatorname{diam}(K, \widehat{R}_p)^{p-1}\rho_{v(i)}^{-1}\Gamma_{\mathcal{E}}\langle u\rangle(K_{v(i)}).
\end{aligned}$$

Hence

$$\begin{aligned}
&|u(y) - u(z)|^p \\
&\le (2M_* + 1)^{p-1}\left(|u(y) - u(x_1)|^p + \sum_{i=1}^{2M_*-1}|u(x_i) - u(x_{i+1})|^p + |u(x_{2M_*}) - u(z)|^p\right) \\
&\overset{(6.3)}{\le} \bigl((2M_* + 1)\operatorname{diam}(K, \widehat{R}_p)\bigr)^{p-1}\sum_{i=1}^{2M_*+1}\rho_{v(i)}^{-1}\Gamma_{\mathcal{E}}\langle u\rangle(K_{v(i)}) \\
&\le C_2 s^{p-1}\Gamma_{\mathcal{E}}\langle u\rangle\left(\bigcup_{i=1}^{2M_*+1}K_{v(i)}\right) \le C_2 s^{p-1}\Gamma_{\mathcal{E}_p}\langle u\rangle(B_{\widehat{R}}(x, \alpha_1^{-1}\alpha_2 s)),
\end{aligned}$$

where $C_2 \coloneqq \bigl((2M_* + 1)\alpha_1^{-1}\operatorname{diam}(K, \widehat{R}_p)\bigr)^{p-1}$. This proves (7.16) for $s \in (0, \alpha_1]$. $\square$

From (7.16), we easily obtain the following (p, p)-Poincaré inequality.

Proposition 7.12 ((p, p)-Poincaré inequality)**.** *Assume that there exist $\alpha_1, \alpha_2 \in (0, \infty)$ such that (7.15) holds for any $(x, s) \in K \times (0, 1]$. Let μ be a Radon measure on K with $\operatorname{supp}_K[\mu] = K$. Then there exist $C, A \in (0, \infty)$ with $A \ge 1$ such that for any $(x, s) \in K \times (0, \infty)$ and any $u \in \mathcal{F}_{\mathrm{loc}}\bigl(B_{\widehat{R}_p}(x, As)\bigr)$,*

$$\fint_{B_{\widehat{R}_p}(x,s)}\left|u - \fint_{B_{\widehat{R}_p}(x,s)} u\,d\mu\right|^p d\mu \le Cs^{p-1}\Gamma_{\mathcal{E}}\langle u\rangle\bigl(B_{\widehat{R}_p}(x, As)\bigr). \tag{7.17}$$

Proof. This is immediate from (7.16) and the obvious inequality

$$\fint_{B_{\widehat{R}_p}(x,s)} \left| u - \fint_{B_{\widehat{R}_p}(x,s)} u \, d\mu \right|^p d\mu \leq \sup_{y,z \in B_{\widehat{R}_p}(x,s)} |u(y) - u(z)|^p . \qquad \square$$

Remark 7.13. In [Cap07, Theorem 2.4], Capitanelli obtained an oscillation estimate like (7.16) from the (p,p)-Poincaré inequality [Cap07, (2.4)] under a suitable volume growth condition for the measure μ. This implication can be seen by a well-known telescopic sum argument (see, e.g., [HK98, Proof of Lemma 5.17]).

As shown in [KS24+, Lemma 6.7 and Proposition 6.9], if $\mathcal{L}$ is a p.-c.f. self-similar structure, then the condition (7.15) and the capacity upper estimate hold. Furthermore by [KS24+, Lemma 6.8], there exists a self-similar measure on $\mathcal{L}$ which is Ahlfors regular with respect to $\widehat{R}_p$ (see Definition 8.5-(2)). We record these results in the following proposition.

Proposition 7.14. *Assume that $\mathcal{L}$ is a p.-c.f. self-similar structure.*

(a) *There exist $\alpha_1, \alpha_2 \in (0,\infty)$ such that for any $(s,x) \in (0,1] \times K$,*

$$B_{\widehat{R}_p}(x, \alpha_1 s) \subseteq U_1^{\widehat{R}_p}(x,s) \subseteq B_{\widehat{R}_p}(x, \alpha_2 s). \tag{7.18}$$

(Equivalently, $\widehat{R}_p$ is 1-adapted to the weight function $g(w) := \rho_w^{-1/(p-1)}$; see [Kig20, Definition 2.4.1].)

(b) *Let $d_{\mathrm{f}}(\boldsymbol{\rho}) \in (0,\infty)$ be such that $\sum_{i \in S} \rho_i^{-d_{\mathrm{f}}(\boldsymbol{\rho})/(p-1)} = 1$, and let m be the self-similar measure on $\mathcal{L}$ with weight $\bigl(\rho_i^{-d_{\mathrm{f}}(\boldsymbol{\rho})/(p-1)}\bigr)_{i \in S}$. Then there exist $c_1, c_2 \in (0,\infty)$ such that for any $(x,s) \in K \times (0, 2\operatorname{diam}(K, \widehat{R}_p))$,*

$$c_1 s^{d_{\mathrm{f}}(\boldsymbol{\rho})} \leq m\bigl(B_{\widehat{R}_p}(x,s)\bigr) \leq c_2 s^{d_{\mathrm{f}}(\boldsymbol{\rho})}. \tag{7.19}$$

In particular, $\widehat{R}_p$ is metric doubling. (Recall that a metric space (X,d) is said to be *metric doubling* if and only if there exists $N \in \mathbb{N}$ such that any $(x,r) \in X \times (0,\infty)$ satisfies $B_d(x,r) \subseteq \bigcup_{i=1}^N B_d(x_i, r/2)$ for some $\{x_i\}_{i=1}^N \subseteq X$.)

(c) *There exists $C \in (0,\infty)$ such that for any $(x,s) \in K \times (0,\infty)$,*

$$\inf\bigl\{ \mathcal{E}(u) \bigm| u \in \mathcal{F},\, u|_{B_{\widehat{R}_p}(x,\alpha_1 s)} = 1,\, \operatorname{supp}[u] \subseteq B_{\widehat{R}_p}(x, 2\alpha_2 s) \bigr\} \leq C s^{-(p-1)}, \tag{7.20}$$

where α_1, α_2 are the constants in (7.18).

Proof. Although the proof is the same as [KS24+, Lemmas 6.7, 6.8 and Proposition 6.9], we recall the proof below for the reader's convenience. Throughout this proof, we set $\Lambda_s := \Lambda_s^{\widehat{R}_p}$ for ease of notation. Note that $K \neq \overline{V_0}^K$ since $\#V_0 < \infty$ and K is connected.

(a): By (7.1), we have $\operatorname{diam}(K_w, \widehat{R}_p) \leq \rho_w^{-1/(p-1)} \operatorname{diam}(K, \widehat{R}_p)$ for any $w \in W_*$, which implies the latter inclusion in (7.18) with $\alpha_2 \in (2\operatorname{diam}(K, \widehat{R}_p), \infty)$ arbitrary. (In particular, $\operatorname{diam}(K_w, \widehat{R}_p) < \alpha_2 s$ for any $w \in \Lambda_s$.) We will show the former inclusion in (7.18). It suffices to prove that there exists $\alpha_1 \in (0,\infty)$ such that $\widehat{R}_p(x,y) \geq \alpha_1 s$ for any $s \in (0,1]$, any $w,v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x,y) \in K_w \times K_v$. Let $\psi_q := h^{\mathcal{E}}_{V_0}\bigl[\mathbb{1}^{V_0}_q\bigr]$ for any $q \in V_0$. Fix $w \in \Lambda_s$ and let $u_w \in C(K)$ be such that, for $\tau \in \Lambda_s$,

$$u_w \circ F_\tau = \begin{cases} 1 & \text{if } \tau = w, \\ \sum_{q \in V_0; F_\tau(q) \in F_w(V_0)} \psi_q & \text{if } \tau \neq w \text{ and } K_\tau \cap K_w \neq \emptyset, \\ 0 & \text{if } K_\tau \cap K_w = \emptyset. \end{cases} \tag{7.21}$$

By the self-similarity of $(\mathcal{E},\mathcal{F})$, we have $u_w \in \mathcal{F}$ and

$$\mathcal{E}(u_w) = \sum_{\tau \in \Lambda_s} \rho_\tau \mathcal{E}(u_w \circ F_\tau) = \sum_{\tau \in \Lambda_s \setminus \{w\}; K_\tau \cap K_w \neq \emptyset} \rho_\tau \mathcal{E}\left(\sum_{q \in V_0; F_\tau(q) \in F_w(V_0)} \psi_q \right). \tag{7.22}$$

(Note that Λ_s is a partition of Σ.) Set $\overline{\rho} := \max_{i \in S} \rho_i \in (1,\infty)$ and $c_1 := \max_{q \in V_0} \mathcal{E}(\psi_q) \in (0,\infty)$. Then $\rho_\tau^{-1} \geq \overline{\rho}^{-1} s^{p-1}$ for any $\tau \in \Lambda_s$. Since $\#\{\tau \in \Lambda_s \mid K_\tau \cap K_w \neq \emptyset\} \leq (\#\mathcal{C}_{\mathcal{L}})(\#V_0)$ by [Kig01, Lemma 4.2.3], (7.22) together with Hölder's inequality implies that

$$\mathcal{E}(u_w) \leq (\#\mathcal{C}_{\mathcal{L}})(\#V_0)\overline{\rho} s^{-p+1}(\#V_0)^{p-1} c_1 =: (\alpha_1 s)^{-(p-1)}. \tag{7.23}$$

For any $v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x,y) \in K_w \times K_v$, we clearly have $u_w(x) = 1$ and $u_w(y) = 0$. Hence

$$\widehat{R}_p(x,y) \geq \mathcal{E}(u)^{-1/(p-1)} \geq \alpha_1 s,$$

which proves the desired result.

(b): This is immediate from (7.18), $\#\{\tau \in \Lambda_s \mid K_\tau \cap K_w \neq \emptyset\} \leq (\#\mathcal{C}_{\mathcal{L}})(\#V_0)$ ([Kig01, Lemma 4.2.3]) and $m(K_w) = \rho_w^{-d_{\mathrm{f}}(\boldsymbol{\rho})/(p-1)}$ (Proposition 5.6).

(c): It suffices to consider the case of $s \in (0,1]$ since $B_{\widehat{R}_p}(x, 2\alpha_2 s) = K$ for any $(x,s) \in K \times (1,\infty)$ and $\mathcal{E}^{-1}(0) = \mathbb{R}\mathbb{1}_K$ by (RF1)$_p$. Let $u_w \in \mathcal{F}$ be the same function as in the proof of (a) for each $w \in \Lambda_s$. Then $\varphi := \max_{w \in \Lambda_{s,1}(x)} u_w$ satisfies $\varphi|_{U_1^{\widehat{R}_p}(x,s)} = 1$. Since $\operatorname{diam}(K_w, \widehat{R}_p) < \alpha_2 s$, we see from (7.18) that $\operatorname{supp}[\varphi] \subseteq B_{\widehat{R}_p}(x, 2\alpha_2 s)$. By (2.6) for $(\mathcal{E},\mathcal{F})$, (7.23) and [Kig01, Lemma 4.2.3], we have $\varphi \in \mathcal{F}$ and

$$\mathcal{E}(\varphi) \leq \sum_{w \in \Lambda_{s,1}(x)} \mathcal{E}(u_w) \leq (\alpha_1 s)^{-(p-1)}(\#\mathcal{C}_{\mathcal{L}})(\#V_0) =: C s^{-(p-1)}. \qquad \square$$

Combining Propositions 7.11, 7.14 and Theorem 6.36, we obtain the elliptic Harnack inequality for self-similar p-resistance forms on p.-c.f. self-similar structures.

Theorem 7.15. *Assume that $\mathcal{L}$ is a p.-c.f. self-similar structure. Then there exist $C_{\mathrm{H}} \in (0,\infty)$ and $\delta_{\mathrm{H}} \in (0,1)$ such that for any $(x,s) \in K \times (0,\infty)$ with $B_{\widehat{R}_p}(x, \delta_{\mathrm{H}}^{-1} s) \neq K$ and any $u \in \mathcal{F}$ such that $u \geq 0$ on K and u is $\mathcal{E}$-superharmonic on $B_{\widehat{R}_p}(x, \delta_{\mathrm{H}}^{-1} s)$, it holds that*

$$\sup_{B_{\widehat{R}_p}(x,s)} u \leq C_{\mathrm{H}} \inf_{B_{\widehat{R}_p}(x,s)} u. \tag{7.24}$$

Proof. We have Theorem 6.36-(i),(ii),(iii) with $\Upsilon(x,s) \coloneqq s^{-(p-1)}$ by Propositions 7.11 and 7.14, and Theorem 6.36-(iv) holds by $\mathcal{F} \subseteq C(K)$ and Theorem 5.12. Since $\Gamma_{\mathcal{E}}\langle u \rangle(K) = \mathcal{E}(u)$ for any $u \in \mathcal{F}$ by Proposition 5.10-(a), the desired estimate (7.24) follows from Theorem 6.36. □

Remark 7.16. The results in this subsection, Propositions 7.11, 7.12, 7.14 and Theorem 7.15, are applicable to a large class of p.-c.f. self-similar structures. Indeed, their assumptions are all satisfied in the situation of Theorem 8.43, which summarizes the construction of regular self-similar p-resistance forms on p.-c.f. self-similar structures due to [CGQ22], and the assumptions of Theorem 8.43 in turn hold for strongly symmetric p.-c.f. self-similar sets (see Framework 8.46 and Definition 8.47) as proved in Theorem 8.50 below.

8 Constructions of p-energy forms satisfying the generalized p-contraction property

In the preceding sections, we have established fundamental results on p-energy forms satisfying the generalized p-contraction property (GC)$_p$, in particular p-Clarkson's inequality (Cla)$_p$. In this section, we would like to describe how to get a good p-energy form satisfying these properties in a few settings inspired by [Kig23] and [CGQ22]. (See also [KS24+] for another approach toward such a construction.)

8.1 p-Energy forms on p-conductively homogeneous compact metric spaces

In this subsection, we verify that p-energy forms on *p-conductively homogeneous* compact metric spaces constructed in [Kig23] satisfy (GC)$_p$. We mainly follow the notation and terminology of [Kig23] in this and the next subsections. We refer to [Kig23, Chapter 2] and [Kig20, Chapters 2 and 3] for further details.

Throughout this subsection, we fix a locally finite, non-directed infinite tree (T, E_T) in the usual sense (see [Kig23, Definition 2.1] for example), and fix a *root* $\phi \in T$ of T. (Here T is the set of vertices and E_T is the set of edges.) For any $w \in T \setminus \{\phi\}$, we use $\overline{\phi w}$ to denote the unique simple path in (T, E_T) from ϕ to w.

Definition 8.1 ([Kig23, Definition 2.2])**.** (1) For $w \in T$, define $\pi\colon T \to T$ by

$$\pi(w) \coloneqq \begin{cases} w_{n-1} & \text{if } w \neq \phi \text{ and } \overline{\phi w} = (w_0, \dots, w_n), \\ \phi & \text{if } w = \phi. \end{cases}$$

Set $S(w) \coloneqq \{v \in T \mid \pi(v) = w\} \setminus \{w\}$. Moreover, for $k \in \mathbb{N}$, we define $S^k(w)$ inductively as

$$S^{k+1}(w) = \bigcup_{v \in S(w)} S^k(v).$$

For $A \subseteq T$, define $S^k(A) := \bigcup_{w\in A} S^k(A)$.

(2) For $w \in T$ and $n \in \mathbb{N} \cup \{0\}$, define $|w| := \min\{n \geq 0 \mid \pi^n(w) = \phi\}$ and $T_n := \{w \in T \mid |w| = n\}$.

(3) Define $\Sigma := \{(\omega_n)_{n\geq 0} \mid \omega_n \in T_n \text{ and } \omega_n = \pi(\omega_{n+1}) \text{ for all } n \in \mathbb{N}\cup\{0\}\}$. For $\omega = (\omega_n)_{n\geq 0} \in \Sigma$, we write $[\omega]_n$ for $\omega_n \in T_n$. Define $\Sigma_w := \{(\omega_n)_{n\geq 0} \in \Sigma \mid \omega_{|w|} = w\}$ for $w \in T$, and $\Sigma_A := \bigcup_{w\in A} \Sigma_w$ for $A \subseteq T$.

Let us recall the definition of a partition parametrized by a rooted tree.

Definition 8.2 (Partition parametrized by a tree; [Kig20, Definition 2.2.1] and [Sas23, Lemma 3.6])**.** Let K be a compact metrizable topological space without isolated points. A family of non-empty compact subsets $\{K_w\}_{w\in T}$ of K is called a *partition of K parametrized by the rooted tree* (T, E_T, ϕ) if and only if it satisfies the following conditions:

(P1) $K_\phi = K$ and for any $w \in T$, $\#K_w \geq 2$ and $K_w = \bigcup_{v\in S(w)} K_v$.

(P2) For any $w \in \Sigma$, $\bigcap_{n\geq 0} K_{[\omega]_n}$ is a single point.

In the rest of this subsection, we fix a compact metrizable topological space without isolated points K, a locally finite rooted tree (T, E_T, ϕ) satisfying $\#\{v \in T \mid \{v, w\} \in E_T\} \geq 2$ for any $w \in T$, a partition $\{K_w\}_{w\in T}$ parametrized by (T, E_T, ϕ), a metric d on K with $\operatorname{diam}(K, d) = 1$, and a Borel probability measure m on K. Now we introduce a graph approximation $\{(T_n, E_n^*)\}_{n\in\mathbb{N}\cup\{0\}}$ of K.

Definition 8.3 ([Kig23, Proposition 2.8 and Definition 2.5-(3)])**.** For $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$, define

$$E_n^* := \bigl\{\{v, w\} \bigm| v, w \in T_n, v \neq w, K_v \cap K_w \neq \emptyset\bigr\},$$

and $E_n^*(A) = \bigl\{\{v, w\} \in E_n^* \bigm| v, w \in A\bigr\}$. Let d_n be the graph distance of (T_n, E_n^*). For $M \in \mathbb{N} \cup \{0\}$ and $w \in T_n$, define

$$\Gamma_M(w) := \{v \in T_n \mid d_n(v, w) \leq M\} \quad \text{and} \quad U_M(x; n) := \bigcup_{w\in T_n; x\in K_w} \bigcup_{v\in\Gamma_M(w)} K_v.$$

To state geometric assumptions in [Kig23], we need the following definition.

Definition 8.4 ([Kig20, Definitions 2.2.1 and 3.1.15])**.** (1) The partition $\{K_w\}_{w\in T}$ is said to be *minimal* if and only if $K_w \setminus \bigcup_{v\in T_{|w|}\setminus\{w\}} \neq \emptyset$ for any $w \in T$.

(2) The partition $\{K_w\}_{w\in T}$ is said to be *uniformly finite* if and only if $\sup_{w\in T} \#\Gamma_1(w) < \infty$. We set $L_* := \sup_{w\in T} \#\Gamma_1(w)$.

We also recall the following standard notion on metric measure spaces; see, e.g., [Hei, Kig20, MT] for further background.

Definition 8.5. (1) The measure m is said to be *volume doubling* with respect to the metric d if and only if there exists $C_{\mathrm{D}} \in (0, \infty)$ such that

$$m(B_d(x, 2r)) \leq C_{\mathrm{D}}\, m(B_d(x, r)) \quad \text{for any } (x, r) \in K \times (0, \infty). \tag{8.1}$$

The constant C_{D} is called the doubling constant of m.

(2) Let $Q \in (0,\infty)$. The measure m is said to be *Q-Ahlfors regular* with respect to the metric d if and only if there exists $C_{\mathrm{AR}} \in [1,\infty)$ such that

$$C_{\mathrm{AR}}^{-1} r^{Q} \leq m(B_d(x,r)) \leq C_{\mathrm{AR}} r^{Q} \quad \text{for any } (x,r) \in K \times (0, 2\operatorname{diam}(K,d)). \tag{8.2}$$

The measure m is simply said to be *Ahlfors regular* (with respect to d) if there exists $Q \in (0,\infty)$ such that m is Q-Ahlfors regular. Also, the metric d is said to be Q-Ahlfors regular if there exists a Borel measure μ on K which is Q-Ahlfors regular with respect to d.

(3) A metric ρ on K is said to be *quasisymmetric* to d, $\rho \underset{\mathrm{QS}}{\sim} d$ for short, if and only if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{\rho(x,b)}{\rho(x,a)} \leq \eta\left(\frac{d(x,b)}{d(x,a)}\right) \quad \text{for any } x,a,b \in K \text{ with } x \neq a.$$

(4) The *Ahlfors regular conformal dimension* of (K,d) is the value $\dim_{\mathrm{ARC}}(K,d)$ defined as

$$\dim_{\mathrm{ARC}}(K,d) := \inf\left\{ Q > 0 \;\middle|\; \begin{array}{l} \text{there exists a metric } \rho \text{ on } K \text{ such that} \\ \rho \underset{\mathrm{QS}}{\sim} d \text{ and } \rho \text{ is } Q\text{-Ahlfors regular} \end{array} \right\}.$$

If m is Ahlfors regular, then it is clearly volume doubling. It is well known that the existence of a Q-Ahlfors regular m on (K,d) implies that the Hausdorff dimension of (K,d) is Q.

Now we recall basic geometric conditions in [Kig23]. The conditions (1), (2) and (3) below are important to follow the rest of this paper.

Assumption 8.6 ([Kig23, Assumption 2.15])**.** Let $(K,\mathcal{O})$ be a connected compact metrizable space, $\{K_w\}_{w\in T}$ a partition parametrized by the rooted tree (T,ϕ), let d be a metric on K that is compatible with the topology $\mathcal{O}$ and $\operatorname{diam}(K,d) = 1$ and let m be a Borel probability measure on K. There exist $M_* \in \mathbb{N}$ and $r_* \in (0,1)$ such that the following conditions (1)–(5) hold.

(1) K_w is connected for any $w \in T$, $\{K_w\}_{w\in T}$ is minimal and uniformly finite, and $\inf_{m\geq 0}\min_{w\in T_m} \#S(w) \geq 2$.

(2) There exist $c_i \in (0,\infty)$, $i \in \{1,\dots,5\}$, such that the following conditions (2A)-(2C) are true.

(2A) For any $w \in T$,

$$c_1 r_*^{|w|} \leq \operatorname{diam}(K_w,d) \leq c_2 r_*^{|w|}. \tag{8.3}$$

(2B) For any $n \in \mathbb{N}$ and any $x \in K$,

$$B_d(x, c_3 r_*^{n}) \subseteq U_{M_*}(x;n) \subseteq B_d(x, c_4 r_*^{n}). \tag{8.4}$$

(In [Kig20], the metric d is called *M_*-adapted* if the condition (8.4) holds.)

(2C) For any $n \in \mathbb{N}$ and $w \in T_n$, there exists $x_w \in K_w$ satisfying

$$K_w \supseteq B_d(x_w, c_5 r_*^{n}). \tag{8.5}$$

(3) There exist $m_1 \in \mathbb{N}$, $\gamma_1 \in (0,1)$ and $\gamma \in (0,1)$ such that

$$m(K_w) \geq \gamma\, m(K_{\pi(w)}) \quad \text{for any } w \in T, \tag{8.6}$$

and

$$m(K_v) \leq \gamma_1\, m(K_w) \quad \text{for any } w \in T \text{ and any } v \in S^{m_1}(w). \tag{8.7}$$

Furthermore, m is volume doubling with respect to d and

$$m(K_w) = \sum_{v \in S(w)} m(K_v) \quad \text{for any } w \in T. \tag{8.8}$$

(4) There exists $M_0 \geq M_*$ such that for any $w \in T$, any $k \geq 1$ and any $v \in S^k(w)$,

$$\Gamma_{M_*}(v) \cap S^k(w) \subseteq \left\{ v' \in T_{|v|} \,\middle|\, \begin{array}{l} \text{there exist } l \leq M_0 \text{ and } (v_0, \ldots, v_l) \in S^k(w)^{l+1} \\ \text{such that } (v_{j-1}, v_j) \in E^*_{|v|} \text{ for any } j \in \{1, \ldots, l\} \end{array} \right\}.$$

(5) For any $w \in T$, $\pi(\Gamma_{M_*+1}(w)) \subseteq \Gamma_{M_*}(\pi(w))$.

We record a simple consequence of (8.8) in the next proposition.

Proposition 8.7. *Assume that the Borel probability measure m satisfies* (8.8)*. Then $m(K_v \cap K_w) = 0$ for any $v, w \in T$ with $v \neq w$ and $|v| = |w|$.*

Proof. Let $n \in \mathbb{N} \cup \{0\}$ and $v, w \in T_n$ satisfy $v \neq w$. Enumerate T_n as $\{z(1), z(2), \ldots, z(l_n)\}$ such that $z(1) = v$ and $z(2) = w$, where $l_n = \#T_n$. Inductively, define $\widetilde{K}_{z(j)}$ by

$$\widetilde{K}_{z(1)} = K_{z(1)}$$

and

$$\widetilde{K}_{z(j+1)} = K_{z(j+1)} \setminus \left(\bigcup_{i=1}^{k} \widetilde{K}_{z(i)} \right).$$

Then $\{\widetilde{K}_{z(j)}\}_{j=1}^{l_n}$ is a disjoint family of Borel sets and $\bigcup_{j=1}^{l_n} \widetilde{K}_{z(j)} = K$. Therefore,

$$1 = m(K) = \sum_{j=1}^{l_n} m\Big(\widetilde{K}_{z(j)}\Big).$$

On the other hand,(8.8) implies that

$$1 = m(K_\phi) = \sum_{j=1}^{l_n} m\big(K_{z(j)}\big).$$

Therefore, we conclude that $m\big(K_{z(j)} \setminus \widetilde{K}_{z(j)}\big) = 0$ for any $j \in \{1, \ldots, l_n\}$. In particular,

$$0 = m\Big(K_{z(2)} \setminus \widetilde{K}_{z(2)}\Big) = m\Big(K_w \setminus \big(K_w \setminus (K_v \cap K_w)\big)\Big) = m(K_v \cap K_w),$$

which completes the proof. $\square$

Next we introduce conductance, neighbor disparity constants and the notion of p-conductive homogeneity in Definitions 8.10, 8.8 and 8.11, following [Kig23, Sections 2.2, 2.3 and 3.3]. We will state some definitions and statements below for any $p \in (0,\infty)$ or $p \in [1,\infty)$, but on each such occasion we will explicitly declare that we let $p \in (0,\infty)$ or $p \in [1,\infty)$. Our main interest lies in the case $p \in (1,\infty)$.

Definition 8.8 ([Kig23, Definitions 2.17 and 3.4])**.** Let $p \in (0,\infty)$, $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$.

(1) Define $\mathcal{E}^n_{p,A} \colon \mathbb{R}^A \to [0,\infty)$ by

$$\mathcal{E}^n_{p,A}(f) \coloneqq \sum_{\{u,v\} \in E^*_n(A)} |f(u) - f(v)|^p, \quad f \in \mathbb{R}^A.$$

We write $\mathcal{E}^n_p(f)$ for $\mathcal{E}^n_{p,T_n}(f)$.

(2) For $A_0, A_1 \subseteq A$, define $\mathrm{cap}^n_p(A_0, A_1; A)$ by

$$\mathrm{cap}^n_p(A_0, A_1; A) \coloneqq \inf\bigl\{\mathcal{E}^n_{p,A}(f) \bigm| f \in \mathbb{R}^A, f|_{A_i} = i \text{ for } i \in \{0,1\}\bigr\}.$$

(3) (Conductance constant) For $A_1, A_2 \subseteq A$ and $k \in \mathbb{N} \cup \{0\}$, define

$$\mathcal{E}_{p,k}(A_1, A_2, A) \coloneqq \mathrm{cap}^{n+k}_p\bigl(S^k(A_1), S^k(A_2); S^k(A)\bigr).$$

For $M \in \mathbb{N}$, define $\mathcal{E}_{M,p,k} \coloneqq \sup_{w \in T} \mathcal{E}_{p,k}(\{w\}, T_{|w|} \setminus \Gamma_M(w), T_{|w|})$.

Let us recall the notion of *covering system*, which will be used to define neighbor disparity constants and the notion of conductive homogeneity.

Definition 8.9 ([Kig23, Definitions 2.26-(3) and 2.29])**.** Let $N_T, N_E \in \mathbb{N}$.

(1) Let $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$. A collection $\{G_i\}_{i=1}^k$ with $G_i \subseteq T_n$ is called a *covering of* $(A, E^*_n(A))$ *with covering numbers* (N_T, N_E) if and only if $A = \bigcup_{i=1}^k G_k$, $\max_{x \in A} \#\{i \mid x \in G_i\} \le N_T$ and for any $(u,v) \in E^*_n(A)$, there exists $l \le N_E$ and $\{w(1), \dots, w(l+1)\} \subseteq A$ such that $w(1) = u$, $w(l+1) = v$ and $(w(i), w(i+1)) \in \bigcup_{j=1}^k E^*_n(G_j)$ for any $i \in \{1, \dots, l\}$.

(2) Let $\mathscr{J} \subseteq \bigcup_{n \in \mathbb{N} \cup \{0\}} \{A \mid A \subseteq T_n\}$. The collection $\mathscr{J}$ is called a *covering system with covering number* (N_T, N_E) if and only if the following conditions are satisfied:
   - (i) $\sup_{A \in \mathscr{J}} \#A < \infty$.
   - (ii) For any $w \in T$ and any $k \in \mathbb{N}$, there exists a finite subset $\mathscr{N} \subseteq \mathscr{J} \cap T_{|w|+k}$ such that $\mathscr{N}$ is a covering of $\bigl(S^k(w), E^*_{|w|+k}(S^k(w))\bigr)$ with covering numbers (N_T, N_E).
   - (iii) For any $G \in \mathscr{J}$ and any $k \in \mathbb{N} \cup \{0\}$, if $G \subseteq T_n$, then there exists a finite subset $\mathscr{N} \subseteq \mathscr{J} \cap T_{n+k}$ such that $\mathscr{N}$ is a covering of $\bigl(S^k(G), E^*_{n+k}(S^k(G))\bigr)$ with covering numbers (N_T, N_E).

   The collection $\mathscr{J}$ is simply said to be a *covering system* if and only if there exist $(N_T, N_E) \in \mathbb{N}^2$ such that $\mathscr{J}$ is a covering system with covering number (N_T, N_E).

Definition 8.10 ([Kig23, Definitions 2.26-(1),(2) and 2.29])**.** Let $p \in (0,\infty)$, $n \in \mathbb{N}$ and $A \subseteq T_n$.

(1) For $k \in \mathbb{N} \cup \{0\}$ and $f\colon T_{n+k} \to \mathbb{R}$, define $P_{n,k}f\colon T_n \to \mathbb{R}$ by

$$(P_{n,k}f)(w) := \frac{1}{\sum_{v \in S^k(w)} m(K_v)} \sum_{v \in S^k(w)} f(v)m(K_v), \quad w \in T_n.$$

(Note that $P_{n,k}f$ depends on the measure m.)

(2) (Neighbor disparity constant) For $k \in \mathbb{N} \cup \{0\}$, define

$$\sigma_{p,k}(A) := \sup_{f\colon S^k(A) \to \mathbb{R}} \frac{\mathcal{E}^n_{p,A}(P_{n,k}f)}{\mathcal{E}^{n+k}_{p,S^k(A)}(f)}.$$

(3) Let $\mathscr{J} \subseteq \bigcup_{n \geq 0}\{A \mid A \subseteq T_n\}$ be a covering system. Define

$$\sigma^{\mathscr{J}}_{p,k,n} := \max\{\sigma_{p,k}(A) \mid A \in \mathscr{J}, A \subseteq T_n\} \quad \text{and} \quad \sigma^{\mathscr{J}}_{p,k} := \sup_{n \in \mathbb{N} \cup \{0\}} \sigma^{\mathscr{J}}_{p,k,n}.$$

Definition 8.11 ([Kig23, Definition 3.4])**.** Let $p \in [1,\infty)$. The compact metric space K (with a partition $\{K_w\}_{w \in T}$ and a measure m) is said to be *p-conductively homogeneous* if and only if there exists a covering system $\mathscr{J}$ such that

$$\sup_{k \in \mathbb{N} \cup \{0\}} \sigma^{\mathscr{J}}_{p,k} \mathcal{E}_{M_*,p,k} < \infty. \tag{8.9}$$

When we would like to clarify which partition is considered, we also say that K is p-conductively homogeneous with respect to $\{K_w\}_{w \in T}$.

For our purposes, the next consequence of (8.9) is more important than the original definition of the p-conductive homogeneity.

Theorem 8.12 (Part of [Kig23, Theorem 3.30])**.** *Let $p \in [1,\infty)$ and assume that Assumption 8.6 holds. If K is p-conductively homogeneous, then there exist $\alpha_0, \alpha_1 \in (0,\infty)$, $\sigma_p \in (0,\infty)$ and a covering system $\mathscr{J}$ such that for any $k \in \mathbb{N} \cup \{0\}$,*

$$\alpha_0 \sigma_p^{-k} \leq \mathcal{E}_{M_*,p,k} \leq \alpha_1 \sigma_p^{-k} \quad \textit{and} \quad \alpha_0 \sigma_p^{k} \leq \sigma^{\mathscr{J}}_{p,k} \leq \alpha_1 \sigma_p^{k}. \tag{8.10}$$

In particular, the constant σ_p is determined by the following limit:

$$\sigma_p = \lim_{k \to \infty} \bigl(\mathcal{E}_{M_*,p,k}\bigr)^{-1/k}. \tag{8.11}$$

Remark 8.13. The existence of the limit in (8.11) is true without the p-conductive homogeneity. Indeed, if $(K, d, \{K_w\}_{w \in T})$ satisfies the conditions Assumption 8.6-(1),(2),(4),(5), then [Kig23, Theorem 2.23] together with Fekete's lemma implies the existence of the limit in (8.11) for *any* $p \in (0,\infty)$. For convenience, we call σ_p the *p-scaling factor* of $(K, d, \{K_w\}_{w \in T})$.

We also recall the "Sobolev space" $\mathcal{W}^p$ introduced in [Kig23, Lemma 3.13].

Definition 8.14. Let $p \in [1,\infty)$. Assume that Assumption 8.6-(1),(2),(4),(5) hold and let σ_p be the constant in (8.11).

(1) For $n \in \mathbb{N} \cup \{0\}$, define $P_n \colon L^1(K,m) \to \mathbb{R}$ by $P_n f(w) := \fint_{K_w} f\,dm$, $w \in T_n$.
(2) Define $\mathcal{N}_p \colon L^p(K,m) \to [0,\infty]$ and a linear subspace $\mathcal{W}^p$ of $L^p(K,m)$ by

$$\mathcal{N}_p(f) := \left(\sup_{n \in \mathbb{N}\cup\{0\}} \sigma_p^n \mathcal{E}_p^n(P_n f)\right)^{1/p}, \quad f \in L^p(K,m),$$
$$\mathcal{W}^p := \bigl\{ f \in L^p(K,m) \bigm| \mathcal{N}_p(f) < \infty \bigr\},$$

and we equip $\mathcal{W}^p$ the norm $\|\cdot\|_{\mathcal{W}^p}$ defined by

$$\|f\|_{\mathcal{W}^p} := \Bigl(\|f\|_{L^p(K,m)}^p + \mathcal{N}_p(f)^p\Bigr)^{1/p}, \quad f \in \mathcal{W}^p.$$

(3) For a linear subspace $\mathcal{D}$ of $\mathcal{W}^p$, we define

$$\mathcal{U}_p(\mathcal{D}) := \left\{ \mathscr{E} \colon \mathcal{D} \to [0,\infty) \;\middle|\; \begin{array}{l} \mathscr{E}^{1/p} \text{ is a seminorm on } \mathcal{D}, \text{ there exist } \alpha_0, \alpha_1 \in (0,\infty) \\ \text{such that } \alpha_0 \mathcal{N}_p(f) \le \mathscr{E}(f)^{1/p} \le \alpha_1 \mathcal{N}_p(f) \text{ for any } f \in \mathcal{D} \end{array} \right\}.$$

For ease of notation, set $\mathcal{U}_p := \mathcal{U}_p(\mathcal{W}^p)$.
(4) For $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$, we define $\widetilde{\mathcal{E}}_{p,A}^n \colon L^p(K,m) \to [0,\infty)$ by

$$\widetilde{\mathcal{E}}_{p,A}^n(f) := \sigma_p^n \mathcal{E}_{p,A}^n(P_n f), \quad f \in L^p(K,m).$$

We also set $\widetilde{\mathcal{E}}_p^n(f) := \widetilde{\mathcal{E}}_{p,T_n}^n(f)$.

We have the following property on $\mathcal{N}_p$ thanks to the connectedness of K and Assumption 8.6-(3).

Proposition 8.15. *Let $p \in [1,\infty)$. Assume that Assumption 8.6 holds. Then $\mathcal{N}_p(f) = 0$ if and only if there exists $c \in \mathbb{R}$ such that $f(x) = c$ for m-a.e. $x \in K$.*

Proof. It is clear that $\mathcal{N}_p(f) = 0$ if f is constant. Assume that $\mathcal{N}_p(f) = 0$. Note that (T_n, E_n^*) is a connected graph for each $n \in \mathbb{N} \cup \{0\}$ ([Kig23, Proposition 2.8]). Therefore, $\mathcal{N}_p(f) = 0$ implies that for each $n \in \mathbb{N} \cup \{0\}$ there exists $c_n \in \mathbb{R}$ such that $P_n f(w) = c_n$ for any $w \in T_n$. By (8.8), we have $c_n = c_{n+1}$ and hence there exists $c \in \mathbb{R}$ such that $c_n = c$ for any $n \in \mathbb{N} \cup \{0\}$. Now we let $\mathscr{L}_f \subseteq K$ denote the set of *Lebesgue points of* f, i.e.,

$$\mathscr{L}_f := \left\{ x \in K \;\middle|\; \lim_{r \downarrow 0} \fint_{B_d(x,r)} |f(x) - f(y)|\, m(dy) = 0 \right\}. \tag{8.12}$$

Then, by the volume doubling property of m and the Lebesgue differentiation theorem (see, e.g., [Hei, Theorem 1.8]), we have $\mathscr{L}_f \in \mathcal{B}(K)$ and $m(K \setminus \mathscr{L}_f) = 0$. For any $x \in \mathscr{L}_f$ and any $n \in \mathbb{N} \cup \{0\}$, by Proposition 8.7 and Assumption 8.6-(2),(3),

$$|f(x) - c| = \left| f(x) - \fint_{U_{M_*}(x;n)} f\,dm \right| \le \frac{1}{m(U_{M_*}(x;n))} \int_{B_d(x, c_4 r_*^n)} |f(x) - f(y)|\, m(dy)$$

$$\leq C \fint_{B_d(x,c_4 r_*^n)} |f(x) - f(y)|\, m(dy),$$

where we used (8.4) and the volume doubling property of m in the last inequality. Here $C \in (0,\infty)$ is a constant that is independent of x, f and n. By letting $n \to \infty$ in the estimate above, we obtain $f(x) = c$ for any $x \in \mathscr{L}_f$, which completes the proof. □

As shown in [Shi24, Kig23], $\mathcal{W}^p$ is a nice Banach space embedded in $C(K)$ if K is p-conductively homogeneous and $p > \dim_{\mathrm{ARC}}(K,d)$. More generally, we can show the following theorem.

Theorem 8.16. *Let $p \in [1,\infty)$. Assume that $(K, d, \{K_w\}_{w\in T}, m)$ satisfies Assumption 8.6 and that K is p-conductively homogeneous. Then $\mathcal{W}^p$ is a Banach space and $\mathcal{W}^p$ is dense in $L^p(K,m)$. If $p \in (1,\infty)$, then $\mathcal{W}^p$ is reflexive and separable. Moreover, if in addition $p > \dim_{\mathrm{ARC}}(K,d)$, then $\mathcal{W}^p$ is a dense linear subspace of $(C(K), \|\cdot\|_{\sup})$.*

Remark 8.17. By [Kig20, Theorem 4.6.9], the condition $p > \dim_{\mathrm{ARC}}(K,d)$ is equivalent to $\sigma_p > 1$.

Proof of Theorem 8.16. Note that $\mathcal{W}^p$ is a Banach space by [Kig23, Lemma 3.24] and that $\mathcal{W}^p$ is dense in $L^p(K,m)$ by [Kig23, Lemma 3.28].

In the rest of this proof, we assume that $p \in (1,\infty)$. Let us show that $\mathcal{W}^p$ is reflexive. Theorem 8.12 and [Kig23, Lemma 2.27] together imply that there exists a constant $C \in (0,\infty)$ such that for any $k, l \in \mathbb{N}$, any $A \subseteq T_k$ and any $f \in \mathbb{R}^{S^l(A)}$,

$$\widetilde{\mathcal{E}}_{p,A}^{k}(P_{k,l}f) \leq C\widetilde{\mathcal{E}}_{p,S^l(A)}^{k+l}(f). \tag{8.13}$$

The rest of the proof is very similar to [MS25+, Proof of Theorem 6.17(ii)], so we give only a sketch (see also [Shi24, Theorem 5.9] and the proof of Theorem 8.19-(a) below). Define $\|\cdot\|_{p,n} \coloneqq \Bigl(\|\cdot\|_{L^p(K,m)}^p + \widetilde{\mathcal{E}}_p^n(\,\cdot\,)\Bigr)^{1/p}$, which can be regarded as the L^p-norm on $K \sqcup E_n^*$. Also, we consider $\widetilde{\mathcal{E}}_p^n$ as a $[0,\infty]$-valued functional on $L^p(K,m)$. From [Dal, Theorem 8.5 and Proposition 11.6], by extracting a subsequence of $\{\widetilde{\mathcal{E}}_p^n\}_{n\in\mathbb{N}}$ if necessary, we can assume that $\{\widetilde{\mathcal{E}}_p^n\}_{n\in\mathbb{N}}$ Γ-converges to some p-homogeneous functional $E_p\colon L^p(K,m) \to [0,\infty]$ as $n \to \infty$. Then $\{\|\cdot\|_{p,n}\}_{n\in\mathbb{N}}$ Γ-converges to $|\!|\!|\cdot|\!|\!| \coloneqq \bigl(\|\cdot\|_{L^p(K,m)}^p + E_p\bigr)^{1/p}$ as $n \to \infty$, and hence $(|\!|\!|\cdot|\!|\!|^p, \mathcal{W}^p)$ is a p-energy form on (K,m) satisfying $(\mathrm{Cla})_p$. By using (8.13) and noting that $\lim_{k\to\infty} P_n f_k(w) = P_n f(w)$ for any $n \in \mathbb{N}\cup\{0\}$, any $w \in T_n$ and any $f, f_k \in L^p(K,m)$ with $\lim_{k\to\infty}\|f - f_k\|_{L^p(K,m)} = 0$, we can show that $|\!|\!|\cdot|\!|\!|$ is a norm on $\mathcal{W}^p$ that is equivalent to $\|\cdot\|_{\mathcal{W}^p}$. Thus, $\mathcal{W}^p$ is reflexive by Proposition 3.5 and the Milman–Pettis theorem. The separability of $\mathcal{W}^p$ immediately follows from Corollary 3.16 (see also [AHM23, Proposition 4.1]).

In the case of $p > \dim_{\mathrm{ARC}}(K,d)$, $\mathcal{W}^p$ can be identified with a subspace of $C(K)$ and is dense in $(C(K), \|\cdot\|_{\sup})$ by [Kig23, Lemmas 3.15, 3.16 and 3.19]. □

Let us introduce an important value, p-walk dimension, which will be a main topic in Section 9.

Definition 8.18 (p-Walk dimension)**.** Let $p \in (0, \infty)$. Assume that $(K, d, \{K_w\}_{w \in T})$ satisfies Assumption 8.6-(1),(2),(4),(5). Let $r_* \in (0,1)$ be the constant in (8.4), let σ_p be the p-scaling factor of $(K, d, \{K_w\}_{w \in T})$ (see (8.11) and Remark 8.13). We define $\tau_p \in \mathbb{R}$ by

$$\tau_p \coloneqq \frac{\log \sigma_p}{\log r_*^{-1}}. \tag{8.14}$$

If in addition m is Ahlfors regular with respect to d, then we define $d_{\mathrm{w},p} \in \mathbb{R}$ by

$$d_{\mathrm{w},p} \coloneqq d_{\mathrm{f}} + \tau_p, \tag{8.15}$$

where d_{f} denotes the Hausdorff dimension of (K, d). We call $d_{\mathrm{w},p}$ the *p-walk dimension* of $(K, d, \{K_w\}_{w \in T})$.

Now we prove the main result in this subsection, which is an improvement of [Kig23, Theorem 3.21].

Theorem 8.19. *Let $p \in (1, \infty)$. Assume that $(K, d, \{K_w\}_{w \in T}, m)$ satisfies Assumption 8.6 and that K is p-conductively homogeneous. Then there exist $\widehat{\mathcal{E}}_p \colon \mathcal{W}^p \to [0, \infty)$ and $c \in (0, \infty)$ such that the following hold:*

(a) *$(\widehat{\mathcal{E}}_p)^{1/p}$ is a seminorm on $\mathcal{W}^p$ and*

$$c\mathcal{N}_p(f) \le \widehat{\mathcal{E}}_p(f)^{1/p} \le \mathcal{N}_p(f) \quad \text{for any } f \in \mathcal{W}^p. \tag{8.16}$$

(b) *$(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ is a p-energy form on (K, m) satisfying $(\mathrm{GC})_p$.*

(c) (Invariance) *Let $\mathsf{T} \colon (K, \mathcal{B}(K), m) \to (K, \mathcal{B}(K), m)$ be Borel measurable and satisfy $\widetilde{\mathcal{E}}_p^n(f \circ \mathsf{T}) = \widetilde{\mathcal{E}}_p^n(f)$ for any $n \in \mathbb{N}$ and any $f \in L^p(K, m)$. Then $f \circ \mathsf{T} \in \mathcal{W}^p$ and $\widehat{\mathcal{E}}_p(f \circ \mathsf{T}) = \widehat{\mathcal{E}}_p(f)$ for any $f \in \mathcal{W}^p$.*

(d) *If in addition $p > \dim_{\mathrm{ARC}}(K, d)$, then $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ is a regular p-resistance form on K and there exist $C \in [1, \infty)$ such that*

$$C^{-1} d(x,y)^{\tau_p} \le R_{\widehat{\mathcal{E}}_p}(x,y) \le C d(x,y)^{\tau_p} \quad \text{for any } x, y \in K. \tag{8.17}$$

Proof. The most part of the proof will be very similar to that in [Kig23, Theorem 3.21], but we present the details because we do not assume $p > \dim_{\mathrm{ARC}}(K, d)$ unlike [Kig23, Theorem 3.21]. Let $\widehat{\mathcal{E}}_p$ be a subsequential Γ-limit of $\{\widetilde{\mathcal{E}}_p^n\}_n$ with respect to the topology of $L^p(K, m)$ as in [Kig23, Proof of Theorem 3.21], i.e., there exists a subsequence $\{\widetilde{\mathcal{E}}_p^{n'}\}_{n'}$ Γ-converging to $\widehat{\mathcal{E}}_p$ with respect to $L^p(K, m)$ as $n' \to \infty$. (Note that such a subsequential Γ-limit exists by [Dal, Theorem 8.5].)

(a): $\widehat{\mathcal{E}}_p$ is p-homogeneous by [Dal, Proposition 11.6]. The triangle inequality for $\widehat{\mathcal{E}}_p(\,\cdot\,)^{1/p}$ will be included in the proof of (b), so we shall prove (8.16). From the definition of the Γ-convergence, it is immediate that $\widehat{\mathcal{E}}_p(f) \le \liminf_{n \to \infty} \widetilde{\mathcal{E}}_p^n(f) \le \mathcal{N}_p(f)^p$. Let us show the former inequality in (8.16). Let $f \in \mathcal{W}^p$ and let $\{f_{n'}\}_{n'}$ be a recovery sequence of $\{\widetilde{\mathcal{E}}_p^n\}_{n'}$ at f, i.e., $\lim_{n'\to\infty} \|f - f_{n'}\|_{L^p(K,m)} = 0$ and $\widehat{\mathcal{E}}_p(f) = \lim_{n'\to\infty} \widetilde{\mathcal{E}}_p^{n'}(f_{n'})$. Since $\lim_{n'\to\infty} P_k f_{n'}(w) = P_k f(w)$ for any $k \in \mathbb{N}$ and any $w \in T_k$, by (8.13),

$$\widetilde{\mathcal{E}}_p^k(f) = \lim_{n'\to\infty} \widetilde{\mathcal{E}}_p^k(f_{n'}) \le C \lim_{n'\to\infty} \widetilde{\mathcal{E}}_p^{n'}(f_{n'}) = C\widehat{\mathcal{E}}_p(f),$$

where $C \in (0,\infty)$ is the constant in (8.13). We obtain the desired estimate by taking the supremum over $k \in \mathbb{N} \cup \{0\}$.

(b): Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2). Define $Q_n \colon L^1(K,m) \to L^1(K,m)$ by

$$Q_n f \coloneqq \sum_{w\in T_n} P_n f(w) \mathbb{1}_{K_w} \quad \text{for } f \in L^1(K,m). \tag{8.18}$$

Note that $\|Q_n\|_{L^p(K,m)\to L^p(K,m)} \le 1$ by (8.8) and Hölder's inequality. Let us show $\|f - Q_n f\|_{L^p(K,m)} \to 0$ as $n \to \infty$ for any $f \in L^p(K,m)$. Define the Hardy–Littlewood maximal operator $\mathscr{M} \colon L^p(K,m) \to L^0(K,m)$ by

$$\mathscr{M} f(x) = \sup_{r>0} \fint_{B_d(x,r)} |f(y)|\, m(dy), \quad x \in K.$$

Since m is volume doubling with respect to d by Assumption 8.6-(3), by [HKST, Theorem 3.5.6], there exists a constant $C \in (0,\infty)$ such that $\|\mathscr{M} f\|_{L^p(K,m)} \le C\,\|f\|_{L^p(K,m)}$ for any $f \in L^p(K,m)$. We also easily see that for any $f \in L^p(K,m)$ and any $x \in K$,

$$\begin{aligned}
|Q_n f(x)| \le \sum_{w\in T_n; x\in K_w} |P_n f(w)| &\le \sum_{w\in T_n; x\in K_w} \frac{m(B_d(x, 2c_2 r_*^n))}{m(K_w)} \fint_{B_d(x,2c_2r_*^n)} |f|\, dm \\
&\le \sum_{w\in T_n; x\in K_w} \frac{m(B_d(x, 2c_2 r_*^n))}{m(B_d(x_w, c_5 r_*^n))} \mathscr{M} f(x) \le C_1 \mathscr{M} f(x),
\end{aligned}$$

where $x_w \in K_w$ and c_2, c_5 are the same as in Assumption 8.6-(2) and we used the volume doubling property in the last inequality, and $C_1 \in (0,\infty)$ is a constant depending only on $\sup_{w\in T} \#\Gamma_1(w)$, c_2, c_5 and the doubling constant of m. Let $f \in L^p(K,m)$ and let $\mathscr{L}_f \subseteq K$ denote the set of Lebesgue points of f as in (8.12). Then $\mathscr{L}_f \in \mathcal{B}(K)$ and $m(K \setminus \mathscr{L}_f) = 0$ by the Lebesgue differentiation theorem for a volume doubling metric measure space (see, e.g., [Hei, Theorem 1.8]). Since

$$\begin{aligned}
|f(x) - Q_n f(x)| &\le \sum_{w\in T_n; x\in K_w} \fint_{K_w} |f(x) - f(y)|\, m(dy) \\
&\le C_1 \fint_{B_d(x,2c_2r_*^n)} |f(x) - f(y)|\, m(dy),
\end{aligned}$$

we have $|f(x) - Q_n f(x)| \to 0$ as $n \to \infty$ for any $x \in \mathscr{L}_f$. Now the dominated convergence theorem implies $\|f - Q_n f\|_{L^p(K,m)} \to 0$.

Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in (\mathcal{W}^p)^{n_1}$ and choose a recovery sequence $\{u_{k,n'}\}_{n'}$ of $\{\widetilde{\mathcal{E}}_p^{n'}\}_{n'}$ at u_k for each $k \in \{1, \dots, n_1\}$. For brevity, we write $\boldsymbol{u}_{n'} = (u_{1,n'}, \dots, u_{n_1,n'})$ and

$$\begin{aligned} P_{n'}\boldsymbol{u}_{n'}(v) &= \bigl(P_{n'}u_{1,n'}(v), \dots, P_{n'}u_{n_1,n'}(v)\bigr) \in \mathbb{R}^{n_1}, \quad v \in T_{n'}, \\ Q_{n'}\boldsymbol{u}_{n'}(v) &= \bigl(Q_{n'}u_{1,n'}(v), \dots, Q_{n'}u_{n_1,n'}(v)\bigr) \in \mathbb{R}^{n_1}, \quad v \in T_{n'}. \end{aligned}$$

Note that $\|u_{n'} - Q_{n'}u_{k,n'}\|_{L^p(K,m)} \to 0$ as $n' \to \infty$ by the fact proved in the previous paragraph. Similar to an argument in [Kig23, p. 46], by using $\|Q_n\|_{L^p(K,m)\to L^p(K,m)} \leq 1$ and the estimate (2.20), we have

$$\|T_l(\boldsymbol{u}) - T_l(Q_{n'}\boldsymbol{u}_{n'})\|_{L^p(K,m)} \xrightarrow[n'\to\infty]{} 0 \quad \text{for any } l \in \{1, \dots, n_2\}. \tag{8.19}$$

Also, by Proposition 8.7, we note that

$$P_{n'}\bigl(T_l(Q_{n'}\boldsymbol{u}_{n'})\bigr) = T_l(P_{n'}\boldsymbol{u}_{n'}) \in \mathbb{R}^{T_{n'}} \quad \text{for any } l \in \{1, \dots, n_2\}. \tag{8.20}$$

With these preparations, we prove $(\mathrm{GC})_p$ for $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$. We consider the case of $q_2 < \infty$ since the case of $q_2 = \infty$ is similar. By (8.19) and (8.20), we see that

$$\begin{aligned} \sum_{l=1}^{n_2} \widehat{\mathcal{E}}_p\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} &\overset{(8.19)}{\leq} \sum_{l=1}^{n_2} \liminf_{n'\to\infty} \widetilde{\mathcal{E}}_p^{n'}\bigl(T_l(Q_{n'}\boldsymbol{u}_{n'})\bigr)^{q_2/p} \\ &\overset{(8.20)}{\leq} \liminf_{n'\to\infty} \sum_{l=1}^{n_2} \left[\frac{\sigma_p^{n'}}{2} \sum_{(v,w)\in E_{n'}^*} |T_l(P_{n'}\boldsymbol{u}_{n'}(v)) - T_l(P_{n'}\boldsymbol{u}_{n'}(w))|^{q_2\cdot\frac{p}{q_2}}\right]^{q_2/p} \\ &\overset{(2.18)}{\leq} \liminf_{n'\to\infty} \left(\frac{\sigma_p^{n'}}{2} \sum_{(v,w)\in E_{n'}^*} \|T(P_{n'}\boldsymbol{u}_{n'}(v)) - T(P_{n'}\boldsymbol{u}_{n'}(v))\|_{\ell^{q_2}}^p\right)^{q_2/p} \\ &\overset{(2.2)}{\leq} \liminf_{n'\to\infty} \left(\frac{\sigma_p^{n'}}{2} \sum_{(v,w)\in E_{n'}^*} \|P_{n'}\boldsymbol{u}(v) - P_{n'}\boldsymbol{u}(v)\|_{\ell^{q_1}}^p\right)^{q_2/p} \\ &\leq \liminf_{n'\to\infty} \left(\frac{\sigma_p^{n'}}{2} \sum_{(v,w)\in E_{n'}^*} \left[\sum_{k=1}^{n_1} |P_{n'}u_{k,n'}(v) - P_{n'}u_{k,n'}(w)|^{p\cdot\frac{q_1}{p}}\right]^{p/q_1}\right)^{q_2/p} \\ &\overset{(*)}{\leq} \liminf_{n'\to\infty} \left(\sum_{k=1}^{n_1} \left[\frac{\sigma_p^{n'}}{2} \sum_{(v,w)\in E_{n'}^*} |P_{n'}u_{k,n'}(v) - P_{n'}u_{k,n'}(w)|^{p}\right]^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} \\ &\leq \left(\sum_{k=1}^{n_1} \limsup_{n'\to\infty} \widetilde{\mathcal{E}}_p^{n'}(u_{k,n'})^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} \leq \left(\sum_{k=1}^{n_1} \widehat{\mathcal{E}}_p(u_k)^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}}, \end{aligned} \tag{8.21}$$

where we used the triangle inequality for the ℓ^{p/q_1}-norm on E_n^* in $(*)$. Hence $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ satisfies $(\mathrm{GC})_p$.

(c): This is clear from the definitions of $\mathcal{W}^p$ and of $\widehat{\mathcal{E}}_p$.

(d): In the case of $p > \dim_{\mathrm{ARC}}(K,d)$, a combination of (b), [Kig23, Lemmas 3.13, 3.16, 3.19 and Theorem 3.21] and Theorem 8.16 implies that $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ is a regular p-resistance form on K. Then the estimate (8.17) is exactly the same as [Kig23, (3.21) in Lemma 3.34], so we complete the proof. □

Remark 8.20. The construction of $\mathcal{E}_p^{\Gamma}$ in [MS25+, Theorem 6.22] is very similar to that of $\widehat{\mathcal{E}}_p$ in the proof above although the setting and assumption on a partition in [MS25+] is slightly different from ours. Thanks to Proposition 8.7, the operators M_n and J_n defined in [MS25+, (6.8) and (6.9)] correspond to P_n and Q_n respectively. In particular, (8.19) and (8.20) for M_n and J_n are also true. Hence we can easily see that the p-energy form $(\mathcal{E}_p^{\Gamma}, \mathcal{F}_p)$ in [MS25+, Theorem 6.22] also satisfies $(\mathrm{GC})_p$.

Before concluding this subsection, we deal with the capacity upper estimate and a Poincaré-type inequality under the additional assumption on the Ahlfors regularity of m. In addition to the density of $\mathcal{W}^p$ in $C(K)$, we can obtain the following capacity upper bound under the p-conductive homogeneity of K if $p > \dim_{\mathrm{ARC}}(K,d)$ and m is Ahlfors regular.

Proposition 8.21 (Capacity upper estimate)**.** *Let $p \in (1,\infty)$ and $\lambda \in (1,\infty)$. Assume that Assumption 8.6 holds, that K is p-conductively homogeneous, that $p > \dim_{\mathrm{ARC}}(K,d)$ and that m is Ahlfors regular. Then there exists $C \in (0,\infty)$ such that for any $(x,r) \in K \times (0,1]$,*

$$\inf\bigl\{\mathcal{N}_p(u)^p \bigm| u \in \mathcal{W}^p,\, u|_{B_d(x,r)} = 1,\, \mathrm{supp}_K[u] \subseteq B_d(x,\lambda r)\bigr\} \le C\frac{m(B_d(x,r))}{r^{d_{\mathrm{w},p}}}. \tag{8.22}$$

Proof. Let $r_* \in (0,1)$ and $M_* \in \mathbb{N}$ be the constants in Assumption 8.6. For $r \in (0,1]$, choose $n \in \mathbb{N}$ as the minimal positive integer such that $c_2(M_*+1)r_*^n < (\lambda-1)r$, where c_2 is the constant in (8.3). Let $x \in K$ and set $T_n(x,r) \coloneqq T_n[B_d(x,r)]$ for ease of notation. Then, by the metric doubling property of (K,d), there exists $N \in \mathbb{N}$ which is independent of x and r such that $\#T_n(x,r) \le N$. By [Kig23, Lemma 3.18] and its proof, for any $w \in T_n(x,r)$ there exists $h_{M_*,w} \in \mathcal{W}^p$ such that $h_{M_*,w}|_{K_w} = 1$, $\mathrm{supp}_K[h_{M_*,w}] \subseteq U_{M_*}(w)$ and $\mathcal{N}_p(h_{M_*,w})^p \lesssim \sigma_p^n$. Now we define $\psi_{x,r} \coloneqq \sum_{w \in T_n(x,r)} h_{M_*,w} \in \mathcal{W}^p$. Then $\psi_{x,r}|_{B_d(x,r)} \ge 1$, $\mathrm{supp}_K[\psi_{x,r}] \subseteq B_d(x,\lambda r)$ and

$$\mathcal{N}_p(\psi_{x,r})^p \le N^{p-1}\max_{w \in T_n(x,r)}\mathcal{N}_p(h_{M_*,w})^p \lesssim \sigma_p^n = r_*^{n(d_{\mathrm{f}}-d_{\mathrm{w},p})} \lesssim r^{d_{\mathrm{f}}-d_{\mathrm{w},p}}.$$

Since m is Ahlfors regular and $\mathcal{N}_p(\psi_{x,r} \wedge 1) \lesssim \mathcal{N}_p(\psi_{x,r})$ by [Kig23, Theorem 3,21], we obtain (8.22). □

The following Poincaré-type inequality for cells is easy.

Lemma 8.22. *Let $p \in (1,\infty)$. Assume that Assumption 8.6 holds, that K is p-conductively homogeneous, and that m is Ahlfors regular. Then there exists a constant $C \in (0,\infty)$ such that for any $f \in L^p(K,m)$ and any $w \in T$,*

$$\int_{K_w}\left|f(x) - \fint_{K_w} f\,dm\right|^p m(dx) \le Cr_*^{|w|d_{\mathrm{w},p}}\liminf_{n\to\infty}\widetilde{\mathcal{E}}_{p,S^n(w)}^{n+|w|}(f). \tag{8.23}$$

Proof. Set $k := |w|$. Recall that $\lim_{n\to\infty}\|Q_n f - f\|_{L^p(K,m)} = 0$ as shown in the proof of Theorem 8.19-(b). Hence, for any $n \in \mathbb{N}$, we see that

$$\begin{aligned}
&\frac{1}{m(K_w)}\sum_{v\in S^n(w)}|P_{n+k}f(v) - P_k f(w)|^p\, m(K_v)\\
&= \frac{1}{m(K_w)}\sum_{v\in S^n(w)}\int_{K_v}|Q_{n+k}f(x) - P_k f(w)|^p\, m(dx)\\
&= \fint_{K_w}|Q_{n+k}f(x) - P_k f(w)|^p\, m(dx) \xrightarrow[n\to\infty]{} \fint_{K_w}|f(x) - P_k f(w)|^p\, m(dx),
\end{aligned} \tag{8.24}$$

where we used Proposition 8.7 in the second equality. By [Kig23, (5.11) in Theorem 5.11] and (8.10), there exists $C \in (0,\infty)$ which is independent of f and n such that

$$\frac{1}{m(K_w)}\sum_{v\in S^n(w)}|P_{n+k}f(v) - P_k f(w)|^p\, m(K_v) \leq C r_*^{k(d_{\mathrm{w},p}-d_{\mathrm{f}})}\widetilde{\mathcal{E}}^{n+k}_{p,S^n(w)}(f). \tag{8.25}$$

We obtain (8.23) by combining (8.24), (8.25), (8.5) and the Ahlfors regularity of m. □

To upgrade (8.23) to a Poincaré inequality for metric balls in K, we need the following standard fact.

Lemma 8.23 ([BB, Lemma 4.17])**.** *Let $q \in [1,\infty)$ and let $(Y,\mathcal{A},\mu)$ be a measure space. For any $f \in L^1(Y,\mu)$ and any $E \in \mathcal{A}$ with $\mu(E) \in (0,\infty)$,*

$$\fint_E\left|f - \fint_E f\, d\mu\right|^q d\mu \leq 2^q \inf_{a\in\mathbb{R}}\fint_E |f-a|^q\, d\mu. \tag{8.26}$$

Now we prove a Poincaré-type inequality in terms of discrete p-energy forms.

Proposition 8.24. *Let $p \in (1,\infty)$. Assume that Assumption 8.6 holds, that K is p-conductively homogeneous, and that m is Ahlfors regular. Then there exist $C,\alpha \in (0,\infty)$ such that for any $(x,r) \in K \times (0,1]$ and any $f \in L^p(K,m)$,*

$$\int_{B_d(x,r)}\left|f - \fint_{B_d(x,r)} f\, dm\right|^p dm \leq C r^{d_{\mathrm{w},p}}\liminf_{k\to\infty}\widetilde{\mathcal{E}}^k_{p,T_k[B_d(x,\alpha r)]}(f). \tag{8.27}$$

Proof. Throughout this proof, $M_* \in \mathbb{N}$ and $r_* \in (0,1)$ are the same constants as in Assumption 8.6. Let $(x,r) \in K \times (0,1]$. We first consider the case of $r \in (c_3 r_*, 1]$, where c_3 is the constant in (8.4). By Lemma 8.22 with $w = \phi$,

$$\begin{aligned}
\int_{B_d(x,r)}\left|f - \fint_{B_d(x,r)} f\, dm\right|^p dm &\overset{(8.26)}{\leq} 2^p\int_{B_d(x,r)}\left|f - \fint_K f\, dm\right|^p dm\\
&\leq \int_K\left|f - \fint_K f\, dm\right|^p dm \leq C\liminf_{n\to\infty}\widetilde{\mathcal{E}}^n_p(f),
\end{aligned}$$

where $C \in (0,\infty)$ is the constant in (8.23). Since $\mathrm{diam}(K,d) = 1$, this shows (8.27) for any $A \geq (c_3 r_*)^{-1}$. Hence it suffices to consider the remaining case, i.e., the case of $r \in (0, c_3 r_*]$. Let $n \in \mathbb{N}$ satisfy $c_3 r_*^n \geq r > c_3 r_*^{n+1}$. Set $\Gamma_{M_*}(x;n) \coloneqq \{v \in T \mid v \in \Gamma_{M_*}(w) \text{ for some } w \in T_n \text{ such that } x \in K_w\}$. Then we see that

$$
\begin{aligned}
&\int_{U_{M_*}(x;n)} \left| f(y) - \fint_{U_{M_*}(x;n)} f\,dm \right|^p m(dy) \\
&\leq 2^{p-1} \sum_{w \in \Gamma_{M_*}(x;n)} \left(\int_{K_w} |f(y) - P_n f(w)|^p \, m(dy) + m(K_w) \left| P_n f(w) - \fint_{U_{M_*}(x;n)} f\,dm \right|^p \right) \\
&\lesssim \sum_{w \in \Gamma_{M_*}(x;n)} \left(r^{d_{\mathrm{w},p}} \liminf_{k\to\infty} \widetilde{\mathcal{E}}^{n+k}_{p,S^k(w)}(f) + r^{d_{\mathrm{f}}} \left| P_n f(w) - \fint_{U_{M_*}(x;n)} f\,dm \right|^p \right). \qquad (8.28)
\end{aligned}
$$

Note that, by Proposition 8.7,

$$
P_n f(w) - \fint_{U_{M_*}(x;n)} f\,dm = \frac{1}{m(U_{M_*}(x;n))} \sum_{v \in \Gamma_{M_*}(x;n)} (P_n f(w) - P_n f(v)) m(K_v). \qquad (8.29)
$$

For any $w \in \Gamma_{M_*}(x;n)$, by choosing $w' \in \Gamma_{M_*}(x;n) \setminus \{w\}$ so that $P_n f(w) - P_n f(w') = \max_{v \in \Gamma_{M_*}(x;n)} |P_n f(w) - P_n f(v)|$, we have from (8.29) and Proposition 8.7 that

$$
\left| P_n f(w) - \fint_{U_{M_*}(x;n)} f\,dm \right| \leq |P_n f(w) - P_n f(w')| .
$$

Hence, by Hölder's inequality, (8.10) and [Kig23, (2.17)],

$$
\begin{aligned}
\left| P_n f(w) - \fint_{U_{M_*}(x;n)} f\,dm \right|^p &\leq (2M_* + 1)^{p-1} \mathcal{E}^n_{p,\Gamma_{M_*}(x;n)}(f) \\
&\lesssim r^{d_{\mathrm{w},p} - d_{\mathrm{f}}} \liminf_{k\to\infty} \widetilde{\mathcal{E}}^{n+k}_{p,S^k(\Gamma_{M_*}(x;n))}(f). \qquad (8.30)
\end{aligned}
$$

Note that $\#\Gamma_{M_*}(x;n) \leq L_*^{M_*+2}$ by Assumption 8.6-(1) and that $S^k(\Gamma_{M_*}(x;n)) \subseteq T_{n+k}[B_d(x, c_4 r_*^n)] \subseteq T_{n+k}[B_d(x, c_3^{-1} r_*^{-1} c_4 r)]$ by Assumption 8.6-(2), where c_4 is the same as in (8.4). Now we set $A \coloneqq (1 \vee c_4) c_3^{-1} r_*^{-1}$. Then, by (8.28) and (8.30),

$$
\begin{aligned}
&\int_{U_{M_*}(x;n)} \left| f(y) - \fint_{U_{M_*}(x;n)} f\,dm \right|^p m(dy) \\
&\overset{(8.30)}{\lesssim} r^{d_{\mathrm{w},p}} \liminf_{k\to\infty} \sum_{w \in \Gamma_{M_*}(x;n)} \widetilde{\mathcal{E}}^{n+k}_{p,S^k(\Gamma_{M_*}(x;n))}(f) \leq L_*^{M_*+2} r^{d_{\mathrm{w},p}} \liminf_{k\to\infty} \widetilde{\mathcal{E}}^{k}_{p,T_k[B_d(x,Ar)]}(f).
\end{aligned}
$$

Since

$$
\int_{B_d(z,s)} \left| f(y) - \fint_{B_d(x,r)} f\,dm \right|^p m(dy) \overset{(8.26)}{\leq} 2^p \int_{B_d(z,s)} \left| f(y) - \fint_{U_{M_*}(x;n)} f\,dm \right|^p m(dy)
$$

$$\stackrel{(8.4)}{\leq} 2^p \int_{U_{M_*}(x;n)} \left| f(y) - \fint_{U_{M_*}(x;n)} f\,dm \right|^p m(dy),$$

we obtain (8.27). □

8.2 Self-similar p-energy forms on p-conductively homogeneous self-similar structures

In this subsection, we construct a self-similar p-resistance form on self-similar structures under suitable assumptions. Our main result in this subsection, Theorem 8.30, implies that self-similar p-energy forms constructed in [Kig23, Theorem 4.6] satisfy $(\mathrm{GC})_p$.

We start with some preparations before constructing self-similar p-resistance forms. In the following definition, we introduce a good partition parametrized by a rooted tree.

Definition 8.25 ([Kig23, Definition 4.2])**.** Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure, let $r \in (0,1)$ and let $(j_s)_{s\in S} \in \mathbb{N}^S$. Define

$$j(w) := \sum_{i=1}^{n} j_{w_i} \quad \text{and} \quad g(w) := r^{j(w)} \quad \text{for } w = w_1 \ldots w_n \in W_n.$$

Define $\widetilde{\pi}(w_1 \cdots w_n) := w_1 \cdots w_{n-1}$ for $w = w_1 \ldots w_n \in W_n$ and

$$\Lambda^g_{r^k} := \{ w = w_1 \cdots w_n \in W_* \mid g(\widetilde{\pi}(w)) > r^k \geq g(w) \}.$$

Set $T^{(r)}_k := \{ (k,w) \mid w \in \Lambda^g_{r^k} \}$, $T^{(r)} := \bigcup_{k\in\mathbb{N}\cup\{0\}} T^{(r)}_k$ and define $\iota \colon T^{(r)} \to W_*$ as $\iota(k,w) = w$. Moreover, define $E_{T^{(r)}} \subseteq T^{(r)} \times T^{(r)}$ by

$$E_{T^{(r)}} := \left\{ ((k,v),(k+1,w)) \in T^{(r)}_k \times T^{(r)}_{k+1} \;\middle|\; k \in \mathbb{N}\cup\{0\}, v = w \text{ or } v = \widetilde{\pi}(w) \right\},$$

so that $(T^{(r)}, E_{T^{(r)}})$ is a rooted tree ([Kig23, Proposition 4.3]).

In the rest of this subsection, we presume the following assumption on the geometry of our self-similar structure.

Assumption 8.26. Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure with $\#S \geq 2$ and K connected. Set $K_w := K_{\iota(w)}$ for $w \in T^{(r_*)}_*$ for simplicity. There exist $r_* \in (0,1)$ and a metric d giving the original topology of K with $\mathrm{diam}(K,d) = 1$ such that $(K, d, \{K_w\}_{w\in T^{(r_*)}}, m)$ satisfies Assumption 8.6, where m is the self-similar measure on K with weight $(r_*^{j_s d_{\mathrm{f}}})_{s\in S}$ and d_{f} is the unique number satisfying $\sum_{s\in S} r_*^{j_s d_{\mathrm{f}}} = 1$.

Under Assumption 8.26, we have the d_{f}-Ahlfors regularity of m as follows.

Proposition 8.27 ([Kig23, Proposition 4.5])**.** *The value d_{f} coincides with the Hausdorff dimension of (K,d) and m is d_{f}-Ahlfors regular with respect to d.*

To obtain a self-similar p-energy form on $\mathcal{L}$, we first discuss the self-similarity of $\mathcal{W}^p$ (recall (5.5)). The following lemma can be shown in exactly the same way as [Kig23, Theorem 4.6-(1)] although the condition $p > \dim_{\mathrm{ARC}}(K, d)$ is assumed in [Kig23, Theorem 4.6].

Lemma 8.28. *For any $u \in L^p(K, m)$, any $k \in \mathbb{N} \cup \{0\}$ and any $n \in \mathbb{N} \cup \{0\}$ with $n \geq \max_{w \in W_k} j(w)$,*

$$\sum_{w \in W_k} \mathcal{E}_p^{n-j(w)}(P_{n-j(w)}(u \circ F_w)) \leq \mathcal{E}_p^n(P_n u). \tag{8.31}$$

In particular, if in addition K is p-conductively homogeneous (with respect to $\{K_w\}_{w \in T^{(r_*)}}$), then $u \circ F_w \in \mathcal{W}^p$ for any $u \in \mathcal{W}^p$ and any $w \in W_*$, and hence

$$\mathcal{W}^p \cap C(K) \subseteq \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\}. \tag{8.32}$$

Similar to the case of $p = 2$ (see, e.g., [Kig00, KZ92]), we will obtain a self-similar p-energy form on $(\mathcal{L}, m)$ with weight $\boldsymbol{\sigma}_p := (\sigma_p^{j_s})_{s \in S}$ as a fixed point obtained by applying Theorem 5.22. To this end, we need the converse inclusion of (8.32) and uniform estimates on $\mathcal{S}_{\boldsymbol{\sigma_p},n}(E)$ for any/some $E \in \mathcal{U}_p(\mathcal{W}^p \cap C(K))$; recall the definition of $\mathcal{S}_{\boldsymbol{\sigma}_p,n}$ in Definition 5.21. These conditions are true if K is p-conductively homogeneous and $p > \dim_{\mathrm{ARC}}(K, d)$ as described in the following proposition. (This result is essentially proved in [Kig23, Proof of Theorem 4.6].)

Proposition 8.29. *Let $p \in (1, \infty)$ and assume that K is p-conductively homogeneous (with respect to $\{K_w\}_{w \in T^{(r_*)}}$). If $p > \dim_{\mathrm{ARC}}(K, d)$, then*

$$\mathcal{W}^p = \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\}, \tag{8.33}$$

and there exists $C \in [1, \infty)$ such that for any $E \in \mathcal{U}_p$, any $u \in \mathcal{W}^p$ and any $n \in \mathbb{N}$,

$$C^{-1}\mathcal{N}_p(u)^p \leq \mathcal{S}_{\boldsymbol{\sigma_p},n}(E)(u) \leq C\mathcal{N}_p(u)^p. \tag{8.34}$$

Proof. The uniform estimate (8.34) follows from [Kig23, (4.6) and (4.8)]. (In the proof of [Kig23], the assumption $p > \dim_{\mathrm{ARC}}(K, d)$ is used to obtain [Kig23, (4.8)].) In the rest of the proof, we prove

$$\mathcal{W}^p \supseteq \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\} =: \mathcal{W}_S^p.$$

(The converse inclusion is proved in Lemma 8.28.) We note that the following estimate in [Kig23, lines 8-9 in p. 61] is true for every $u \in \mathcal{W}_S^p$: there exists a constant $C' \in (0, \infty)$ such that

$$\widetilde{\mathcal{E}}_p^n(u) \leq C' \sum_{w \in W_n} \sigma_p^{j(w)} \mathcal{N}_p(u \circ F_w)^p = C'\mathcal{S}_{\boldsymbol{\sigma}_p,n}(\mathcal{N}_p^p)(u) \quad \text{for any } n \in \mathbb{N},\, u \in \mathcal{W}_S^p. \tag{8.35}$$

(We need $p > \dim_{\mathrm{ARC}}(K, d)$ to obtain (8.35) by following the argument in [Kig23, p. 61].) Taking the supremum over $n \in \mathbb{N}$ in the left-hand side of (8.35), we have $\mathcal{W}_S^p \subseteq \mathcal{W}^p$. □

Now we can obtain a desired self-similar p-energy form. The following theorem is a generalization of [Kig23, Theorem 4.6] taking into account the case of $p \leq \dim_{\mathrm{ARC}}(K, d)$.

Theorem 8.30. *Let $p \in (1, \infty)$. Assume that Assumption 8.26 holds, that K is p-conductively homogeneous (with respect to $\{K_w\}_{w \in T^{(r_*)}}$) and that the following* pre-self-similarity conditions *hold:*

$$\mathcal{W}^p \cap C(K) = \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\}. \tag{8.36}$$

$$\text{There exists } C \in [1, \infty) \text{ such that (8.34) holds for any } u \in \mathcal{W}^p \cap C(K),\ n \in \mathbb{N}. \tag{8.37}$$

Let σ_p be the constant in (8.11), set $\boldsymbol{\sigma}_p := (\sigma_p^{j_s})_{s \in S}$, let $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ be any p-energy form on (K, m) given in Theorem 8.19 and set $\mathcal{F}_p := \overline{\mathcal{W}^p \cap C(K)}^{\mathcal{W}^p}$. Then there exists $\{n_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ with $n_k < n_{k+1}$ for any $k \in \mathbb{N}$ such that the following limit exists in $[0, \infty)$ for any $u \in \mathcal{F}_p$:

$$\mathcal{E}_p(u) := \lim_{k \to \infty} \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \mathcal{S}_{\boldsymbol{\sigma}_p, j}(\widehat{\mathcal{E}}_p)(u). \tag{8.38}$$

Moreover, the following properties hold:

(a) *$(\mathcal{E}_p, \mathcal{F}_p)$ is a self-similar p-energy form on $(\mathcal{L}, m)$ with weight $\boldsymbol{\sigma}_p$, and there exist $\alpha_0, \alpha_1 \in (0, \infty)$ such that $\alpha_0 \mathcal{N}_p(u)^p \leq \mathcal{E}_p(u) \leq \alpha_1 \mathcal{N}_p(u)^p$ for any $u \in \mathcal{F}_p$.*
(b) *$(\mathcal{E}_p, \mathcal{F}_p)$ satisfies (GC)$_p$.*
(c) *$(\mathcal{E}_p, \mathcal{F}_p)$ satisfies the strong local property (SL1).*
(d) *If in addition $p > \dim_{\mathrm{ARC}}(K, d)$, then $\mathcal{F}_p = \mathcal{W}^p$ and $(\mathcal{E}_p, \mathcal{F}_p)$ is a regular self-similar p-resistance form on $\mathcal{L}$ with weight $\boldsymbol{\sigma}_p$ and there exist $\alpha_0, \alpha_1 \in (0, \infty)$ such that*

$$\alpha_0\, d(x, y)^{\tau_p} \leq R_{\mathcal{E}_p}(x, y) \leq \alpha_1\, d(x, y)^{\tau_p} \quad \text{for any } x, y \in K. \tag{8.39}$$

Remark 8.31. (1) In the case of $p > \dim_{\mathrm{ARC}}(K, d)$, the pre-self-similarity conditions, (8.36) and (8.37), can be dropped by virtue of Proposition 8.29.

(2) On p.-c.f. self-similar structures, self-similar p-energy forms have been constructed also in [CGQ22]; we show in Subsection 8.3 below that the self-similar p-energy forms considered in [CGQ22] are all p-resistance forms (on V_*, and ones on K if the weight $\rho = (\rho_i)_{i \in S}$ of the form satisfies $\min_{i \in S} \rho_i > 1$). Note that *any* $p \in (1, \infty)$ is allowed in the framework of [CGQ22] unlike that of [Kig23] (see (d) above). While it is extremely hard to determine the value $\dim_{\mathrm{ARC}}(K, d)$ in general, for a p.-c.f. self-similar set K typically $\dim_{\mathrm{ARC}}(K, d) = 1$ (see [CP14, Theorem 1.2] for a sufficient condition for $\dim_{\mathrm{ARC}}(K, d) = 1$). In Appendix B.2, we prove that the Ahlfors regular conformal dimension of any *strongly symmetric p.-c.f. self-similar set* (see Framework 8.46 and Definition 8.47 below) is one when it is equipped with the p-resistance metric of a nice self-similar p-resistance form; the proof is based on the existence of self-similar p-resistance forms on strongly symmetric p.-c.f. self-similar sets proved in Theorem 8.50 as an extension of [CGQ22, Theorem 6.3].

Proof of Theorem 8.30. The existence of the limit in (8.38) and its properties (a), (b) and (c) are immediate from (8.36), (8.34), Lemma 5.16, Theorem 5.22, Propositions 5.23-(a) and 5.24. Let us verify (d). Recall that $\mathcal{W}^p \subseteq C(K)$ by $p > \dim_{\mathrm{ARC}}(K,d)$ (Theorem 8.16), whence $\mathcal{F}_p = \mathcal{W}^p$. A similar argument as in the proof of Theorem 8.19-(d) shows that $(\mathcal{E}_p, \mathcal{W}^p)$ is a regular p-resistance form on K satisfying (8.39). This completes the proof. $\square$

Similar to Theorem 7.9, we can obtain the monotonicity of $\sigma_p^{1/(p-1)}$ in $p > \dim_{\mathrm{ARC}}(K,d)$. Note that the following result is *not* restricted to p.-c.f. self-similar structures.

Theorem 8.32. *Assume that Assumption 8.26 holds. Let $p, q \in (\dim_{\mathrm{ARC}}(K,d), \infty)$ satisfy $p \le q$, and assume that K is s-conductively homogeneous (with respect to $\{K_w\}_{w \in T^{(r_*)}}$) for each $s \in \{p,q\}$. Then*

$$\sigma_p^{1/(p-1)} \le \sigma_q^{1/(q-1)}. \tag{8.40}$$

Proof. The proof is very similar to that of Theorem 7.9. By Proposition 8.29, (8.36) and (8.34) with $s \in \{p,q\}$ in place of p hold. Let $(\mathcal{E}_s, \mathcal{W}^s)$ be a self-similar s-resistance form on $\mathcal{L}$ given in Theorem 8.30 for each $s \in \{p,q\}$. Fix two distinct points $x_0, y_0 \in K$, set $B := \{x_0, y_0\}$ and define $h_p := h_B^{\mathcal{E}_p}\bigl[\mathbb{1}_{x_0}^B\bigr] \in \mathcal{W}^p$. Then $0 \le h_p \le 1$ by the weak comparison principle (Proposition 6.26) and we can find $w \in W_*$ satisfying $K_w \cap B = \emptyset$ and $h_{p,w} := h_p \circ F_w \notin \mathbb{R}\mathbb{1}_K$. Similar to (7.14), by using (6.34) and (7.1), we can show that for any $\{u,v\} \in E_n^*$,

$$|P_n h_{q,w}(u) - P_n h_{q,w}(v)|^{q-p} \le C r_*^{n(d_{\mathrm{w},p} - d_{\mathrm{f}})\frac{q-p}{p-1}},$$

where $C \in (0,\infty)$ is independent of n. Hence we have

$$\mathcal{E}_q^n(h_{p,w}) = \sum_{\{u,v\} \in E_n^*} |P_n h_{q,w}(u) - P_n h_{q,w}(v)|^q \le C r_*^{n(d_{\mathrm{w},p} - d_{\mathrm{f}})\frac{q-p}{p-1}} \mathcal{E}_p^n(h_{p,w}),$$

which implies that

$$\left(\sigma_q^{-1}\sigma_p^{(q-1)/(p-1)}\right)^n \widetilde{\mathcal{E}}_q^n(h_{p,w}) \le C \widetilde{\mathcal{E}}_p^n(h_{p,w}) \le C \mathcal{N}_p(h_{p,w})^p. \tag{8.41}$$

By (8.13), there exists $C_q \in (0,\infty)$ such that $\mathcal{N}_q(f)^q \le C_q \liminf_{n\to\infty} \widetilde{\mathcal{E}}_q^n(f)$ for any $f \in L^q(K,m)$. This together with (8.41) implies that

$$\mathcal{N}_q(h_{p,w})^q \limsup_{n\to\infty} \left(\sigma_q^{-1}\sigma_p^{(q-1)/(p-1)}\right)^n \le C' \mathcal{N}_p(h_{p,w})^p < \infty.$$

Since $\mathcal{N}_q(h_{p,w}) > 0$, we obtain $\sigma_q^{-1}\sigma_p^{(q-1)/(p-1)} \le 1$, which means (8.40). $\square$

We conclude this subsection by applying Theorem 6.36 (elliptic Harnack inequality) to the p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ in Theorem 8.30 in the case of $p > \dim_{\mathrm{ARC}}(K,d)$. We immediately obtain the following corollary by combining Propositions 5.10, 7.11, 8.21, 8.27, (8.4), (8.39) and Theorem 6.36.

Corollary 8.33 (Elliptic Harnack inequality for self-similar p-resistance form)**.** *Let $p \in (1,\infty)$. Assume that Assumption 8.26 holds, that K is p-conductively homogeneous (with respect to $\{K_w\}_{w\in T^{(r_*)}}$) and that $p > \dim_{\mathrm{ARC}}(K,d)$. Then $(\mathcal{E}_p, \mathcal{W}^p)$ and $\{\Gamma_{\mathcal{E}_p}\langle u\rangle\}_{u \in \mathcal{W}^p}$ given in Theorem 8.30 and in (5.11) respectively satisfy the assumptions, and thereby the property in the conclusion, of Theorem 6.36 with $K, m, \frac{m(B_d(x,s))}{s^{d_{\mathrm{w},p}}}$ in place of $X, \mu, \Upsilon(x,s)$.*

8.3 Self-similar p-resistance forms on p.-c.f. self-similar structures

In this subsection, under the condition $(\mathbf{R})$ in [CGQ22, p. 18], we see that the construction of p-energy forms on p.-c.f. self-similar structures constructed due to [CGQ22] yields p-resistance forms. The framework in [CGQ22] focuses only on p.-c.f. self-similar structures, but allows any $p \in (1, \infty)$ throughout, and also the choice of the weights of self-similar p-resistance forms is flexible there so that non-arithmetic weights can arise unlike Theorem 8.30; see Subsection B.1 for a proof that non-arithmetic weights do arise in the framework of Subsection 8.4 under a mild condition on the p.-c.f. self-similar structure $\mathcal{L}$.

In the following definitions, we recall some classes of p-energy forms on finite sets considered in [CGQ22].

Definition 8.34 ([CGQ22, Definition 2.1])**.** Let A be a finite set with $\#A \geq 2$. Let $E\colon \mathbb{R}^A \to [0, \infty)$ and consider the following conditions.

(i) $E(tf + (1-t)g) \leq tE(f) + (1-t)E(g)$ for any $f, g \in \mathbb{R}^A$ and any $t \in [0, 1]$.
(ii) $E(tf) = |t|^p E(f)$ for any $f \in \mathbb{R}^A$ and any $t \in \mathbb{R}$.
(iii) $E(f + t\mathbb{1}_A) = E(f)$ for any $f \in \mathbb{R}^A$ and any $t \in \mathbb{R}$.
(iv) $E(f^+ \wedge 1) \leq E(f)$ for any $f \in \mathbb{R}^A$.
(v) $\{f \in \mathbb{R}^A \mid E(f) = 0\} = \mathbb{R}\mathbb{1}_A$.

We define $\mathcal{M}_p(A)$ and $\widetilde{\mathcal{M}}_p(A)$ by

$$\mathcal{M}_p(A) \coloneqq \{E\colon \mathbb{R}^A \to [0, \infty) \mid E \text{ satisfies (i)-(v)}\}, \tag{8.42}$$

$$\widetilde{\mathcal{M}}_p(A) \coloneqq \{E\colon \mathbb{R}^A \to [0, \infty) \mid E \text{ satisfies (i)-(iv)}\}. \tag{8.43}$$

Definition 8.35 ([CGQ22, Definition 2.8])**.** Let A be a finite set with $\#A \geq 2$. For $E_1, E_2 \in \widetilde{\mathcal{M}}_p(A)$, define a metric $d_{\widetilde{\mathcal{M}}_p(A)}$ on $\widetilde{\mathcal{M}}_p(A)$ by

$$\begin{aligned} d_{\widetilde{\mathcal{M}}_p(A)}(E_1, E_2) &\coloneqq \sup\Bigl\{|E_1(u) - E_2(u)| \Bigm| u \in \mathbb{R}^A \setminus \mathbb{R}\mathbb{1}_A, \operatorname*{osc}_A[u] = 1\Bigr\} \\ &= \sup\Bigl\{|E_1(u) - E_2(u)| \Bigm| u \in \mathbb{R}^A, \operatorname*{osc}_A[u] \leq 1\Bigr\}. \end{aligned} \tag{8.44}$$

For ease of notation, we set $|E|_{\widetilde{\mathcal{M}}_p(A)} \coloneqq d_{\widetilde{\mathcal{M}}_p(A)}(E, 0)$ for $E \in \widetilde{\mathcal{M}}_p(A)$.

(1) We define $\mathcal{S}_p(A) \subseteq \mathcal{M}_p(A)$ by

$$\mathcal{S}_p(A) \coloneqq \left\{ E \in \mathcal{M}_p(A) \,\middle|\, \begin{array}{l} \text{there exists } (c_{xy})_{x,y\in A} \subseteq [0,\infty) \text{ such that} \\ E(f) = \sum_{x,y\in A} |f(x) - f(y)|^p c_{xy} \text{ for } f \in \mathbb{R}^A \end{array} \right\}. \tag{8.45}$$

Note that any functional in $\mathcal{S}_p(A)$ is a p-resistance form on A (see Example 6.3-(3)).

(2) We define $\mathcal{Q}'_p(A) \subseteq \mathcal{M}_p(A)$ by

$$\mathcal{Q}'_p(A) \coloneqq \left\{ E \in \mathcal{M}_p(A) \,\middle|\, \begin{array}{l} \text{there exist } B \supseteq A \text{ and } \widetilde{E} \in \mathcal{S}_p(B) \text{ such that} \\ \widetilde{E}\big|_A = E, \text{ where } \widetilde{E}\big|_A \text{ is the trace of } \widetilde{E} \text{ on } A \end{array} \right\}. \tag{8.46}$$

Let $\mathcal{Q}_p(A)$ be the closure of $\mathcal{Q}'_p(A)$ in $(\mathcal{M}_p(A), d_{\widetilde{\mathcal{M}}_p(A)})$, i.e.,

$$\mathcal{Q}_p(A) := \left\{ E \in \mathcal{M}_p(A) \;\middle|\; \begin{array}{l} \text{there exists } \{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{Q}'_p(A) \text{ such} \\ \text{that } \lim_{n\to\infty} d_{\widetilde{\mathcal{M}}_p(A)}(E, E_n) = 0 \end{array} \right\}. \tag{8.47}$$

Then we can show that any functional in $\mathcal{Q}_p(A)$ is a p-resistance form on A.

Proposition 8.36. *Let A be a finite set with $\#A \geq 2$ and let $E \in \mathcal{Q}_p(A)$. Then E is a p-resistance form on A.*

Proof. Note that (RF1)$_p$-(RF4)$_p$ for $E \in \mathcal{Q}_p(A)$ are clear, so we shall prove (RF5)$_p$, i.e., (GC)$_p$. Let $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{Q}'_p(A)$ satisfy $\lim_{n\to\infty} d_{\widetilde{\mathcal{M}}_p(A)}(E, E_n) = 0$. Then it is easy to see that $\lim_{n\to\infty} E_n(u) = E(u)$ for any $u \in \mathbb{R}^A$ (see also [CGQ22, Lemma A.1]). Since E_n satisfies (GC)$_p$ for any $n \in \mathbb{N}$, we have (GC)$_p$ for E by Proposition 2.10-(b). □

Next we introduce renormalization operators playing central roles in the construction of p-energy forms on p.-c.f. self-similar structures. In the rest of this subsection, we always assume that $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a p.-c.f. self-similar structure with $\#S \geq 2$ and K connected.

Definition 8.37 (Renormalization operator; [CGQ22, Definition 3.1])**.** Let $\boldsymbol{\rho}_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$ and $k \in \mathbb{N} \cup \{0\}$. For a p-resistance form E on V_k, define p-resistance forms $\Lambda_{\boldsymbol{\rho}_p}(E) \colon \mathbb{R}^{V_{k+1}} \to [0,\infty)$ and $\mathcal{R}_{\boldsymbol{\rho}_p}(E) \colon \mathbb{R}^{V_k} \to [0,\infty)$ by[12]

$$\Lambda_{\boldsymbol{\rho}_p}(E)(u) := \sum_{i\in S} \rho_{p,i} E(u \circ F_i), \quad u \in \mathbb{R}^{V_{k+1}}, \quad \text{and} \quad \mathcal{R}_{\boldsymbol{\rho}_p}(E) := \Lambda_{\boldsymbol{\rho}_p}(E)\big|_{V_k} \tag{8.48}$$

(recall Proposition 7.8 and Theorem 6.13). To be precise, $\Lambda_{\boldsymbol{\rho}_p}, \mathcal{R}_{\boldsymbol{\rho}_p}$ depend on k, but we omit it for ease of the notation. By [CGQ22, Lemma 3.2-(b)], we have $\Lambda^n_{\boldsymbol{\rho}_p}(E)\big|_{V_k} = \mathcal{R}^n_{\boldsymbol{\rho}_p}(E)$ for any $n \in \mathbb{N} \cup \{0\}$, i.e.,

$$\mathcal{R}^n_{\boldsymbol{\rho}_p}(E)(u) = \inf\left\{ \sum_{w\in W_n} \rho_{p,w}\, E(v \circ F_w) \;\middle|\; v \in \mathbb{R}^{V_{n+k}}, v|_{V_k} = u \right\}, \quad u \in \mathbb{R}^{V_k}.$$

The following theorem, which is an adaptation of [CGQ22, Theorem 4.2], gives a necessary and sufficient condition for the existence of an eigenform with respect to $\mathcal{R}_{\boldsymbol{\rho}_p}$. This theorem can be shown by combining [CGQ22, Lemma 4.4 and Proof of Theorem 4.2] with Proposition 8.36, so we omit the proof.

Theorem 8.38 (Condition for the existence of an eigenform; cf. [CGQ22, Theorem 4.2])**.** *Let $\boldsymbol{\rho}_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$. Let us consider the following condition* (**A**)*: there exist $c \in (0,\infty)$ and a p-resistance form E on V_0 such that*

$$\min_{x,y\in V_0; x\neq y} R_{\mathcal{R}^n_{\boldsymbol{\rho}_p}(E)}(x,y) \geq c \max_{x,y\in V_0; x\neq y} R_{\mathcal{R}^n_{\boldsymbol{\rho}_p}(E)}(x,y) \quad \textit{for any } n \in \mathbb{N} \cup \{0\}. \tag{\textbf{A}}$$

[12] We use different symbols from [CGQ22].

(a) *Assume that* (**A**) *holds. Then there exists a unique number* $\lambda = \lambda(\boldsymbol{\rho}_p) \in (0,\infty)$ *such that the following hold: for any* $E' \in \mathcal{M}_p(V_0)$*, there exists* $C \in [1,\infty)$ *such that*

$$C^{-1}\lambda^n E'(u) \le \mathcal{R}^n_{\boldsymbol{\rho}_p}(E')(u) \le C\lambda^n E'(u) \quad \text{for any } n \in \mathbb{N}\cup\{0\} \text{ and any } u \in \mathbb{R}^{V_0}. \tag{8.49}$$

(b) *Assume that* (**A**) *holds. Let* $E_0 \in \mathcal{S}_p(V_0)$*. For* $n \in \mathbb{N}$*, define* $E_n \in \mathcal{Q}'_p(V_0)$ *by*

$$E_n(u) \coloneqq \inf\left\{ \frac{1}{n+1}\sum_{j=0}^{n} \lambda^{-j}\Lambda^j_{\boldsymbol{\rho}_p}(E_0)(v|_{V_j}) \,\middle|\, v \in \mathbb{R}^{V_n}, v|_{V_0} = u \right\}, \quad u \in \mathbb{R}^{V_0}, \tag{8.50}$$

where λ *is the number given in* (a)*. Then there exists a subsequence* $\{E_{n_k}\}_{k\in\mathbb{N}}$ *such that it converges in the topology induced by* $d_{\widetilde{\mathcal{M}}_p}$*. In particular, there exists* $E_* \in \mathcal{Q}_p(V_0)$ *such that*

$$E_*(u) = \lim_{k\to\infty} E_{n_k}(u), \quad u \in \mathbb{R}^{V_0}. \tag{8.51}$$

(c) *Assume that* (**A**) *holds. Let* $E_0 \in \mathcal{S}_p(V_0)$*, let* $E_* \in \mathcal{Q}_p(V_0)$ *be given by* (8.51) *and let* λ *be the number given in* (a)*. Then* $\{\lambda^{-l}\mathcal{R}^l_{\boldsymbol{\rho}_p}(E_*)(u)\}_{l\in\mathbb{N}\cup\{0\}}$ *is non-decreasing for any* $u \in \mathbb{R}^{V_0}$ *and* $\mathcal{R}_{\boldsymbol{\rho}_p}(\mathcal{E}^{(0)}_p) = \lambda\mathcal{E}^{(0)}_p$*, where* $\mathcal{E}^{(0)}_p \in \mathcal{Q}_p(V_0)$ *is given by*

$$\mathcal{E}^{(0)}_p(u) \coloneqq \lim_{l\to\infty} \lambda^{-l}\mathcal{R}^l_{\boldsymbol{\rho}_p}(E_*)(u), \quad u \in \mathbb{R}^{V_0}. \tag{8.52}$$

(d) *Assume that there exist* $\lambda \in (0,\infty)$ *and a* p*-resistance form* E *on* V_0 *such that* $\mathcal{R}_{\boldsymbol{\rho}_p}(E) = \lambda E$*. Then* (**A**) *holds.*

Remark 8.39. (1) If $\boldsymbol{\rho}_p$ satisfies (**A**) for some p-resistance form E on V_0, then by [CGQ22, Lemma 4.4-(a)], for any p-resistance form $\widetilde{E}$ on V_0 there exists $\tilde{c} \in (0,\infty)$ such that (**A**) with $\widetilde{E}, \tilde{c}$ in place of E, c holds. Hence (**A**) is a condition relying only on $\boldsymbol{\rho}_p$.

(2) The assertion $\mathcal{E}^{(0)}_p \in \mathcal{Q}_p(V_0)$ follows from (8.52). Indeed, for any $n, l \in \mathbb{N}\cup\{0\}$, by (5.2) and Proposition 6.15, one can see that

$$\mathcal{R}^l_{\boldsymbol{\rho}_p}(E_n) = \left(\frac{1}{n+1}\sum_{j=0}^{n} \lambda^{-j}\Lambda^{l+j}_{\boldsymbol{\rho}_p}(E_0) \right)\Bigg|_{V_0} \in \mathcal{Q}'_p(V_0).$$

Let $\varepsilon > 0$. Then for all large enough $k \in \mathbb{N}$, we have

$$|E'(u) - E_{n_k}(u)| \le \varepsilon \quad \text{whenever } u \in \mathbb{R}^{V_0} \text{ satisfies } \operatorname*{osc}_{V_0}[u] \le 1.$$

For such $k \in \mathbb{N}$ and $u \in \mathbb{R}^{V_0}$, since (6.32) implies

$$\operatorname*{osc}_{V_l}\left[h^{\Lambda^l_{\boldsymbol{\rho}_p}(E')}_{V_0}[u]\right] \le \operatorname*{osc}_{V_0}[u], \quad \operatorname*{osc}_{V_l}\left[h^{\Lambda^l_{\boldsymbol{\rho}_p}(E_{n_k})}_{V_0}[u]\right] \le \operatorname*{osc}_{V_0}[u],$$

we have

$$\left|\mathcal{R}^l_{\boldsymbol{\rho}_p}(E')(u) - \mathcal{R}^l_{\boldsymbol{\rho}_p}(E_{n_k})(u)\right| \le \varepsilon \sum_{w\in W_l} \rho_{p,w} \quad \text{whenever } u \in \mathbb{R}^{V_0} \text{ satisfies } \operatorname*{osc}_{V_0}[u] \le 1.$$

This shows that $d_{\widetilde{\mathcal{M}}_p(V_0)}(\mathcal{R}^l_{\boldsymbol{\rho}_p}(E'), \mathcal{R}^l_{\boldsymbol{\rho}_p}(E_{n_k})) \to 0$ as $k \to \infty$, whence it follows that $\mathcal{R}^l_{\boldsymbol{\rho}_p}(E') \in \mathcal{Q}_p(V_0)$. Therefore, $\mathcal{E}^{(0)}_p \in \mathcal{Q}_p(V_0)$.

In the rest of this subsection, we fix $\boldsymbol{\rho}_p = (\rho_{p,i})_{i \in S} \in (0,\infty)^S$. Let us introduce two important conditions on $\boldsymbol{\rho}_p$, following [CGQ22, Section 5]:

(**A'**) There exists a p-resistance form $\mathcal{E}_p^{(0)}$ on V_0 such that $\mathcal{R}_{\boldsymbol{\rho}_p}(\mathcal{E}_p^{(0)}) = \mathcal{E}_p^{(0)}$.
(**R**) ((**A'**) holds and) $\min_{i \in S} \rho_{p,i} > 1$.

Note that by Theorem 8.38, (**A'**) implies (**A**), and (**A**) implies that $\lambda^{-1}\boldsymbol{\rho}_p$ satisfies (**A'**) for some $\lambda \in (0,\infty)$.

The following proposition is important to construct a self-similar p-resistance form as an "inductive limit" of discrete p-resistance forms as presented in [CGQ22, Proposition 5.3], which is an adaptation of the relevant pieces of the theory of resistance forms due to [Kig01, Sections 2.2, 2.3 and 3.3].

Proposition 8.40. *Assume that* (**A'**) *holds. Define* $\mathcal{E}_p^{(n)} \coloneqq \Lambda_{\boldsymbol{\rho}_p}^n(\mathcal{E}_p^{(0)})$*, i.e.,*

$$\mathcal{E}_p^{(n)}(u) \coloneqq \sum_{w \in W_n} \rho_{p,w}\mathcal{E}_p^{(0)}(u \circ F_w), \quad u \in \mathbb{R}^{V_n}. \tag{8.53}$$

Then $\mathcal{E}_p^{(n)}$ *is a p-resistance form on* V_n *and* $\mathcal{E}_p^{(n+m)}|_{V_n} = \mathcal{E}_p^{(n)}$ *for any* $n, m \in \mathbb{N} \cup \{0\}$*, i.e.,* $\bigl\{(V_n, \mathcal{E}_p^{(n)})\bigr\}_{n \geq 0}$ *is a compatible sequence of p-resistance forms.*

Proof. We will show $\mathcal{E}_p^{(n+m)}|_{V_n} = \mathcal{E}_p^{(n)}$. (See [Kig01, Proposition 3.1.3] for the case of $p = 2$.) It suffices to prove $\mathcal{E}_p^{(n+1)}|_{V_n} = \mathcal{E}_p^{(n)}$ for any $n \in \mathbb{N} \cup \{0\}$ by virtue of Proposition 6.15. Note that the case of $n = 0$ is true by $\mathcal{R}_{\boldsymbol{\rho}_p}(\mathcal{E}_p^{(0)}) = \mathcal{E}_p^{(0)}$, and that

$$\mathcal{E}_p^{(n+1)}(u) = \sum_{i \in S} \rho_{p,i}\, \mathcal{E}_p^{(n)}(u \circ F_i), \quad \text{for any } n \in \mathbb{N} \cup \{0\} \text{ and } u \in \mathbb{R}^{V_{n+1}}. \tag{8.54}$$

Assume that $\mathcal{E}_p^{(m)}|_{V_{m-1}} = \mathcal{E}_p^{(m-1)}$ for some $m \in \mathbb{N}$. Then for any $u \in \mathbb{R}^{V_m}$,

$$\begin{aligned}
\mathcal{E}_p^{(m)}(u) &\overset{(8.54)}{=} \sum_{i \in S} \rho_{p,i}\, \mathcal{E}_p^{(m-1)}(u \circ F_i) \\
&= \sum_{i \in S} \rho_{p,i} \min\Bigl\{\mathcal{E}_p^{(m)}(v \circ F_i) \Bigm| v \in \mathbb{R}^{K_i \cap V_{m+1}}, v|_{K_i \cap V_m} = u|_{K_i}\Bigr\} \\
&\overset{(5.2)}{=} \min\Biggl\{\sum_{i \in S} \rho_{p,i}\, \mathcal{E}_p^{(m)}(v \circ F_i) \Biggm| v \in \mathbb{R}^{V_{m+1}}, v|_{V_m} = u\Biggr\} \\
&\overset{(8.54)}{=} \min\Bigl\{\mathcal{E}_p^{(m+1)}(v) \Bigm| v \in \mathbb{R}^{V_{m+1}}, v|_{V_m} = u\Bigr\} = \mathcal{E}_p^{(m+1)}|_{V_m}(u),
\end{aligned}$$

which completes the proof. $\square$

We can naturally construct a p-resistance form as an inductive limit on the countable set V_* as described in the following proposition.

Proposition 8.41. *Assume that* (**A**') *holds and let* $\{(V_n, \mathcal{E}_p^{(n)})\}_{n\geq 0}$ *be the compatible sequence of* p*-resistance forms given in Proposition 8.40. Define a linear subspace* $\mathcal{F}_{p,*}$ *of* $\mathbb{R}^{V_*}$ *and* $\mathcal{E}_{p,*}\colon \mathcal{F}_{p,*} \to [0,\infty)$ *by*

$$\mathcal{F}_{p,*} \coloneqq \Bigl\{ u \in \mathbb{R}^{V_*} \Bigm| \lim_{n\to\infty} \mathcal{E}_p^{(n)}(u|_{V_n}) < \infty \Bigr\}, \quad \textit{and} \tag{8.55}$$

$$\mathcal{E}_{p,*}(u) \coloneqq \lim_{n\to\infty} \mathcal{E}_p^{(n)}(u|_{V_n}), \quad u \in \mathcal{F}_{p,*}. \tag{8.56}$$

Then $(\mathcal{E}_{p,*}, \mathcal{F}_{p,*})$ *is a* p*-resistance form on* V_* *satisfying* $\mathcal{E}_{p,*}|_{V_n} = \mathcal{E}_p^{(n)}$ *for any* $n \in \mathbb{N}\cup\{0\}$*. Moreover, the following self-similarity properties hold:*

$$\mathcal{F}_{p,*} = \bigl\{ u \in \mathbb{R}^{V_*} \bigm| u \circ F_i \in \mathcal{F}_{p,*} \textit{ for any } i \in S \bigr\}, \tag{8.57}$$

$$\mathcal{E}_{p,*}(u) = \sum_{i\in S} \rho_{p,i}\, \mathcal{E}_{p,*}(u \circ F_i) \quad \textit{for any } u \in \mathcal{F}_{p,*}. \tag{8.58}$$

If in addition (**R**) *holds, then for any* $u \in \mathcal{F}_{p,*}$ *there exists a unique* $\widehat{u} \in C(K)$ *such that* $\widehat{u}|_{V_*} = u$*, and* $\{\widehat{u} \mid u \in \mathcal{F}_{p,*}\}$ *is dense in* $C(K)$*.*

Proof. It is immediate from Theorem 6.21 that $(\mathcal{E}_{p,*}, \mathcal{F}_{p,*})$ is a p-resistance form on V_* with $\mathcal{E}_{p,*}|_{V_n} = \mathcal{E}_p^{(n)}$. By the definition in (8.53), it is easy to see that for any $n, k \in \mathbb{N}\cup\{0\}$ and any $u \in \mathbb{R}^{V_*}$,

$$\mathcal{E}_p^{(n+k)}(u|_{V_{n+k}}) = \sum_{w\in W_k} \rho_{p,w}\, \mathcal{E}_p^{(n)}(u \circ F_w|_{V_n}).$$

This immediately implies (8.57) and (8.58). The existence of unique continuous extensions of functions in $\mathcal{F}_{p,*}$ under (**R**) is proved in [CGQ22, Theorem 5.1-(b)]. A standard argument using the Stone–Weierstrass theorem shows that $\mathscr{C} \coloneqq \{\widehat{u} \mid u \in \mathcal{F}_{p,*}\}$ is dense in $C(K)$. Indeed, $\mathscr{C}$ is an algebra since $\mathcal{F}_{p,*}$ is also an algebra by Proposition 2.3-(d). For any $x, y \in K$ with $x \neq y$, choose $n \in \mathbb{N}$ and $v, w \in W_n$ so that $x \in K_v, y \in K_w$ and $K_v \cap K_w = \emptyset$. (Such n, v, w exist by (5.3).) Then, by setting $v \coloneqq h_{V_n}^{\mathcal{E}_{p,*}}[\mathbb{1}_{F_v(V_0)}]$, we see that $\varphi_{xy} \coloneqq \widehat{v} \in \mathscr{C}$ satisfies $\varphi_{xy}(x) = 1$ and $\varphi_{xy}(y) = 0$, so we can use the Stone–Weierstrass theorem to conclude that $\mathscr{C}$ is dense in $C(K)$. □

To extend $(\mathcal{E}_{p,*}, \mathcal{F}_{p,*})$ to a p-energy form defined on K, we need to specify how to regard functions in $\mathcal{F}_{p,*}$ as functions defined on K, which is indeed a delicate problem and discussed in [CGQ22, Theorems 5.1 and 5.2]. In this paper, we are mainly interested in the case where $\mathcal{F}_{p,*}$ can be embedded into $C(K)$. In other words, we always assume that (**R**) holds. (See [CGQ22, Theorem 5.2] and [KS+a, Appendix] for details on a situation when we can identify a function $u \in \mathbb{R}^{V_*}$ satisfying $\lim_{n\to\infty} \mathcal{E}_p^{(n)}(u|_{V_n}) < \infty$ with a function on K without (**R**).) To state a construction of self-similar p-resistance forms under (**R**), we need the following lemma.

Lemma 8.42. *Assume that* (**A**') *and* (**R**) *hold. Let* $(\mathcal{E}_{p,*}, \mathcal{F}_{p,*})$ *be the* p*-resistance form on* V_* *given in Proposition 8.41. Then* $\mathrm{id}_{V_*}\colon (V_*, R_{\mathcal{E}_{p,*}}^{1/p}) \to K$ *is uniquely extended to the completion, which gives a homeomorphism.*

Proof. The proof is very similar to arguments in [Kig01, Proposition 3.3.2, Lemma 3.3.5 and Theorem 3.3.4]. Let $(\widehat{K}, \widehat{d})$ be the completion of $(V_*, R_{p,\mathcal{E}_{p,*}}^{1/p})$ and let $(\widehat{\mathcal{E}}_{p,*}, \widehat{\mathcal{F}}_{p,*})$ be the p-resistance form on $\widehat{K}$ defined by (6.27) and (6.28), where we choose $\mathcal{S} = \{(V_n, \mathcal{E}_p^{(n)})\}_{n\in\mathbb{N}\cup\{0\}}$. Also, we fix a metric d on K which gives the original topology of K. Recall that $R_{\widehat{\mathcal{E}}_{p,*}}^{1/p} = \widehat{d}$ by Corollary 6.23. For $n \in \mathbb{N}$, we define

$$\delta_n := \min_{v,w\in W_n; K_v\cap K_w=\emptyset}\left(\inf_{x\in F_v(V_*), y\in F_w(V_*)} R_{\mathcal{E}_{p,*}}(x,y)\right).$$

Then $\delta_n > 0$ since $R_{\mathcal{E}_{p,*}}(x,y) \geq \mathcal{E}_{p,*}\bigl(h_{V_n}^{\mathcal{E}_{p,*}}[\mathbb{1}_{F_w(V_0)}]\bigr)^{-1}$ for any $(x,y) \in F_v(V_*)\times F_w(V_*)$. Let $\{x_n\}_{n\geq 0}$ be a Cauchy sequence in $(V_*, R_{\mathcal{E}_{p,*}}^{1/p})$. For each $n \in \mathbb{N}$, choose $N(n) \in \mathbb{N}$ so that

$$\sup_{k,l\geq N(n)} R_{\mathcal{E}_{p,*}}(x_k, x_l) < \delta_n.$$

Then there exists $w \in W_n$ such that $\{x_k\}_{k\geq N(n)} \subseteq \bigcup_{v\in W_n; K_v\cap K_w\neq\emptyset} F_v(V_*) =: A_{n,w}$. Since $\lim_{n\to\infty}\max_{w\in W_n}\operatorname{diam}(A_{n,w}, d) = 0$ by (5.3), we conclude that $\mathrm{id}_{V_*}\colon (V_*, R_{\mathcal{E}_{p,*}}^{1/p}) \to (V_*, d|_{V_*\times V_*})$ is uniformly continuous. Now we define $\theta\colon (\widehat{K}, \widehat{d}) \to (K, d)$ as the unique continuous map satisfying $\theta|_{V_*} = \mathrm{id}_{V_*}$. Let us show that θ is injective. Assume that $x, y \in \widehat{K}$ satisfy $\theta(x) = \theta(y)$. Let $\{x_n\}_{n\geq 0}, \{y_n\}_{n\geq 0}$ be Cauchy sequences in $(V_*, R_{\mathcal{E}_{p,*}}^{1/p})$ satisfying $\lim_{n\to\infty}\widehat{d}(x, x_n) = \lim_{n\to\infty}\widehat{d}(y, y_n) = 0$. Then $\lim_{n\to\infty} d(\theta(x), x_n) = \lim_{n\to\infty} d(\theta(y), y_n) = 0$ since θ is continuous. For any $u \in \widehat{\mathcal{F}}_{p,*}$, let $\widehat{u}_n \in C(K)$ be the unique function satisfying $\widehat{u}_n|_{V_*} = h_{V_n}^{\mathcal{E}_{p,*}}[u|_{V_n}]$, which exists by Proposition 8.41. Also, let $v_n \in C(\widehat{K})$ be the unique function satisfying $v_n|_{V_*} = h_{V_n}^{\mathcal{E}_{p,*}}[u|_{V_n}]$; recall the proof of Theorem 6.22. Then we see that

$$v_n(x) = \lim_{k\to\infty} h_{V_n}^{\mathcal{E}_{p,*}}[u](x_k) = \widehat{u}_n(\theta(x)) = \widehat{u}_n(\theta(y)) = \lim_{k\to\infty} h_{V_n}^{\mathcal{E}_{p,*}}[u](y_k) = v_n(y). \tag{8.59}$$

Let us fix $o \in V_0 \subseteq V_n$. By (6.3) for $(\widehat{\mathcal{E}}_{p,*}, \widehat{\mathcal{F}}_{p,*})$,

$$|u(x) - v_n(x)|^p \leq R_{\widehat{\mathcal{E}}_{p,*}}(x,o)\widehat{\mathcal{E}}_{p,*}(u - \check{u}_n) = R_{\widehat{\mathcal{E}}_{p,*}}(x,o)\mathcal{E}_{p,*}\bigl(u|_{V_*} - h_{V_n}^{\mathcal{E}_{p,*}}[u|_{V_n}]\bigr),$$

which together with (6.18) and (8.59) implies that

$$u(x) = \lim_{n\to\infty} v_n(x) = \lim_{n\to\infty} v_n(y) = u(y).$$

Since $u \in \widehat{\mathcal{F}}_{p,*}$ is arbitrary, we conclude that $R_{\widehat{\mathcal{E}}_{p,*}}(x,y) = 0$ and hence $x = y$. This means that θ is injective.

Next we see that $\{F_i\}_{i\in S}$ yields a family of contractions on the complete (non-empty) metric space $(\widehat{K}, \widehat{d})$. By virtue of (8.58), similarly to the proof of (7.1), one can show that for any $w \in W_*$ and any $x, y \in V_*$,

$$\widehat{d}(F_w(x), F_w(y))^p = R_{\widehat{\mathcal{E}}_{p,*}}(F_w(x), F_w(y)) \leq \rho_{p,w}^{-1} R_{\widehat{\mathcal{E}}_{p,*}}(x,y) = \rho_{p,w}^{-1}\widehat{d}(x,y)^p.$$

In particular, $F_w|_{V_*}\colon (V_*, \widehat{d}) \to (V_*, \widehat{d})$ is uniformly continuous, and hence there exists a unique continuous map $F_w^{\widehat{K}}\colon \widehat{K} \to \widehat{K}$ such that $F_w^{\widehat{K}}\big|_{V_*} = F_w|_{V_*}$. Then it is clear that

$$\widehat{d}\left(F_w^{\widehat{K}}(x), F_w^{\widehat{K}}(y)\right) \le \rho_{p,w}^{-1/p}\, \widehat{d}(x,y) \quad \text{for any } x, y \in \widehat{K}, \tag{8.60}$$

and that $\theta \circ F_w^{\widehat{K}} = F_w \circ \theta$. Now, by (**R**) and (8.60), $\bigl\{F_i^{\widehat{K}}\bigr\}_{i\in S}$ is a family of contractions on $(\widehat{K}, \widehat{d})$. By [Kig01, Theorem 1.1.4], there exists a unique non-empty compact subset $\widehat{K}_0$ of $\widehat{K}$ such that $\widehat{K}_0 = \bigcup_{i\in S} F_i^{\widehat{K}}(\widehat{K}_0)$. Let us fix $o \in \widehat{K}_0$ and set $A \coloneqq \bigcup_{w\in W_*} F_w^{\widehat{K}}(o) \subseteq \widehat{K}_0$. Then $\theta(A) = \bigcup_{w\in W_*} F_w(\theta(o))$ is dense in (K,d) by (5.3). Since $\theta(A) \subseteq \theta(\widehat{K}_0) \subseteq K$ and $\theta(\widehat{K}_0)$ is compact by the continuity of θ, we have $\theta(\widehat{K}_0) = K$ and thus $\theta(\widehat{K}) = K$. Then $\widehat{K}$ turns out to be compact since $\widehat{K} = \widehat{K}_0$ by the injectivity of θ. Now θ turns out to be a homeomorphism between $\widehat{K}$ and K. From the uniqueness of θ, we conclude that $\widehat{K} = K$ and $\theta = \mathrm{id}_K$. We complete the proof. □

The following theorem describes a construction of a self-similar p-resistance form as the inductive limit of $\{\mathcal{E}_p^{(n)}\}_{n\ge 0}$ under the condition (**R**).

Theorem 8.43. *Assume that (**A'**) and (**R**) hold. Let $\bigl\{(V_n, \mathcal{E}_p^{(n)})\bigr\}_{n\ge 0}$ be the compatible sequence of p-resistance forms given in Proposition 8.40, and define*

$$\mathcal{F}_p \coloneqq \left\{ u \in C(K) \;\middle|\; \lim_{n\to\infty} \mathcal{E}_p^{(n)}(u|_{V_n}) < \infty \right\}, \tag{8.61}$$

$$\mathcal{E}_p(u) \coloneqq \lim_{n\to\infty} \mathcal{E}_p^{(n)}(u), \quad u \in \mathcal{F}_p. \tag{8.62}$$

Then $(\mathcal{E}_p, \mathcal{F}_p)$ is a regular self-similar p-resistance form on $\mathcal{L}$ with weight $\boldsymbol{\rho}_p$, $\mathcal{E}_p|_{V_n} = \mathcal{E}_p^{(n)}$ for any $n \in \mathbb{N} \cup \{0\}$, and $R_{\mathcal{E}_p}$ is compatible with the original topology of K.

Remark 8.44. Similar to Proposition 5.23, by choosing a suitable $E_0 \in \mathcal{S}_p(V_0)$ in Theorem 8.38, we can verify nice properties like the *symmetry-invariance* (see (9.7) for details) of E_* in (8.51), $\mathcal{E}_p^{(0)}$ in (8.52) and $\mathcal{E}_p$. See also Theorem 8.51.

Proof of Theorem 8.43. By Lemma 8.42 and Corollary 6.23, $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on K. The self-similarity conditions, (5.5) and (5.6), for $(\mathcal{E}_p, \mathcal{F}_p)$ are obvious from Proposition 8.40. By Lemma 8.42 and Proposition 8.41, $R_{\mathcal{E}_p}$ is compatible with the original topology of K and $(\mathcal{E}_p, \mathcal{F}_p)$ is regular (recall Definition 6.5). □

Let us recall the following proposition, which is useful to verify (**R**) for concrete examples.

Proposition 8.45 ([CGQ22, Lemma 5.4])**.** *Assume that (**A'**) holds. If $w \in W_* \setminus \{\emptyset\}$ satisfies $w^{\infty} \coloneqq www\ldots \in \mathcal{P}_{\mathcal{L}}$, then $\rho_{p,w} > 1$.*

8.4 Existence of eigenforms on strongly symmetric p.-c.f. self-similar sets

Let us conclude this section by showing (**A**) for a special class of p.-c.f. self-similar sets called *affine nested fractals*, which was introduced in [FHK94] as a generalization of the class called nested fractals introduced by Liondstrøm [Lin90]. More precisely, we will work in a wider class called *strongly symmetric p.-c.f. self-similar sets*. The proof of (**A**) for affine nested fractals was given in [CGQ22, Theorem 6.3], but their description on the group of symmetries in the paper [CGQ22] is not sophisticated[13], so we provide the details of the proof for (**A**) and improve the assumptions in [CGQ22, Theorem 6.3] simultaneously in Theorem 8.50.

We start with recalling the definitions of a group of symmetries, affine nested fractals and strongly symmetric p.-c.f. self-similar sets. See, e.g., [Kig01, Section 3.8] for details.

Framework 8.46. Let $D \in \mathbb{N}$ and let S be a non-empty finite set with $\#S \geq 2$. Let $\{c_i\}_{i \in S} \subseteq (0,1)$, $\{a_i\}_{i \in S} \subseteq \mathbb{R}^D$ and $\{U_i\}_{i \in S} \subseteq O(D)$, where $O(D)$ is the collection of orthogonal transformations of $\mathbb{R}^D$. Define $f_i \colon \mathbb{R}^D \to \mathbb{R}^D$ by $f_i(x) \coloneqq c_i U_i x + a_i$ for each $i \in S$. Let K be the self-similar set associated with $\{f_i\}_{i \in S}$, set $F_i \coloneqq f_i|_K$ for each $i \in S$ and assume that $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is a p.-c.f. self-similar structure. We also assume that K is connected, $M \coloneqq \#(V_0) < \infty$ and $\sum_{i=1}^{M} q_i = 0$, where $q_i \in \mathbb{R}^D$ is defined so that $V_0 = \{q_i\}_{i=1}^{M}$. Let $d \colon K \times K \to [0,\infty)$ be the Euclidean metric on K given by $d(x,y) \coloneqq |x - y|$.

Definition 8.47 ([Kig01, Definitions 3.8.3 and 3.8.4])**.** (1) We define

$$\mathcal{G}_{\mathrm{sym}}(\mathcal{L}) \coloneqq \mathcal{G}_{\mathrm{sym}} \coloneqq \left\{ g|_K \;\middle|\; \begin{array}{l} g \in O(D), \text{ for any } n \in \mathbb{N} \cup \{0\} \text{ and any} \\ w \in W_n \text{ there exists } w' \in W_n \text{ such that} \\ g(K_w) = K_{w'} \text{ and } g(F_w(V_0)) = F_{w'}(V_0) \end{array} \right\},$$

where $O(D)$ denotes the orthogonal group in dimension D. Note that for any $g \in \mathcal{G}_{\mathrm{sym}}$ and any $w \in W_*$, a word $w' \in W_*$ satisfying $|w| = |w'|$ and $g(K_w) = K_{w'}$ is uniquely determined. In particular, the map $\tau_g \colon W_* \to W_*$ defined by $\tau_g(w) \coloneqq w'$ gives a bijection such that $|\tau_g(w)| = |w|$ for any $w \in W_*$.

(2) For $x, y \in \mathbb{R}^D$ with $x \neq y$, let $g_{xy} \colon \mathbb{R}^D \to \mathbb{R}^D$ be the reflection in the hyperplane $\{z \in \mathbb{R}^D \mid |x - z| = |y - z|\}$.

(3) Let $m_* \coloneqq \#\{|x - y| \mid x, y \in V_0, x \neq y\}$ and $l_0 \coloneqq \min\{|x - y| \mid x, y \in V_0, x \neq y\}$. We define $\{l_i\}_{i=0}^{m_*-1}$ inductively by $l_{i+1} \coloneqq \min\{|x - y| \mid x, y \in V_0, |x - y| > l_i\}$.

(4) Let $m \in \mathbb{N} \cup \{0\}$ and $(x_i)_{i=1}^{n} \in (V_m)^n$. Then $(x_i)_{i=1}^{n}$ is called an *m-walk* (between x_1 and x_n) if and only if there exist $w^1, \dots, w^n \in W_m$ such that $\{x_i, x_{i+1}\} \subseteq F_{w^i}(V_0)$ for all $i \in \{1, 2, \dots, n-1\}$. A 0-walk $(x_i)_{i=1}^{n}$ is called a *strict 0-walk* (between x_1 and x_n) if and only if $|x_i - x_{i+1}| = l_0$ for any $i \in \{1, 2, \dots, n-1\}$.

[13] For a group of symmetries, say $\mathcal{G}$, one of the essential properties that is needed to prove the $\mathcal{G}$-invariance of the resulting self-similar p-energy form is Proposition 8.49-(2). We have to be careful whether this property holds for $\mathcal{G}$, but this point is not taken care of in [CGQ22].

(5) $\mathcal{L}$ is called a *strongly symmetric p.-c.f. self-similar set* if and only if it satisfies the following four conditions:
   (SS1) For any $x, y \in V_0$ with $x \neq y$, there exists a strict 0-walk between x and y.
   (SS2) If $x, y, z \in V_0$ and $|x-y| = |x-z|$, then there exists $g \in \mathcal{G}_{\mathrm{sym}}$ such that $g(x) = x$ and $g(y) = z$.
   (SS3) For any $i \in \{0, \dots, m_* - 2\}$, there exist $x, y, z \in V_0$ such that $|x-y| = l_i$, $|x-z| = l_{i+1}$ and $g_{yz}|_K \in \mathcal{G}_{\mathrm{sym}}$.
   (SS4) V_0 is $\mathcal{G}_{\mathrm{sym}}$-transitive, i.e., for any $x, y \in V_0$ with $x \neq y$, there exists $g \in \mathcal{G}_{\mathrm{sym}}$ such that $g(x) = y$.

(6) $\mathcal{L}$ is called an *affine nested fractal* if $g_{xy}|_K \in \mathcal{G}_{\mathrm{sym}}(\mathcal{L})$ for any $x, y \in V_0$ with $x \neq y$.

Remark 8.48. In [Kig01, Definitions 3.8.3 and 3.8.4], the following group of symmetries $\mathcal{G}_s$ is used instead of $\mathcal{G}_{\mathrm{sym}}$:

$$\mathcal{G}_s := \left\{ g|_K \;\middle|\; \begin{array}{l} g \in O(D), \text{ for any } n \in \mathbb{N} \cup \{0\} \text{ and any } w \in W_n \\ \text{there exists } w' \in W_n \text{ such that } g(F_w(V_0)) = F_{w'}(V_0) \end{array} \right\};$$

note that $\mathcal{G}_{\mathrm{sym}} \subseteq \mathcal{G}_s$. Under the assumption that

$$\#(F_i(V_0) \cap F_j(V_0)) \leq 1 \quad \text{for any } i, j \in S \text{ with } i \neq j, \tag{8.63}$$

we know that $\mathcal{G}_{\mathrm{sym}} = \mathcal{G}_s$ by [Kig01, Proposition 3.8.19]. The difference between $\mathcal{G}_{\mathrm{sym}}$ and $\mathcal{G}_s$ does not affect the arguments in the parts of [CGQ22, Kig01] (Proposition 8.49 and Theorem 8.50 below) that we need.

Let us recall a few properties of $\mathcal{G}_{\mathrm{sym}}$ and of affine nested fractals in the following proposition, which can be shown in the same ways as in [Kig01, Section 3.8]. (Let us emphasize that we do not assume (8.63) unlike [Kig01, Section 3.8].)

Proposition 8.49 ([Kig01, Propositions 3.8.7, 3.8.20 and Lemma 3.8.23])**.** (1) *If $\mathcal{L}$ is an affine nested fractal, then it is a strongly symmetric self-similar set.*

(2) *Let $w \in W_*$, $g \in \mathcal{G}_{\mathrm{sym}}$ and set*

$$U_{g,w} := F_{w'}^{-1} \circ g \circ F_w,$$

where $w' \in W_*$ is the unique word satisfying $F_{w'}(V_0) = g(F_w(V_0))$. Then $U_{g,w} \in \mathcal{G}_{\mathrm{sym}}$.

(3) *Let $a, b \in V_0$ and assume that $g_{ab}|_K \in \mathcal{G}_{\mathrm{sym}}$. If $x, y \in F_w(V_0)$ for some $w \in W_*$, $|x-b| < |x-a|$ and $|y-b| > |y-a|$, then $g_{ab}(K_w) = K_w$.*

Now we can present the following theorem proving the existence of an eigenform on V_0 for strongly symmetric self-similar sets and improving [CGQ22, Theorem 6.3]. Note that the case $p = 2$ corresponds to the existence of a harmonic structure on $\mathcal{L}$ in [Kig01, Theorem 3.8.10].

Theorem 8.50. *Assume that $\mathcal{L}$ is strongly symmetric. If*

$$\rho_{p,i} = \rho_{p,i'} \quad \text{for any } i \in S \text{ and any } g \in \mathcal{G}_{\mathrm{sym}}, \tag{8.64}$$

*where $i' \in S$ is the unique element satisfying $F_{i'}(V_0) = g(F_i(V_0))$, then $\boldsymbol{\rho}_p$ satisfies (**A**). In particular, if there exists $\rho_p \in (0,\infty)$ such that $\rho_{p,i} = \rho_p$ for any $i \in S$, then (**A**') and (**R**) with $\bigl(\lambda(\boldsymbol{\rho}_p)^{-1}\rho_p\bigr)_{i\in S}$ in place of $\boldsymbol{\rho}_p$ hold, where $\lambda(\boldsymbol{\rho}_p)$ is the number given in Theorem 8.38-(a).*

Proof. The proof is essentially the same as [CGQ22, Proof of Theorem 6.3], but we give the details of it since we weaken the assumption of [CGQ22, Theorem 6.3]. For $n \in \mathbb{N} \cup \{0\}$, define $E_{p,n} \in \mathcal{S}_p(V_n)$ by

$$E_{p,n}(y) \coloneqq \sum_{w\in W_n} \rho_{p,w} \sum_{x,y\in V_0;|x-y|=l_0} |u(F_w(x)) - u(F_w(y))|^p, \quad u \in \mathbb{R}^{V_n}.$$

Note that, by Proposition 8.49-(2) and (8.64), $E_{p,n}$ is $\mathcal{G}_{\mathrm{sym}}$-invariant, i.e., $E_{p,n}(u \circ g|_{V_n}) = E_{p,n}(u)$ for any $u \in \mathbb{R}^{V_n}$ and any $g \in \mathcal{G}_{\mathrm{sym}}$. We fix $a_1, a_2 \in V_0$ that satisfy $|a_1 - a_2| = l_0$ and claim that for any $n \in \mathbb{N}$ and any $x, y \in V_0$ with $x \neq y$,

$$\frac{1}{2}R_{E_{p,n}}(a_1,a_2) \le R_{E_{p,n}}(x,y) \le (\#V_0)^p R_{E_{p,n}}(a_1,a_2), \tag{8.65}$$

which implies (**A**) for $\boldsymbol{\rho}_p$ with $c = 2(\#V_0)^{-p}$.

We first show the upper estimate in (8.65). Let $(x_i)_{i=0}^k \in (V_0)^{k+1}$ be a strict 0-walk between x and y. Then, by (SS2), (SS4) and the $\mathcal{G}_{\mathrm{sym}}$-invariance of $E_{p,n}$, we have $R_{E_{p,n}}(x_i, x_{i+1}) = R_{E_{p,n}}(a_1,a_2)$ for any $i \in \{0,\dots,k-1\}$. Hence we see that

$$R_{E_{p,n}}(x,y)^{1/p} \le \sum_{i=0}^{k-1} R_{E_{p,n}}(x_i,x_{i+1})^{1/p} = kR_{E_{p,n}}(a_1,a_2)^{1/p} \le (\#V_0)R_{E_{p,n}}(a_1,a_2)^{1/p},$$

which shows the desired estimate.

Next we prove the lower estimate in (8.65). The case of $|x-y| = l_0$ is clear by (SS2), (SS4) and the $\mathcal{G}_{\mathrm{sym}}$-invariance of $E_{p,n}$, so we assume that $|x-y| > l_0$. By (SS1), there exists $z \in V_0$ such that $|x-z| = l_0$. Define $u \in \mathbb{R}^{V_n}$ by

$$u(a) \coloneqq \begin{cases} h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x](a) & \text{if } a \in V_n \text{ satisfies } |a-z| \le |a-y|, \\ h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x](g_{yz}(a)) & \text{if } a \in V_n \text{ satisfies } |a-z| \ge |a-y|, \end{cases}$$

which is well-defined since Theorem 6.13 implies $h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x](a) = 1/2$ whenever $|a-z| = |a-y|$. Since $|x-z| = l_0 < |x-y|$, we have $u(x) = h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x](x) = 1$. Also, $u(y) = h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x](g_{yz}(y)) = 0$. Hence $R_{E_{p,n}}(x,y) \ge E_{p,n}(u)^{-1}$. Now we define $H_{1,n} \coloneqq \{a \in V_n \mid |a-z| \le |a-y|\}$, $H_{2,n} \coloneqq \{a \in V_n \mid |a-z| \ge |a-y|\}$ and we see that

$$E_{p,n}(u) = \left(\sum_{\substack{w\in W_n;\\ F_w(V_0)\subseteq H_{1,n}}} + \sum_{\substack{w\in W_n;\\ F_w(V_0)\subseteq H_{2,n}}} + \sum_{\substack{w\in W_n;\\ F_w(V_0)\not\subseteq H_{i,n}}}\right) \rho_{p,w} E_{p,0}(u \circ F_w|_{V_0})$$

$$= 2 \sum_{\substack{w \in W_n;\\ F_w(V_0) \subseteq H_{1,n}}} \rho_{p,w} E_{p,0}\Bigl(h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x] \circ F_w|_{V_0}\Bigr) + \sum_{\substack{w \in W_n;\\ F_w(V_0) \not\subseteq H_{i,n}}} \rho_{p,w} E_{p,0}(u \circ F_w|_{V_0}).$$

To estimate the second term in the right-hand side in the above equality, let $a, b \in V_0$ satisfy $|a-b| = l_0$, $|F_w(a) - z| < |F_w(a) - y|$ and $|F_w(b) - z| > |F_w(b) - y|$. Then we have $g_{yz}(F_w(V_0)) = F_w(V_0)$ by Proposition 8.49-(3). This along with the minimality of l_0 implies that $g_{yz}(F_w(a)) = F_w(b)$, whence it follows that $u(F_w(a)) = u(F_w(b))$. Hence

$$\begin{aligned}
\sum_{\substack{w \in W_n;\\ F_w(V_0) \not\subseteq H_{i,n}}} \rho_{p,w} E_{p,0}(u \circ F_w|_{V_0}) &= \sum_{\substack{w \in W_n;\\ F_w(V_0) \not\subseteq H_{i,n}}} \rho_{p,w} \sum_{\substack{a,b \in V_0; |a-b| = l_0,\\ \{F_w(a), F_w(b)\} \subseteq H_{1,n}\\ \text{or} \{F_w(a), F_w(b)\} \subseteq H_{2,n}}} |u(F_w(a)) - u(F_w(b))|^p \\
&\le 2 \sum_{\substack{w \in W_n;\\ F_w(V_0) \not\subseteq H_{i,n}}} \rho_{p,w} E_{p,0}\Bigl(h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x] \circ F_w|_{V_0}\Bigr),
\end{aligned}$$

and we deduce that

$$R_{E_{p,n}}(x,y) \ge E_{p,n}(u)^{-1} \ge \frac{1}{2} E_{p,n}\Bigl(h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x]\Bigr)^{-1} = \frac{1}{2} R_{E_{p,n}}(a_1, a_2),$$

completing the proof. □

The following theorem gives symmetry-invariant self-similar p-resistance forms on strongly symmetric self-similar sets.

Theorem 8.51. *Assume that $\mathcal{L}$ is a strongly symmetric p.-c.f. self-similar set and that (**A**'), (**R**) and (8.64) hold. Then there exists a self-similar p-resistance form $(\mathcal{E}_p, \mathcal{F}_p)$ on $\mathcal{L}$ with weight $\boldsymbol{\rho}_p$ such that $u \circ g \in \mathcal{F}_p$ and $\mathcal{E}_p(u \circ g) = \mathcal{E}_p(u)$ for any $u \in \mathcal{F}_p$ and any $g \in \mathcal{G}_{\mathrm{sym}}$.*

Proof. Define $E_0 \in \mathcal{S}_p(V_0)$ by $E_0(u) \coloneqq \sum_{x,y \in V_0} |u(x) - u(y)|^p$ for $u \in \mathbb{R}^{V_0}$. Then $E_0(u) = E_0(u \circ g)$ for any $u \in \mathbb{R}^{V_0}$ and $g \in \mathcal{G}_{\mathrm{sym}}$. By Theorem 8.50 and explicit expressions (8.50), (8.51) and (8.52), there exists a p-resistance form $\mathcal{E}^{(0)}_p$ on V_0 such that $\mathcal{R}_{\boldsymbol{\rho}_p}(\mathcal{E}^{(0)}_p) = \mathcal{E}^{(0)}_p$ and $\mathcal{E}^{(0)}_p(u) = \mathcal{E}^{(0)}_p(u \circ g)$ for any $u \in \mathbb{R}^{V_0}$ and any $g \in \mathcal{G}_{\mathrm{sym}}$. The desired symmetry-invariance for $(\mathcal{E}_p, \mathcal{F}_p)$ is immediate from (8.64), Proposition 8.49-(2), the fact that $\tau_g|_{W_n} \colon W_n \to W_n$ is a bijection for any $n \in \mathbb{N} \cup \{0\}$, and the expressions (8.61), (8.62). □

9 p-Walk dimension of Sierpiński carpets/gaskets

In this section, we prove the strict inequality $d_{\mathrm{w},p} > p$ for the generalized Sierpiński carpets and the D-dimensional level-l Sierpiński gasket as an application of the nonlinear potential theory developed in Sections 6 and 7. In particular, we remove the *planarity* in the hypothesis of the previous result [Shi24, Theorem 2.27].

9.1 Generalized Sierpiński carpets

Following [Kaj23, Section 2], we recall the definition of the generalized Sierpiński carpets.

Framework 9.1. Let $D, l \in \mathbb{N}$, $D \geq 2$, $l \geq 3$ and set $Q_0 := [0,1]^D$. Let $S \subsetneq \{0,1,\dots,l-1\}^D$ be non-empty, define $f_i \colon \mathbb{R}^D \to \mathbb{R}^D$ by $f_i(x) := l^{-1}i + l^{-1}x$ for each $i \in S$ and set $Q_1 := \bigcup_{i \in S} f_i(Q_0)$, so that $Q_1 \subsetneq Q_0$. Let K be the self-similar set associated with $\{f_i\}_{i \in S}$. Note that $K \subsetneq Q_0$. Set $F_i := f_i|_K$ for each $i \in S$ and $\mathrm{GSC}(D,l,S) := (K, S, \{F_i\}_{i \in S})$. Let $d \colon K \times K \to [0,\infty)$ be the Euclidean metric on K normalized so that $\mathrm{diam}(K,d) = 1$, set $d_{\mathrm{f}} := \log_l(\#S)$, and let m be the self-similar measure on $\mathrm{GSC}(D,l,S)$ with uniform weight $(1/\#S)_{i \in S}$.

Recall that d_{f} is the Hausdorff dimension of (K,d) and that m is a constant multiple of the d_{f}-dimensional Hausdorff measure on (K,d); see, e.g., [Kig01, Proposition 1.5.8 and Theorem 1.5.7]. Note that $d_{\mathrm{f}} < D$ by $S \subsetneq \{0,1,\dots,l-1\}^D$.

The following definition is due to Barlow and Bass [BB99, Section 2], except that the non-diagonality condition in [BB99, Hypotheses 2.1] has been strengthened later in [BBKT] to fill a gap in [BB99, Proof of Theorem 3.19]; see [BBKT, Remark 2.10-1.] for some more details of this correction.

Definition 9.2 (Generalized Sierpiński carpet)**.** $\mathrm{GSC}(D,l,S)$ is called a *generalized Sierpiński carpet* if and only if the following four conditions are satisfied:

(GSC1) (Symmetry) $f(Q_1) = Q_1$ for any isometry f of $\mathbb{R}^D$ with $f(Q_0) = Q_0$.

(GSC2) (Connectedness) Q_1 is connected.

(GSC3) (Non-diagonality) $\mathrm{int}_{\mathbb{R}^D}\bigl(Q_1 \cap \prod_{k=1}^{D}[(i_k - \varepsilon_k)l^{-1}, (i_k+1)l^{-1}]\bigr)$ is either empty or connected for any $(i_k)_{k=1}^D \in \mathbb{Z}^D$ and any $(\varepsilon_k)_{k=1}^D \in \{0,1\}^D$.

(GSC4) (Borders included) $[0,1] \times \{0\}^{D-1} \subseteq Q_1$.

See [BB99, Remark 2.2] for a description of the meaning of each of the four conditions (GSC1), (GSC2), (GSC3) and (GSC4) in Definition 9.2. To be precise, (GSC3) is slightly different from the formulation of the non-diagonality condition in [BBKT, Subsection 2.2], but they have been proved to be equivalent to each other in [Kaj10, Theorem 2.4]; see [Kaj10, Section 2] for some other equivalent formulations of the non-diagonality condition.

In this subsection, we assume that $\mathrm{GSC}(D,l,S) = (K,S,\{F_i\}_{i \in S})$ as introduced in Framework 9.1 is a generalized Sierpiński carpet as defined in Definition 9.2.

We next ensure the existence of a symmetry-invariant p-resistance form on $\mathrm{GSC}(D,l,S)$ for $p > \dim_{\mathrm{ARC}}(K,d)$ by applying Theorem 8.30.

Definition 9.3. We define

$$\mathcal{G}_0 := \{ f|_K \mid f \text{ is an isometry of } \mathbb{R}^D,\ f(Q_0) = Q_0 \}, \tag{9.1}$$

which forms a finite subgroup of the group of homeomorphisms of K by virtue of (GSC1).

Corollary 9.4. *Let $p \in (\dim_{\mathrm{ARC}}(K,d), \infty)$. Then Assumption 8.26 holds with $r_* = l^{-1}$, K is p-conductively homogeneous, and there exists a regular self-similar p-resistance form*

$(\mathcal{E}_p, \mathcal{W}^p)$ on $\mathrm{GSC}(D, l, S)$ *with weight $(\sigma_p)_{i \in S}$ such that it satisfies the conditions (a)-(d) of Theorem 8.30. Moreover, $(\mathcal{E}_p, \mathcal{W}^p)$ has the following property:*

$$\text{If } u \in \mathcal{W}^p \text{ and } g \in \mathcal{G}_0 \text{ then } u \circ g \in \mathcal{W}^p \text{ and } \mathcal{E}_p(u \circ g) = \mathcal{E}_p(u). \tag{9.2}$$

Proof. Assumption 8.26 and the p-conductive homogeneity for the generalized Sierpiński carpets in the case $p \in (d_{\mathrm{ARC}}, \infty)$ follow from [Kig23, Theorem 4.13] or [Shi24, Proposition 4.5 and Theorem 4.14]. Let $(\mathcal{E}_p, \mathcal{W}^p)$ be a self-similar p-resistance form given in Theorem 8.30. Then the desired properties except for (9.2) are already proved. We can easily see that $\widetilde{\mathcal{E}}_p^n(f \circ g) = \widetilde{\mathcal{E}}_p^n(f)$ for any $f \in L^p(K, m)$, any $g \in \mathcal{G}_0$ and any $n \in \mathbb{N} \cup \{0\}$, and that the conditions (5.44)-(5.46) with $\mathcal{G}_0$ in place of $\mathscr{T}$ hold. Hence the desired symmetry-invariance (9.2) follows Theorem 8.19-(c), (8.38) and Proposition 5.23-(b). $\square$

Recall that σ_p and $d_{\mathrm{w},p}$ are defined for any $p \in (0, \infty)$ (under Assumption 8.26). We know the following monotonicity on $d_{\mathrm{w},p}/p$ in $p \in (0, \infty)$.

Proposition 9.5. *$d_{\mathrm{w},p}/p \geq d_{\mathrm{w},q}/q$ for any $p, q \in (0, \infty)$ with $p \leq q$.*

Proof. This follows from [Kig20, Lemma 4.7.4] and the fact that $d_{\mathrm{f}} = \log_l(\#S)$. $\square$

The following definition is exactly the same as part of [Kaj23, Definition 3.6].

Definition 9.6. (1) We set $V_0^{\varepsilon} := K \cap (\{\varepsilon\} \times \mathbb{R}^{D-1})$ for each $\varepsilon \in \{0, 1\}$ and $U_0 := K \setminus (V_0^0 \cup V_0^1)$.

(2) We define $g_\varepsilon \in \mathcal{G}_0$ by $g_\varepsilon := \tau_\varepsilon|_K$ for each $\varepsilon = (\varepsilon_k)_{k=1}^D \in \{0, 1\}^D$, where $\tau_\varepsilon \colon \mathbb{R}^D \to \mathbb{R}^D$ is given by $\tau_\varepsilon((x_k)_{k=1}^D) := (\varepsilon_k + (1 - 2\varepsilon_k)x_k)_{k=1}^D$, and define a subgroup $\mathcal{G}_1$ of $\mathcal{G}_0$ by

$$\mathcal{G}_1 := \{g_\varepsilon \mid \varepsilon \in \{0\} \times \{0, 1\}^{D-1}\}. \tag{9.3}$$

In the rest of this subsection, we fix $p \in (d_{\mathrm{ARC}}, \infty)$ and a self-similar p-resistance form $(\mathcal{E}_p, \mathcal{W}^p)$ in Corollary 9.4. Recall Theorem 6.13 and let $h_0 := h^{\mathcal{E}_p}_{V_0^0 \cup V_0^1}[\mathbb{1}_{V_0^1}] \in \mathcal{W}^p$. The strategy to prove $d_{\mathrm{w},p} > p$ is very similar to [Kaj23], that is, we will prove the *non-$\mathcal{E}_p$-harmonicity* on U_0 of $h_2 := \sum_{w \in W_2} (F_w)_*(l^{-2}h_0 + q_1^w \mathbb{1}_K) \in \mathcal{W}^p$, which also satisfies $h_2|_{V_0^i} = i$ $(i = 0, 1)$. (See [Kaj23, Figures 2 and 3] for illustrations of h_0 and of h_2.) Then the strict estimate $d_{\mathrm{w},p} > p$ will follow from $\mathcal{E}_p(h_0) < \mathcal{E}_p(h_2)$ and the self-similarity of $\mathcal{E}_p$. Our arguments will be easier than that of [Kaj23] by virtue of $\mathcal{W}^p \subseteq C(K)$.

The next proposition is a key ingredient. Note that it requires our standing assumption that $S \neq \{0, 1, \ldots, l-1\}^D$, which excludes the case of $K = [0, 1]^D$ from the present framework.

Proposition 9.7. *Let $h_2 := \sum_{w \in W_2} (F_w)_*(l^{-2}h_0 + q_1^w \mathbb{1}_K) \in \mathcal{W}^p$. Then h_2 is not $\mathcal{E}_p$-harmonic on U_0 and $h_2|_{V_0^i} = i$ for each $i \in \{0, 1\}$.*

Proof. The proof is a straightforward modification of [Kaj23, Proposition 3.11] thanks to Theorem 6.13. We present here a self-contained proof for the reader's convenience.

We claim that, if h_2 were $\mathcal{E}_p$-harmonic on U_0, then $h_0 \in \mathcal{W}^p$ would turn out to be $\mathcal{E}_p$-harmonic on $K \setminus V_0^0$, which would imply by combining with Proposition 6.11 that $\mathcal{E}_p(h_0) = \mathcal{E}_p(h_0; h_0) = 0$, which would be a contradiction by (RF1)$_p$ and $\mathcal{W}^p \subseteq C(K)$.

For each $\varepsilon = (\varepsilon_k)_{k=1}^D \in \{1\} \times \{0,1\}^{D-1}$, set $U^\varepsilon := K \cap \prod_{k=1}^D (\varepsilon_k - 1, \varepsilon_k + 1)$ and $K^\varepsilon := K \cap \prod_{k=1}^D [\varepsilon_k - 1/2, \varepsilon_k + 1/2]$. Fix $\varphi_\varepsilon \in \mathcal{W}^p \cap C_c(U^\varepsilon)$ so that $\varphi_\varepsilon|_{K^\varepsilon} = \mathbb{1}_{K^\varepsilon}$, which exists by (8.17), (RF1)$_p$ and (RF5)$_p$. Let $v \in \mathcal{W}^p \cap C_c(K \setminus V_0^0)$ and, taking an enumeration $\{\varepsilon^{(k)}\}_{k=1}^{2^{D-1}}$ of $\{1\} \times \{0,1\}^{D-1}$ and recalling Proposition 2.3-(d), define $v_\varepsilon \in \mathcal{W}^p \cap C_c(U^\varepsilon)$ for $\varepsilon \in \{1\} \times \{0,1\}^{D-1}$ by $v_{\varepsilon^{(1)}} := v\varphi_{\varepsilon^{(1)}}$ and $v_{\varepsilon^{(k)}} := v\varphi_{\varepsilon^{(k)}} \prod_{j=1}^{k-1}(\mathbb{1}_K - \varphi_{\varepsilon^{(j)}})$ for $k \in \{2, \ldots, 2^{D-1}\}$. Then $v - \sum_{\varepsilon \in \{1\}\times\{0,1\}^{D-1}} v_\varepsilon = v \prod_{\varepsilon \in \{1\}\times\{0,1\}^{D-1}} (\mathbb{1}_K - \varphi_\varepsilon) \in \mathcal{W}^p \cap C_c(U_0)$, hence $\mathcal{E}_p(h_0; v) = \sum_{\varepsilon \in \{1\}\times\{0,1\}^{D-1}} \mathcal{E}_p(h_0; v_\varepsilon)$ by Proposition 6.11 (with $Y = K \setminus U_0$). Therefore the desired $\mathcal{E}_p$-harmonicity of h_0 on $K \setminus V_0^0$ would be obtained by deducing that $\mathcal{E}(h_0; v_\varepsilon) = 0$ for any $\varepsilon \in \{1\} \times \{0,1\}^{D-1}$.

To this end, set $\varepsilon^{(0)} := (\mathbb{1}_{\{1\}}(k))_{k=1}^D$, take $i = (i_k)_{k=1}^D \in S$ with $i_1 < l - 1$ and $i + \varepsilon^{(0)} \notin S$, which exists by $\emptyset \neq S \subsetneq \{0, 1, \ldots, l-1\}^D$ and (GSC1), and let $\varepsilon = (\varepsilon_k)_{k=1}^D \in \{1\} \times \{0,1\}^{D-1}$. We will choose $i^\varepsilon \in S$ with $F_{ii^\varepsilon}(\varepsilon) \in F_i(K \cap (\{1\} \times (0,1)^{D-1}))$ and assemble $v_\varepsilon \circ g_w \circ F_w^{-1}$ with a suitable $g_w \in \mathcal{G}_1$ for $w \in W_2$ with $F_{ii^\varepsilon}(\varepsilon) \in K_w$ into a function $v_{\varepsilon,2} \in \mathcal{W}^p \cap C_c(U_0)$. Specifically, set $i^{\varepsilon,\eta} := \bigl((l-1)(\mathbb{1}_{\{1\}}(k) + 1 - \varepsilon_k) + (2\varepsilon_k - 1)\eta_k\bigr)_{k=1}^D$ for each $\eta = (\eta_k)_{k=1}^D \in \{0\} \times \{0,1\}^{D-1}$ and $I^\varepsilon := \{\eta \in \{0\} \times \{0,1\}^{d-1} \mid i^{\varepsilon,\eta} \in S\}$, so that $i^\varepsilon := i^{\varepsilon,\mathbf{0}_D} \in S$ by (GSC4) and (GSC1) and hence $\mathbf{0}_D \in I^\varepsilon$. Thanks to $v_\varepsilon \in \mathcal{W}^p \cap C_c(U^\varepsilon)$ and $i + \varepsilon^{(0)} \notin S$ we can define $v_{\varepsilon,2} \in C(K)$ by setting

$$v_{\varepsilon,2}|_{K_w} := \begin{cases} v_\varepsilon \circ g_\eta \circ F_w^{-1} & \text{if } \eta \in I^\varepsilon \text{ and } w = ii^{\varepsilon,\eta} \\ 0 & \text{if } w \notin \{ii^{\varepsilon,\eta} \mid \eta \in I^\varepsilon\} \end{cases} \quad \text{for each } w \in W_2. \tag{9.4}$$

Then $\operatorname{supp}_K[v_{\varepsilon,2}] \subseteq K_i \setminus V_0^0 \subseteq U_0$ by (9.4) and $i_1 < l - 1$. In addition, $v_{\varepsilon,2} \circ F_w \in \mathcal{W}^p$ for any $w \in W_2$ by (9.4), $v_\varepsilon \in \mathcal{W}^p$ and (9.2). Thus $v_{\varepsilon,2} \in \mathcal{F}_p$ by (5.5) and therefore $v_{\varepsilon,2} \in \mathcal{W}^p \cap C_c(U_0)$. Recall that $h_2 \circ F_w = l^{-2}h_0 + q_1^w \mathbb{1}_K$ for any $w \in W_2$ and note that, by the uniqueness in Theorem 6.13, $h_0 \circ g_\eta = h_0$ for any $\eta \in I^\varepsilon$. Then we have

$$\begin{aligned} \mathcal{E}_p(h_2; v_{\varepsilon,2}) &= \sum_{\eta \in I^\varepsilon} \sigma_p^2 l^{-2(p-1)} \mathcal{E}_p(h_0; v_\varepsilon \circ g_\eta) \\ &= \sum_{\eta \in I^\varepsilon} \sigma_p^2 l^{-2(p-1)} \mathcal{E}_p(h_0 \circ g_\eta; v_\varepsilon) = (\#I^\varepsilon)\sigma_p^2 l^{-2(p-1)} \mathcal{E}_p(h_0; v_\varepsilon). \end{aligned} \tag{9.5}$$

Now, supposing that h_2 were $\mathcal{E}_p$-harmonic on U_0, from (9.5), $\#I^\varepsilon > 0$, $v_{\varepsilon,2} \in \mathcal{F}_p \cap C_c(U_0)$ and Proposition 6.11, we would obtain $\mathcal{E}_p(h_0; v_\varepsilon) = \sigma_p^{-2} l^{2(p-1)} (\#I^\varepsilon)^{-1} \mathcal{E}_p(h_2; v_{\varepsilon,2}) = 0$, which would imply a contradiction as explained in the last two paragraphs. □

Theorem 9.8. *$d_{\mathrm{w},p} > p$ for any $p \in (0, \infty)$.*

Proof. It suffices to prove the case of $p \in (d_{\mathrm{ARC}}, \infty)$ by Proposition 9.5. Let $h_0, h_2 \in \mathcal{W}^p$ be as in Proposition 9.7. By Proposition 9.7, we have $\mathcal{E}_p(h_0) < \mathcal{E}_p(h_2)$. This strict inequality combined with (5.6) shows that

$$\mathcal{E}_p(h_0) < \mathcal{E}_p(h_2) = \bigl(\sigma_p(\#S)l^{-p}\bigr)^2 \mathcal{E}_p(h_0),$$

whence $l^p < \sigma_p(\#S)$. Since $\sigma_p = l^{d_{\mathrm{w},p}-d_{\mathrm{f}}}$ and $d_{\mathrm{f}} = \log(\#S)/\log l$, we get $d_{\mathrm{w},p} = \log\bigl(\sigma_p(\#S)\bigr)/\log l > p$. $\square$

9.2 D-dimensional level-l Sierpiński gasket

Following [Kaj13, Example 5.1], we introduce the D-dimensional level-l Sierpiński gasket.

Framework 9.9 (D-dimensional level-l Sierpiński gasket)**.** Let $D, l \in \mathbb{N}$, $D \geq 2$, $l \geq 2$ and let $\{q_k\}_{k=0}^{D} \subseteq \mathbb{R}^D$ be the set of the vertices of a regular D-dimensional simplex so that $q_0, \dots, q_{D-1} \in \{(x_1, \dots, x_D) \in \mathbb{R}^D \mid x_1 = 0\}$ and $q_D \in \{(x_1, \dots, x_D) \in \mathbb{R}^D \mid x_1 \geq 0\}$. Further let $S \coloneqq \bigl\{(i_k)_{k=1}^{D} \bigm| i_k \in \mathbb{N} \cup \{0\}, \sum_{k=1}^{D} i_k \leq l-1\bigr\}$, and for each $i = (i_k)_{k=1}^{D} \in S$ we set $q_i \coloneqq q_0 + \sum_{k=1}^{D} l^{-1} i_k (q_k - q_0)$ and define $f_i \colon \mathbb{R}^D \to \mathbb{R}^D$ by $f_i(x) \coloneqq q_i + l^{-1}(x - q_0)$. Let K be the self-similar set associated with $\{f_i\}_{i\in S}$ and set $F_i \coloneqq f_i|_K$. Let $\mathrm{SG}(D,l,S) = (K, S, \{F_i\}_{i\in S})$, which is a self-similar structure. Let $d \colon K \times K \to [0,\infty)$ be the Euclidean metric on K normalized so that $\mathrm{diam}(K,d) = 1$, set $d_{\mathrm{f}} \coloneqq \log_l(\#S)$, and let m be the self-similar measure on $\mathrm{SG}(D,l,S)$ with uniform weight $(1/\#S)_{i\in S}$.

$\mathrm{SG}(D,l,S)$ is clearly an affine nested fractal (recall Framework 8.46 and Definition 8.47), and called the *D-dimensional level-l Sierpiński gasket*. In the rest of this subsection, we fix the Sierpiński gasket $\mathrm{SG}(D,l,S)$ and the self-similar measure m as in Framework 9.9. We can easily verify [Kig23, Assumption 2.15] for $\mathrm{SG}(D,l,S)$. In addition, it is well known that m is d_{f}-Ahlfors regular (see [Kig23, Proposition E.7] for example). Similar to Corollary 9.4, we have a symmetry-invariant p-resistance form on $\mathrm{SG}(D,l,S)$ for any $p \in (1,\infty)$. (The Ahlfors regular conformal dimension of (K,d) is 1; see Theorem B.8.)

Definition 9.10. We define

$$\mathcal{G}_0 \coloneqq \{f|_K \mid f \text{ is an isometry of } \mathbb{R}^D,\ f(V_0) = V_0\}, \tag{9.6}$$

which forms a finite subgroup of the group of homeomorphisms of K.

Corollary 9.11. *Let $p \in (1,\infty)$. Then Assumption 8.26 holds with $r_* = l^{-1}$, K is p-conductively homogeneous, and there exists a regular self-similar p-resistance form $(\mathcal{E}_p, \mathcal{W}^p)$ on $\mathrm{SG}(D,l,S)$ with weight $(\sigma_p)_{i\in S}$ such that it satisfies the conditions (a)-(d) in Theorem 8.30. Moreover, $(\mathcal{E}_p, \mathcal{W}^p)$ has the following property:*

$$\textit{If } u \in \mathcal{W}^p \textit{ and } g \in \mathcal{G}_0 \textit{ then } u \circ g \in \mathcal{W}^p \textit{ and } \mathcal{E}_p(u \circ g) = \mathcal{E}_p(u). \tag{9.7}$$

Similar to Proposition 9.5, we have the following monotonicity of $d_{\mathrm{w},p}/p$ in p.

Proposition 9.12. *$d_{\mathrm{w},p}/p \geq d_{\mathrm{w},q}/q$ for any $p, q \in (0,\infty)$ with $p \leq q$.*

We can prove the following main result by using compatible sequences.

Theorem 9.13. *$d_{\mathrm{w},p} > p$ for any $p \in (0,\infty)$.*

Proof. Let $p \in (1, \infty)$ and let $(\mathcal{E}_p, \mathcal{W}^p)$ be a self-similar p-resistance form on $\mathrm{SG}(D, l, S)$ as given in Corollary 9.11. Define $u \in C(K)$ by $u(x_1, \dots, x_D) \coloneqq x_1$ for any $(x_1, \dots, x_D) \in K \subseteq \mathbb{R}^D$. Then $u|_{V_n} \in \mathcal{W}^p|_{V_n}$ for any $n \in \mathbb{N} \cup \{0\}$ by Proposition 6.8. We claim that if $u|_{V_1}$ were $\mathcal{E}_p|_{V_1}$-harmonic on $V_1 \setminus V_0$, then $\mathcal{E}_p|_{V_0}(u|_{V_0}) = 0$, which would contradict (RF1)$_p$.

Suppose that $\mathcal{E}_p|_{V_1}(u|_{V_1}; \varphi) = 0$ for every $\varphi \in \mathbb{R}^{V_1}$ with $\varphi|_{V_0} = 0$. Noting that $(u|_{V_1} \circ F_i)|_{V_0} = l^{-1} u|_{V_0} + c_i \mathbb{1}_{V_0}$ for some constant $c_i \in \mathbb{R}$ and using (7.5), we have

$$\mathcal{E}_p|_{V_1}(u|_{V_1}; \varphi) = \sigma_p \sum_{i \in S} \mathcal{E}_p|_{V_0}(u|_{V_1} \circ F_i; \varphi \circ F_i) = l^{-(p-1)} \sigma_p \sum_{i \in S} \mathcal{E}_p|_{V_0}(u|_{V_0}; \varphi \circ F_i). \tag{9.8}$$

It is easy to see that $(V_1 \setminus V_0) \cap \{(x_1, \dots, x_D) \in \mathbb{R}^D \mid x_1 = 0\} \neq \emptyset$. Let $z \in V_1 \setminus V_0$ with $z \in \{x_1 = 0\}$ and let $\varphi \coloneqq \mathbb{1}^{V_1}_{\{z\}} \in \mathbb{R}^{V_1}$. Since $u \circ g = u$ for any $g \in \mathcal{G}_0$ with $g(\{x_1 = 0\} \cap K) = \{x_1 = 0\} \cap K$, the $\mathcal{G}_0$-invariance (9.7) implies $\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_i\}}\bigr) = \mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_j\}}\bigr)$ for any $i, j \in \{0, \dots, D-1\}$. Since $\varphi \circ F_j = 0 \in \mathbb{R}^{V_0}$ for any $j \in S$ with $z \notin K_j$, we have from (9.8) that

$$\begin{aligned} 0 = \mathcal{E}_p|_{V_1}(u|_{V_1}; \varphi) &= l^{-(p-1)} \sigma_p \sum_{i \in S; z \in K_i} \mathcal{E}_p|_{V_0}(u|_{V_0}; \varphi \circ F_i) \\ &= l^{-(p-1)} \sigma_p \cdot \bigl(\#\{i \in S \mid z \in K_i\}\bigr) \mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_0\}}\bigr). \end{aligned}$$

Hence we get $\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_j\}}\bigr) = 0$ for every $j \in \{0, \dots, D-1\}$. Therefore,

$$\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_D\}}\bigr) = \mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}_{V_0}\bigr) = \sum_{j=0}^{D-1} \mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_j\}}\bigr) = 0,$$

which yields $\mathcal{E}_p|_{V_0}(u|_{V_0}; v) = 0$ for every $v \in \mathbb{R}^{V_0}$. In particular, $\mathcal{E}_p|_{V_0}(u|_{V_0}) = 0$, which is a contradiction and hence we conclude that $u|_{V_1}$ is *not* $\mathcal{E}_p|_{V_1}$-harmonic on $V_1 \setminus V_0$. Combining with Proposition 6.15, we see that

$$\mathcal{E}_p|_{V_0}(u|_{V_0}) = \mathcal{E}_p|_{V_1}|_{V_0}(u|_{V_0}) = \mathcal{E}_p|_{V_1}\Bigl(h^{\mathcal{E}_p|_{V_1}}_{V_0}[u|_{V0}]\Bigr) < \mathcal{E}_p|_{V_1}(u|_{V_1}). \tag{9.9}$$

Similar to (9.8), we have $\mathcal{E}_p|_{V_1}(u|_{V_1}) = l^{-p} \sigma_p (\#S) \mathcal{E}_p|_{V_0}(u|_{V_0})$. Hence the strict inequality (9.9) yields $1 < l^{-p} l^{d_{\mathrm{w},p} - d_{\mathrm{f}}} (\#S) = l^{d_{\mathrm{w},p} - p}$, which proves $d_{\mathrm{w},p} > p$ for any $p \in (1, \infty)$. By Proposition 9.12, we complete the proof. □

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A Symmetric Dirichlet forms and the generalized 2-contraction property

In this section, we verify the generalized contraction properties for various energy forms resulting from symmetric Dirichlet forms.

A.1 Symmetric Dirichlet forms satisfy the generalized 2-contraction property

In this subsection, we show that any symmetric Dirichlet form satisfies (GC)$_2$. Throughout this subsection, we fix a measure space $(X, \mathcal{B}, m)$.

Let us recall the definition of the notion of symmetric Dirichlet form. See, e.g., [CF, FOT, MR] for details of the theory of Dirichlet forms.

Definition A.1 (Symmetric Dirichlet form)**.** Let $\mathcal{F}$ be a dense linear subspace of $L^2(X, m)$ and let $\mathcal{E} \colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ be a non-negative definite symmetric bilinear form on $\mathcal{F}$. The pair $(\mathcal{E}, \mathcal{F})$ is said to be a *symmetric Dirichlet form* on $L^2(X, m)$ if and only if $\mathcal{F}$ equipped with the inner product $\mathcal{E} + \langle\,\cdot\,,\,\cdot\,\rangle_{L^2(X,m)}$ is a Hilbert space, $u^+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(u^+ \wedge 1, , u^+ \wedge 1) \le \mathcal{E}(u, u)$ for any $u \in \mathcal{F}$.

We can show that a symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies (GC)$_2$ by modifying the proof of [MR, Theorem I.4.12].

Proposition A.2. *Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X, m)$. Then $(\mathcal{E}_2, \mathcal{F})$ given by $\mathcal{E}_2(u) := \mathcal{E}(u, u)$ is a 2-energy form on (X, m) satisfying* (GC)$_2$.

Proof. It is clear that $\mathcal{E}_2^{1/2}$ is a seminorm on $\mathcal{F}$, so we shall prove (GC)$_2$ for $(\mathcal{E}_2, \mathcal{F})$. Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, 2]$, $q_2 \in [2, \infty]$ and $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2). We consider the case of $q_2 < \infty$ (the case of $q_2 = \infty$ is similar). Let $\{G_\alpha\}_{\alpha > 0}$ be the strongly continuous resolvent on $L^2(X, m)$ associated with $(\mathcal{E}, \mathcal{F})$; see, e.g., [MR, Theorem I.2.8]. By [MR, Theorem I.2.13-(ii)], it suffices to prove that for any $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in L^2(X, m)^{n_1}$ and any $\alpha \in (0, \infty)$,

$$\left(\sum_{l=1}^{n_2} \langle (1 - \alpha G_\alpha) T_l(\boldsymbol{u}), T_l(\boldsymbol{u}) \rangle_{L^2(X,m)}^{q_2/2}\right)^{1/q_2} \le \left(\sum_{k=1}^{n_1} \langle (1 - \alpha G_\alpha) u_k, u_k \rangle_{L^2(X,m)}^{q_1/2}\right)^{1/q_1}. \tag{A.1}$$

By the linearity of G_α and (2.2), it is enough to prove (A.1) in the case where u_k is a simple function for each $k \in \{1, \dots, n_1\}$, so we assume that

$$u_k = \sum_{i=1}^{N} \alpha_{ki} \mathbb{1}_{A_i}, \quad k \in \{1, \dots, n_1\}, \tag{A.2}$$

where $N \in \mathbb{N}$, $(\alpha_{ki})_{i=1}^{N} \subseteq \mathbb{R}$, $\{A_i\}_{i=1}^{N} \subseteq \mathcal{B}(X)$ with $m(A_i) < \infty$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Fix $\alpha \in (0, \infty)$ and, for $i, j \in \{1, \dots, N\}$, we define

$$b_{i,j} := \langle (1 - \alpha G_\alpha) \mathbb{1}_{A_i}, \mathbb{1}_{A_j} \rangle_{L^2(X,m)}, \quad \lambda_i := m(A_i) \quad \text{and} \quad a_{ij} := \langle \alpha G_\alpha \mathbb{1}_{A_i}, \mathbb{1}_{A_j} \rangle_{L^2(X,m)}.$$

Then $b_{ij} = \lambda_i \delta_{ij} - a_{ij}$ by a simple calculation, and $a_{ij} = a_{ji}$ since G_α is a symmetric operator on $L^2(X,m)$ (see, e.g., [MR, Theorem I.2.8]). Hence for any $(z_1,\dots,z_N) \in \mathbb{R}^N$,

$$\sum_{i,j=1}^{N} z_i z_j b_{ij} = \sum_{i<j} a_{ij}(z_i - z_j)^2 + \sum_{j=1}^{N} m_j z_j^2, \tag{A.3}$$

where $m_j \coloneqq \lambda_j - \sum_{i=1}^{N} a_{ij}$. Note that $a_{ij} \geq 0$ for any $i,j \in \{1,\dots,N\}$ since $\alpha G_\alpha \mathbb{1}_{A_i} \geq 0$ by [MR, Theorem I.4.4]. We set $A \coloneqq \bigcup_{i=1}^{N} A_i$, and then we have $\alpha G_\alpha(\mathbb{1}_A) \leq 1$ by [MR, Theorem I.4.4] and

$$\sum_{u=1}^{N} a_{ij} = \alpha \int_X \mathbb{1}_A G_\alpha(\mathbb{1}_{A_j})\, dm = \alpha \int_X G_\alpha(\mathbb{1}_A)\mathbb{1}_{A_j}\, dm \leq \int_X \mathbb{1}_{A_j}\, dm = \lambda_j,$$

whence $m_j \geq 0$. With these preparations, we show (A.1) for $\boldsymbol{u}$ defined in (A.2). Set $T_{l,i} \coloneqq T_l(\alpha_{1i},\dots,\alpha_{u_1 i})$ for each $l \in \{1,\dots,n_2\}$. We see that

$$\begin{aligned}
&\sum_{l=1}^{n_2} \langle (1-\alpha G_\alpha)T_l(\boldsymbol{u}), T_l(\boldsymbol{u})\rangle_{L^2(X,m)}^{q_2/2} = \sum_{l=1}^{n_2} \left(\sum_{i,j=1}^{N} T_{l,i}T_{l,j} b_{ij} \right)^{q_2/2} \\
&\overset{(A.3)}{=} \sum_{l=1}^{n_2} \left(\sum_{i<j} a_{ij}(T_{l,i} - T_{l,j})^{q_2\cdot\frac{2}{q_2}} + \sum_{j=1}^{N} m_j T_{l,j}^{q_2\cdot\frac{2}{q_2}} \right)^{q_2/2} \\
&\overset{(2.18)}{\leq} \left(\sum_{i<j} \left(a_{ij}^{q_2/2} \sum_{l=1}^{n_2} (T_{l,i} - T_{l,j})^{q_2} \right)^{2/q_2} + \sum_{j=1}^{N} \left(m_j^{q_2/2} \sum_{l=1}^{n_2} T_{l,j}^{q_2} \right)^{2/q_2} \right)^{q_2/2} \\
&\overset{(2.2)}{\leq} \left(\sum_{i<j} \left(a_{ij}^{q_2/2} \left(\sum_{k=1}^{n_1} (\alpha_{ki} - \alpha_{kj})^{q_1} \right)^{q_2/q_1} \right)^{2/q_2} + \sum_{j=1}^{N} \left(m_j^{q_2/2} \left(\sum_{k=1}^{n_1} \alpha_{kj}^{q_1} \right)^{q_2/q_1} \right)^{2/q_2} \right)^{q_2/2} \\
&= \left(\sum_{i<j} \left(\sum_{k=1}^{n_1} \bigl(a_{ij}(\alpha_{ki} - \alpha_{kj})^2\bigr)^{q_1/2} \right)^{2/q_1} + \sum_{j=1}^{N} \left(\sum_{k=1}^{n_1} (m_j \alpha_{kj}^2)^{q_1/2} \right)^{2/q_1} \right)^{\frac{q_1}{2}\cdot\frac{q_2}{q_1}} \\
&\overset{(*)}{\leq} \left(\left(\sum_{k=1}^{n_1} \left(\sum_{i<j} a_{ij}(\alpha_{ki} - \alpha_{kj})^2 + \sum_{j=1}^{N} m_j \alpha_{kj}^2 \right)^{q_1/2} \right)^{2/q_1} \right)^{\frac{q_1}{2}\cdot\frac{q_2}{q_1}} \\
&= \left(\sum_{k=1}^{n_1} \left(\sum_{i<j} a_{ij}(\alpha_{ki} - \alpha_{kj})^2 + \sum_{j=1}^{N} m_j \alpha_{kj}^2 \right)^{q_1/2} \right)^{q_2/q_1} \\
&\overset{(A.3)}{=} \left(\sum_{k=1}^{n_1} \left(\sum_{i,=1}^{N} \alpha_{ki}\alpha_{kj} b_{ij} \right)^{q_1/2} \right)^{q_2/q_1} = \left(\sum_{k=1}^{n_1} \langle (1-\alpha G_\alpha)u_k, u_k \rangle_{L^2(X,m)}^{q_1/2} \right)^{\frac{2}{q_1}\cdot\frac{q_2}{2}},
\end{aligned}$$

where we used the triangle inequality for ℓ^{2/q_1}-norm in $(*)$. The proof is completed. $\square$

Next we will extend $(\mathrm{GC})_2$ to $(\mathcal{E}_2, \mathcal{F}_e)$, where $\mathcal{F}_e$ is the *extended Dirichlet space*; see Definition A.4 below. (See, e.g., [FOT, Section 1.5] or [CF, Section 1.1] for details on the extended Dirichlet space.) We need to recall the following result.

Proposition A.3 ([Sch99b, Proposition 2] and [Sch99a, Lemma 1][14])**.** *Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X, m)$. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ converges m-a.e. to 0 as $n \to \infty$ and $\lim_{k\wedge l\to\infty} \mathcal{E}(u_k - u_l, u_k - u_l) = 0$, then $\lim_{n\to\infty} \mathcal{E}(u_n, u_n) = 0$.*

Now we define the extended Dirichlet form $(\mathcal{E}, \mathcal{F}_e)$.

Definition A.4 (Extended Dirichlet form)**.** Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X, m)$. We define the *extended Dirichlet form* $(\mathcal{E}, \mathcal{F}_e)$ of $(\mathcal{E}, \mathcal{F})$ by

$$\mathcal{F}_e := \left\{ f \in L^0(X, m) \,\middle|\, \begin{array}{l} \lim_{n\to\infty} f_n = f \ m\text{-a.e. for some } \{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \\ \text{with } \lim_{k\wedge l\to\infty} \mathcal{E}(f_k - f_l, f_k - f_l) = 0 \end{array} \right\}, \tag{A.4}$$

$$\mathcal{E}(f, f) := \lim_{n\to\infty} \mathcal{E}(f_n, f_n), \quad f \in \mathcal{F}_e, \text{ where } \{f_n\}_{n\in\mathbb{N}} \text{ is a sequence as in (A.4)}. \tag{A.5}$$

Each such $\{f_n\}_{n\in\mathbb{N}}$ as in (A.4) is called an *approximating sequence for f*. By Proposition A.3, the limit $\lim_{n\to\infty} \mathcal{E}(f_n, f_n)$ in (A.5) does not depend on a particular choice of $\{f_n\}_{n\in\mathbb{N}}$, and we have $\mathcal{F} = \mathcal{F}_e \cap L^2(X, m)$ by [Sch99b, Proposition 2]; see also [CF, Theorem 1.1.5].

We also need the following proposition, which is proved by utilizing a version [CF, Theorem A.4.1-(ii)] of the Banach–Saks theorem.

Proposition A.5 ([Sch99a, Lemma 2][15])**.** *Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X, m)$, and let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$. If $\liminf_{n\to\infty} \mathcal{E}(u_n, u_n) < \infty$ and $\{u_n\}_{n\in\mathbb{N}}$ converges m-a.e. to $u \in L^0(X, m)$ as $n \to \infty$, then $u \in \mathcal{F}_e$ and $\mathcal{E}(u, u) \leq \liminf_{n\to\infty} \mathcal{E}(u_n, u_n)$.*

Now we can show that the extended Dirichlet form $(\mathcal{E}, \mathcal{F}_e)$ satisfies $(\mathrm{GC})_2$.

Proposition A.6. *Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X, m)$. Then $(\mathcal{E}_2, \mathcal{F}_e)$ given by $\mathcal{E}_2(u) := \mathcal{E}(u, u)$ is a 2-energy form on (X, m) satisfying $(\mathrm{GC})_2$.*

Proof. It is clear that $\mathcal{E}_2^{1/2}$ is a seminorm on $\mathcal{F}_e$. Let us show $(\mathrm{GC})_2$ for $(\mathcal{E}_2, \mathcal{F}_e)$. As in the proof of Proposition A.2, let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, 2]$, $q_2 \in [2, \infty]$ and $T = (T_1, \ldots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2). Let $\boldsymbol{u} = (u_1, \ldots, u_{n_1}) \in \mathcal{F}_e^{n_1}$. For each $k \in \{1, \ldots, n_1\}$, let $\{u_{k,n}\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ be an approximating sequence for u_k. Set $\boldsymbol{u}_n := (u_{1,n}, \ldots, u_{n_1,n})$. Since $T_l \in C(\mathbb{R}^{n_1})$ and $(\mathcal{E}_2, \mathcal{F})$ satisfies $(\mathrm{GC})_2$ (Proposition A.2),

[14] In [Sch99a, Lemma 1], Proposition A.3 is stated and proved for a much wider class of $(\mathcal{E}, \mathcal{F})$. The assumptions made in [Sch99a, Lemma 1] are that $(X, \mathcal{B}, m)$ is an arbitrary measure space, that $\mathcal{F}$ is a linear subspace of $L^0(X, m)$ and that $\mathcal{E}\colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a non-negative definite bilinear form satisfying the *strong sector condition* (see [Sch99a, Definition 1]) and the Fatou property (see [Sch99a, Definition 2]), both of which are satisfied if $(\mathcal{E}, \mathcal{F})$ is a symmetric Dirichlet form. Indeed, the strong sector condition is immediate from the Cauchy–Schwarz inequality for $\mathcal{E}$ and the Fatou property for $(\mathcal{E}, \mathcal{F})$ holds by [Sch99b, Proposition 2].

[15] In [Sch99a, Lemma 2], Proposition A.5 is stated and proved for the same class of bilinear forms $(\mathcal{E}, \mathcal{F})$ as Proposition A.3 is in [Sch99a, Lemma 1].

$\lim_{n\to\infty} T_l(\boldsymbol{u}_n) = T_l(\boldsymbol{u})$ m-a.e. and $\{\mathcal{E}_2(T_l(\boldsymbol{u}_n))\}_{n\in\mathbb{N}}$ is bounded. Then we have $T_l(\boldsymbol{u}) \in \mathcal{F}_e$ and $\mathcal{E}_2(T_l(\boldsymbol{u})) \le \liminf_{n\to\infty} \mathcal{E}_2(T_l(\boldsymbol{u}_n))$ by Proposition A.5, and see by (GC)$_2$ for $(\mathcal{E}_2, \mathcal{F})$ from Proposition A.2 that

$$\begin{aligned}\left\| \left(\mathcal{E}_2(T_l(\boldsymbol{u}))^{1/2}\right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} &\le \left\| \left(\liminf_{n\to\infty} \mathcal{E}_2(T_l(\boldsymbol{u}_n))^{1/2}\right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\ &\le \liminf_{n\to\infty} \left\| \left(\mathcal{E}_2(T_l(\boldsymbol{u}_n))^{1/2}\right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\ &\le \liminf_{n\to\infty} \left\| \left(\mathcal{E}_2(u_{k,n})^{1/2}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} = \left\| \left(\mathcal{E}_2(u_k)^{1/2}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}},\end{aligned}$$

proving that $(\mathcal{E}_2, \mathcal{F}_e)$ satisfies (GC)$_2$. □

A.2 The generalized 2-contraction property for energy measures

In this subsection, under the standard topological assumptions on (X, m), we verify (GC)$_2$ for the (2-)energy measures associated with a regular symmetric Dirichlet form.

Throughout this subsection, we assume that X and m are as specified in (3.26) and (3.27), which are precisely the topological assumption [FOT, (1.1.7)] made almost throughout the book [FOT], and that $(\mathcal{E}, \mathcal{F})$ is a symmetric Dirichlet form on $L^2(X, m)$ which is *regular*, i.e., possesses a core in the sense of Definition 3.26-(1).

A regular symmetric Dirichlet form is known to satisfy the following representation.

Theorem A.7 (Beurling–Deny expression; see, e.g., [FOT, Theorem 3.2.1])**.** *There exist a symmetric bilinear form $\mathcal{E}^{(c)}$ on $\mathcal{F} \cap C_c(X)$ satisfying $\mathcal{E}^{(c)}(u, v) = 0$ for any $u, v \in \mathcal{F} \cap C_c(X)$ with v constant on a neighborhood of* $\mathrm{supp}_X[u]$*, a symmetric Radon measure J on $X \times X \setminus \{(x, x) \mid x \in X\}$ and a Radon measure k on X such that*

$$\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \mathcal{E}^{(j)}(u, v) + \mathcal{E}^{(k)}(u, v) \quad \text{for any } u, v \in \mathcal{F} \cap C_c(X), \tag{A.6}$$

where

$$\mathcal{E}^{(j)}(u, v) := \int_{X\times X} (u(x) - u(y))(v(x) - v(y))\, J(dxdy), \quad \mathcal{E}^{(k)}(u, v) := \int_X u(x)v(x)\, k(dx).$$

Moreover, such $\mathcal{E}^{(c)}$, J and k are uniquely determined by $\mathcal{E}$. We call $\mathcal{E}^{(c)}$ the local part *of $\mathcal{E}$, J the* jumping measure *associated with $\mathcal{E}$ and k the* killing measure *associated with $\mathcal{E}$.*

In the next two propositions, we extend each of $\mathcal{E}^{(c)}, \mathcal{E}^{(j)}, \mathcal{E}^{(k)}$ in (A.6) to $\mathcal{F}_e$ and associate energy measures to each of them; see [FOT, Section 3.2] for their proofs.

Proposition A.8. *Let $u \in \mathcal{F}_e$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ be an approximating sequence for u. Then, for any $\mathcal{E}^{\#} \in \{\mathcal{E}^{(c)}, \mathcal{E}^{(j)}, \mathcal{E}^{(k)}\}$, $\{\mathcal{E}^{\#}(u_n, u_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $[0, \infty)$ and the limit $\lim_{n\to\infty} \mathcal{E}^{\#}(u_n, u_n) \in [0, \infty)$ does not depend on a particular choice of an approximating sequence $\{u_n\}_{n\in\mathbb{N}}$ for u.*

Proposition A.9. *Let $\mathcal{E}^{\#} \in \{\mathcal{E}, \mathcal{E}^{(c)}, \mathcal{E}^{(j)}, \mathcal{E}^{(k)}\}$. For any $u \in \mathcal{F} \cap C_c(X)$, there exists a unique Radon measure $\mu^{\#}_{\langle u \rangle}$ on X such that*

$$\int_X \varphi \, d\mu^{\#}_{\langle u \rangle} = \mathcal{E}^{\#}(u, u\varphi) - \frac{1}{2}\mathcal{E}^{\#}(u^2, \varphi) \quad \text{for any } \varphi \in \mathcal{F} \cap C_c(X). \tag{A.7}$$

Moreover, for any Borel measurable function $\varphi\colon X \to [0,\infty)$ with $\|\varphi\|_{\sup} < \infty$, any $u \in \mathcal{F}_e$ and any approximating sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \cap C_c(X)$ for u, $\bigl\{\int_X \varphi \, d\mu^{\#}_{\langle u_n \rangle}\bigr\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $[0,\infty)$, $\lim_{n\to\infty} \int_X \varphi \, d\mu^{\#}_{\langle u_n \rangle}$ does not depend on the choice of $\{u_n\}_n$, and $\int_X \varphi \, d\mu^{\#}_{\langle u \rangle} = \lim_{n\to\infty} \int_X \varphi \, d\mu^{\#}_{\langle u_n \rangle}$, where $\mu^{\#}_{\langle u_n \rangle}$ is the Radon measure on X defined by $\mu^{\#}_{\langle u \rangle}(A) \coloneqq \lim_{n\to\infty} \mu^{\#}_{\langle u_n \rangle}(A)$ for $A \in \mathcal{B}(X)$.

Definition A.10 (Energy measures)**.** Let $u \in \mathcal{F}_e$. Let $\mu_{\langle u \rangle}$ denote the measure in the above proposition in the case $\mathcal{E}^{\#} = \mathcal{E}$. We call $\mu_{\langle u \rangle}$ the *energy measure* of u. For each $w \in \{c, j, k\}$, let $\mu^{w}_{\langle u \rangle}$ denote the measure in the above proposition in the case $\mathcal{E}^{\#} = \mathcal{E}^{(w)}$. For $u, v \in \mathcal{F}_e$, we also define $\mu^{\#}_{\langle u,v \rangle} \coloneqq \frac{1}{4}\bigl(\mu^{\#}_{\langle u+v \rangle} - \mu^{\#}_{\langle u-v \rangle}\bigr)$, where $\mu^{\#}_{\langle \cdot \rangle} \in \bigl\{\mu_{\langle \cdot \rangle}, \mu^{c}_{\langle \cdot \rangle}, \mu^{j}_{\langle \cdot \rangle}, \mu^{k}_{\langle \cdot \rangle}\bigr\}$.

The following lemma is a Fatou-type property for energy measures.

Lemma A.11. *Let $\varphi\colon X \to [0,\infty)$ be a Borel measurable function with $\|\varphi\|_{\sup} < \infty$ and let $\mu^{\#}_{\langle \cdot \rangle} \in \bigl\{\mu_{\langle \cdot \rangle}, \mu^{c}_{\langle \cdot \rangle}, \mu^{j}_{\langle \cdot \rangle}, \mu^{k}_{\langle \cdot \rangle}\bigr\}$. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $u \in \mathcal{F}_e$ satisfy $\lim_{n\to\infty} u_n = u$ m-a.e. and $\sup_{n\in\mathbb{N}} \mathcal{E}(u_n, u_n) < \infty$, then*

$$\int_X \varphi \, d\mu^{\#}_{\langle u \rangle} \le \liminf_{n\to\infty} \int_X \varphi \, d\mu^{\#}_{\langle u_n \rangle}. \tag{A.8}$$

Proof. By extracting a subsequence of $\{u_n\}_n$ if necessary, we can assume that the limit $\lim_{n\to\infty} \int_X \varphi \, d\mu^{\#}_{\langle u_n \rangle}$ exists. By using a version [CF, Theorem A.4.1-(ii)] of the Banach–Saks theorem, we can find a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $\{v_l\}_{l\in\mathbb{N}} \subseteq \mathcal{F}$ defined by $v_l \coloneqq l^{-1}\sum_{k=1}^{l} u_{n_k}$ satisfies $\lim_{k\wedge l\to\infty} \mathcal{E}(v_k - v_l, v_k - v_l) = 0$. Noting that $\lim_{l\to\infty} v_l = u$ m-a.e. and using Proposition A.3, we have $\lim_{l\to\infty} \mathcal{E}(u - v_l, u - v_l) = 0$. Hence $\lim_{l\to\infty} \int_X \varphi \, d\mu^{\#}_{\langle v_l \rangle} = \int_X \varphi \, d\mu^{\#}_{\langle u \rangle}$ by Proposition A.9. By the triangle inequality for $\left(\int_X \varphi \, d\mu^{\#}_{\langle \cdot \rangle}\right)^{1/2}$,

$$\left(\int_X \varphi \, d\mu^{\#}_{\langle v_l \rangle}\right)^{1/2} \le \frac{1}{l}\sum_{k=1}^{l}\left(\int_X \varphi \, d\mu^{\#}_{\langle u_{n_k} \rangle}\right)^{1/2},$$

which implies (A.8) by letting $l \to \infty$. □

Now we can show that the integrals of non-negative bounded Borel measurable functions with respect to energy measures give 2-energy forms satisfying $(\mathrm{GC})_2$.

Proposition A.12. *Let $\varphi\colon X \to [0,\infty)$ be a Borel measurable function with $\|\varphi\|_{\sup} < \infty$ and let $\mu^{\#}_{\langle \cdot \rangle} \in \bigl\{\mu_{\langle \cdot \rangle}, \mu^{c}_{\langle \cdot \rangle}, \mu^{j}_{\langle \cdot \rangle}, \mu^{k}_{\langle \cdot \rangle}\bigr\}$. Then $(\int_X \varphi \, d\mu^{\#}_{\langle \cdot \rangle}, \mathcal{F}_e)$ is a 2-energy form on (X, m) satisfying $(\mathrm{GC})_2$.*

Proof. Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,2]$, $q_2 \in [2,\infty]$ and $T = (T_1, \dots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2). It suffices to prove that for any $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in (\mathcal{F} \cap C_c(X))^{n_1}$ and any $\varphi \in \mathcal{F} \cap C_c(X)$,

$$\left\| \left(\left(\int_X \varphi \, d\mu^{\#}_{\langle T_l(\boldsymbol{u}) \rangle} \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\left(\int_X \varphi \, d\mu^{\#}_{\langle u_k \rangle} \right)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{A.9}$$

Indeed, we can extend (A.9) to any $\boldsymbol{u} \in \mathcal{F}_e^{n_1}$ and any Borel measurable function $\varphi\colon X \to [0,\infty]$ as follows. Let us start with the case of $\varphi = \mathbb{1}_A$, where $A \in \mathcal{B}(X)$. By [Rud, Theorem 2.18], there exist sequences $\{K_n\}_{n\in\mathbb{N}}$ and $\{U_n\}_{n\in\mathbb{N}}$ such that $K_n \subseteq A \subseteq U_n$, K_n is compact, U_n is open and $\lim_{n\to\infty} \max_{v \in \{T_l(\boldsymbol{u})\}_l \cup \{u_k\}_k} \mu^{\#}_{\langle v \rangle}(U_n \setminus K_n) = 0$. By Urysohn's lemma, we can pick $\varphi_n \in C_c(X)$ so that $0 \le \varphi_n \le 1$, $\varphi_n|_{K_n} = 1$ and $\operatorname{supp}_X[\varphi_n] \subseteq U_n$. By (A.9) with φ_n in place of φ, we obtain $\left\| \left(\mu^{\#}_{\langle T_l(\boldsymbol{u}) \rangle}(K_n)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\mu^{\#}_{\langle u_k \rangle}(U_n)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}$. By letting $n \to \infty$, we get (A.9) with $\varphi = \mathbb{1}_A$, i.e.,

$$\left\| \left(\mu^{\#}_{\langle T_l(\boldsymbol{u}) \rangle}(A)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\mu^{\#}_{\langle u_k \rangle}(A)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{A.10}$$

By the reverse Minkowski inequality on $\ell^{q_1/2}$ and the Minkowski inequality on $\ell^{q_2/2}$ (see also (2.19)), we can extend (A.10) to (A.9) for any non-negative Borel measurable simple function φ on X, By the monotone convergence theorem, (A.9) holds for any Borel measurable function $\varphi\colon X \to [0,\infty]$. Next we will extend (A.9) to $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}_e^{n_1}$. Since $\mathcal{F} \cap C_c(X)$ is dense in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$, there exists an approximating sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \cap C_c(X)$ for u_k for each $k \in \{1, \dots, n_1\}$. Set $\boldsymbol{u}_n := (u_{1,n}, \dots, u_{n_1,n})$. Then, for each $l \in \{1, \dots, n_2\}$, $\lim_{n\to\infty} T_l(\boldsymbol{u}_n) = T_l(\boldsymbol{u})$ m-a.e., $T_l(\boldsymbol{u}_n) \in \mathcal{F}$ and $\sup_{n\in\mathbb{N}} \mathcal{E}(T_l(\boldsymbol{u}_n), T_l(\boldsymbol{u}_n)) < \infty$ by Proposition A.2. Hence $T_l(\boldsymbol{u}) \in \mathcal{F}_e$ by Proposition A.5, and

$$\begin{aligned}
\left\| \left(\left(\int_X \varphi \, d\mu^{\#}_{\langle T_l(\boldsymbol{u}) \rangle} \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} &\le \left\| \left(\left(\liminf_{n\to\infty} \int_X \varphi \, d\mu^{\#}_{\langle T_l(\boldsymbol{u}_n) \rangle} \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\
&\le \liminf_{n\to\infty} \left\| \left(\left(\int_X \varphi \, d\mu^{\#}_{\langle T_l(\boldsymbol{u}_n) \rangle} \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\
&\overset{\text{(A.9)}}{\le} \liminf_{n\to\infty} \left\| \left(\left(\int_X \varphi \, d\mu^{\#}_{\langle u_{k,n} \rangle} \right)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} \\
&= \left\| \left(\left(\int_X \varphi \, d\mu^{\#}_{\langle u_k \rangle} \right)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}},
\end{aligned}$$

where we used Lemma A.11 in the first inequality and Proposition A.9 in the last equality. This implies that $(\int_X \varphi \, d\mu^{\#}_{\langle \cdot \rangle}, \mathcal{F}_e)$ is a 2-energy form on (X, m) satisfying (GC)$_2$.

Let us go back to the proof of (A.9) in the case where $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in (\mathcal{F} \cap C_c(X))^{n_1}$ and $\varphi \in \mathcal{F} \cap C_c(X)$. Fix a metric d on X which is compatible with the given topology of X, an increasing sequence of relatively open sets $\{G_l\}_{l\in\mathbb{N}}$ with $\bigcup_{l\in\mathbb{N}} G_l = X$ and a sequence of positive numbers $\{\delta_l\}_{l\in\mathbb{N}}$ with $\delta_l \downarrow 0$ as $l \to \infty$. Then there exist a sequence of positive numbers $\{\beta_n\}_{n\in\mathbb{N}}$ with $\beta_n \uparrow \infty$ as $n \to \infty$, a family of Radon measures $\{\sigma_\beta\}_{\beta>0}$ on $X \times X$ and a family of Radon measures $\{m_\beta\}_{\beta>0}$ on X with $m_\beta \ll m$ such that for any $v \in \mathcal{F} \cap C_c(X)$,

$$\int_X \varphi\, d\mu_{\langle v\rangle} = \lim_{\beta\to\infty}\left(\frac{\beta}{2}\int_{X\times X}|v(x)-v(y)|^2\,\varphi(x)\,\sigma_\beta(dx,dy) + \frac{\beta}{2}\int_X |v(x)|^2\,\varphi(x)\,m_\beta(dx)\right), \tag{A.11}$$

and

$$\int_X \varphi\, d\mu^c_{\langle v\rangle} = \lim_{l\to\infty}\lim_{n\to\infty}\frac{\beta_n}{2}\int_{\{(x,y)\in G_l\times G_l \mid d(x,y)<\delta_l\}}|v(x)-v(y)|^2\,\varphi(x)\,\sigma_{\beta_n}(dx,dy). \tag{A.12}$$

See [FOT, the equations just before (3.2.13) and (3.2.19)] for details. Note that $T_l(\boldsymbol{u}) \in \mathcal{F} \cap C_c(X)$ for each $l \in \{1,\dots,n_2\}$ by Proposition A.2 and $T_l(0) = 0$. If $q_2 < \infty$, then we have from (A.11) that

$$\begin{aligned}
&\sum_{l=1}^{n_2}\left(\int_X \varphi\, d\mu_{\langle T_l(\boldsymbol{u})\rangle}\right)^{q_2/2} \\
&\overset{(2.18)}{\le} \lim_{\beta\to\infty}\left(\frac{\beta}{2}\int_{X\times X}\|T(\boldsymbol{u}(x))-T(\boldsymbol{u}(y))\|_{\ell^{q_2}}^2\,\varphi(x)\,\sigma_\beta(dx,dy)\right. \\
&\qquad\qquad\qquad\qquad \left. + \frac{\beta}{2}\int_X \|T(\boldsymbol{u}(x))\|_{\ell^{q_2}}^2\,\varphi(x)\,m_\beta(dx)\right)^{q_2/2} \\
&\overset{(2.2)}{\le} \lim_{\beta\to\infty}\left(\frac{\beta}{2}\int_{X\times X}\|\boldsymbol{u}(x)-\boldsymbol{u}(y)\|_{\ell^{q_1}}^2\,\varphi(x)\,\sigma_\beta(dx,dy) + \frac{\beta}{2}\int_X \|\boldsymbol{u}(x)\|_{\ell^{q_1}}^2\,\varphi(x)\,m_\beta(dx)\right)^{q_2/2} \\
&\overset{(*)}{\le} \lim_{\beta\to\infty}\left(\sum_{k=1}^{n_1}\left[\frac{\beta}{2}\int_{X\times X}|u_k(x)-u_k(y)|^2\,\varphi(x)\,\sigma_\beta(dx,dy)\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. + \frac{\beta}{2}\int_X |u_k(x)|^2\,\varphi(x)\,m_\beta(dx)\right]^{q_1/2}\right)^{\frac{2}{q_1}\cdot\frac{q_2}{2}} \\
&= \left(\sum_{k=1}^{n_1}\left(\int_X \varphi\, d\mu_{\langle u_k\rangle}\right)^{q_1/2}\right)^{q_2/q_1},
\end{aligned}$$

where we used the triangle inequality for a suitable L^{2/q_1}-norm on $(X\times X)\sqcup X$ in $(*)$ (here $\sqcup$ denotes the disjoint union). The case of $q_2 = \infty$ is similar, so we obtain the desired estimate (A.9) for $\mu^{\#}_{\langle\cdot\rangle} = \mu_{\langle\cdot\rangle}$. The other case $\mu^{\#}_{\langle\cdot\rangle} \in \{\mu^c_{\langle\cdot\rangle}, \mu^j_{\langle\cdot\rangle}, \mu^k_{\langle\cdot\rangle}\}$ can be shown in a similar way by virtue of the expression in [FOT, (3.2.23)]. □

Next we see that "$|\nabla u|$" also satisfies a similar contraction property. To present the precise definition of the density, we recall the notion of *minimal energy dominant measure*.

Definition A.13 (Minimal energy dominant measure; [Hin10, Definition 2.1])**.** A σ-finite Borel measure μ on X is called a *minimal energy-dominant measure* of $(\mathcal{E},\mathcal{F})$ if and only if the following two conditions hold.

(i) For any $f \in \mathcal{F}$, we have $\mu_{\langle f \rangle} \ll \mu$.
(ii) If another σ-finite Borel measure μ' on X satisfies (i) with μ in place of μ', then $\mu \ll \mu'$.

The existence of a minimal energy-dominant measure is proved in [Nak85, Lemma 2.2] (see also [Hin10, Lemma 2.3]). For any minimal energy-dominant measure μ of $(\mathcal{E}, \mathcal{F})$, the same argument as in [Hin10, Proof of Lemma 2.2] implies that $\mu_{\langle f \rangle} \ll \mu$ for any $f \in \mathcal{F}_e$. In addition, for $\mu^{\#}_{\langle\,\cdot\,\rangle} \in \{\mu_{\langle\,\cdot\,\rangle}, \mu^c_{\langle\,\cdot\,\rangle}, \mu^j_{\langle\,\cdot\,\rangle}, \mu^k_{\langle\,\cdot\,\rangle}\}$, we easily see that $\mu^{\#}_{\langle f,g \rangle} \ll \mu$ for any $f, g \in \mathcal{F}_e$. We define $\Gamma^{\#}_{\mu}(u,v) := \frac{d\mu^{\#}_{\langle u,v \rangle}}{d\mu}$ and $\Gamma^{\#}_{\mu}(u) := \Gamma^{\#}_{\mu}(u,u)$ for $u, v \in \mathcal{F}_e$.

Proposition A.14. *Let μ be a minimal energy-dominant measure of $(\mathcal{E}, \mathcal{F})$ and for each $f \in \mathcal{F}_e$, let $\Gamma_{\mu}(f) := d\mu_{\langle f \rangle}/d\mu$ and $\Gamma^w_{\mu}(f) := d\mu^w_{\langle f \rangle}/d\mu$ for each $w \in \{c, j, k\}$. Let $\Gamma^{\#}_{\mu}(\,\cdot\,) \in \{\Gamma_{\mu}(\,\cdot\,), \Gamma^c_{\mu}(\,\cdot\,), \Gamma^j_{\mu}(\,\cdot\,), \Gamma^k_{\mu}(\,\cdot\,)\}$. Then for any $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,2]$, $q_2 \in [2,\infty]$ and $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.2) and any $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}_e^{n_1}$,*

$$\left\| \bigl(\Gamma^{\#}_{\mu}(T_l(\boldsymbol{u}))(x)^{1/2} \bigr)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \bigl(\Gamma^{\#}_{\mu}(u_k)(x)^{1/2} \bigr)_{k=1}^{n_1} \right\|_{\ell^{q_1}} \quad \text{for } \mu\text{-a.e. } x \in X, \tag{A.13}$$

and in particular, for any $p \in [q_1, q_2] \cap (0,\infty)$ and any Borel measurable function $\varphi \colon X \to [0,\infty]$,

$$\left\| \left(\left(\int_X \varphi \Gamma^{\#}_{\mu}(T_l(\boldsymbol{u}))^{\frac{p}{2}}\, d\mu \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\left(\int_X \varphi \Gamma^{\#}_{\mu}(u_k)^{\frac{p}{2}}\, d\mu \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{A.14}$$

Proof. We first construct a good μ-version of $\Gamma^{\#}_{\mu}(v)$ for each $v \in \mathcal{F}_e$. Fix $\{X_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(X)$ such that $X_n \subseteq X_{n+1}$, $X = \bigcup_{n\in\mathbb{N}} X_n$ and $\mu(X_n) \in (0,\infty)$ for each $n \in \mathbb{N}$. Let $\{A_k\}_{k\in\mathbb{N}}$ be a countable open base for the topology of X. Set $A^0_k := X \setminus A_k$ and $A^1_k := A_k$ for each $k \in \mathbb{N}$, and define a non-decreasing sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of σ-algebras in X in the same way as (4.29). For $v \in \mathcal{F}_e$, $n, k \in \mathbb{N}$, $\alpha \in \{0,1\}^k$, define $\Gamma^{\#}_{\mu}(v)_{n,k} \colon X \to [0,\infty)$ by, for $x \in A^{\alpha}_k$,

$$\Gamma^{\#}_{\mu}(v)_{n,k}(x) := \begin{cases} \mu(A^{\alpha}_k \cap X_n)^{-1} \mu^{\#}_{\langle v \rangle}(A^{\alpha}_k \cap X_n) & \text{if } \mu(A^{\alpha}_k \cap X_n) > 0, \\ 0 & \text{if } \mu(A^{\alpha}_k \cap X_n) = 0. \end{cases} \tag{A.15}$$

We also set $\mu_n := \mu(X_n)^{-1}\mu((\cdot) \cap X_n)$ and $v^{\#}_n := \frac{d\mu^{\#}_{\langle v \rangle}((\cdot)\cap X_n)}{\mu((\cdot)\cap X_n)}$. Then we easily see that $\mathbb{E}_{\mu_n}[v^{\#}_n \,|\, \mathcal{A}_k] = \Gamma^{\#}_{\mu}(v)_{n,k}$ μ-a.e. on X_n and hence $\lim_{k\to\infty} \Gamma^{\#}_{\mu}(v)_{n,k} = v^{\#}_n$ μ-a.e. on X_n by the martingale convergence theorem (see, e.g., [Dud, Theorem 10.5.1]) and the fact that $\bigcup_{k\in\mathbb{N}} \mathcal{A}_k$ generates $\mathcal{B}(X)$. Now we define $\widetilde{\Gamma}^{\#}_{\mu}(v) \colon X \to [0,\infty)$ by $\widetilde{\Gamma}^{\#}_{\mu}(v)(x) := v^{\#}_n(x)$ for $n \in \mathbb{N}$ and $x \in X_n \setminus X_{n-1}$, where $X_0 := \emptyset$. Then $\widetilde{\Gamma}^{\#}_{\mu}(v) = \Gamma^{\#}_{\mu}(v)$ μ-a.e. on X.

Next we show (A.13). Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,2]$, $q_2 \in [2,\infty]$, $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}_e^{n_1}$ and let $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2) with 2 in place of p. By Proposition A.12 and (A.15), for any $n, m \in \mathbb{N}$ and any $x \in X$,

$$\left\| \bigl(\Gamma^{\#}_{\mu}(T_l(\boldsymbol{u}))_{n,m}(x)^{1/2} \bigr)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \bigl(\Gamma^{\#}_{\mu}(u_k)_{n,m}(x)^{1/2} \bigr)_{k=1}^{n_1} \right\|_{\ell^{q_1}}.$$

By letting $m \to \infty$, we obtain

$$\left\| \left(\widetilde{\Gamma}^{\#}_{\mu}(T_l(\boldsymbol{u}))(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left(\widetilde{\Gamma}^{\#}_{\mu}(u_k)(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} \quad \text{for } \mu\text{-a.e. } x \in X,$$

whence (A.13) holds. Lastly, if $p \in [q_1, q_2] \cap (0, \infty)$ and $q_2 < \infty$, then we see that for any Borel measurable function $\varphi \colon X \to [0, \infty]$,

$$\begin{aligned} \sum_{l=1}^{n_2} \left(\int_X \varphi \Gamma^{\#}_{\mu}(T_l(\boldsymbol{u}))^{\frac{p}{2}} \, d\mu \right)^{q_2/p} &\overset{(2.18)}{\leq} \left(\int_X \varphi \left\| \left(\Gamma^{\#}_{\mu}(T_l(\boldsymbol{u}))(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}}^{p} \mu(dx) \right)^{q_2/p} \\ &\overset{(A.13)}{\leq} \left(\int_X \varphi \left\| \left(\Gamma^{\#}_{\mu}(u_k)(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}^{p} \mu(dx) \right)^{q_2/p} \\ &\overset{(*)}{\leq} \left(\sum_{k=1}^{n_1} \left(\int_X \varphi \Gamma^{\#}_{\mu}(u_k)^{\frac{p}{2}} \, d\mu \right)^{q_1/p} \right)^{q_2/q_1}, \end{aligned} \tag{A.16}$$

where we used the triangle inequality for the norm of $L^{p/q_1}(X, \varphi \, d\mu)$ in $(*)$. The case of $q_2 = \infty$ is similar, so we obtain (A.14). □

If $(\mathcal{E}, \mathcal{F})$ is strongly local, then we can show $(\mathrm{GC})_p$ for $(\Gamma_\mu(\,\cdot\,)^{p/2}, \mathcal{F}_e)$. To prove it, we need some preparations. The following proposition is the standard Minkowski integral inequality (see, e.g., [DF, Appendix B5]).

Proposition A.15. *Let $(X_i, \mathcal{B}_i, m_i)$ be a σ-finite measure space for each $i \in \{1, 2\}$. Let $q \in [1, \infty)$ and let $f \colon X_1 \times X_2 \to [-\infty, \infty]$ be measurable with respect to the product σ-algebra of $\mathcal{B}_1$ and $\mathcal{B}_2$. Then*

$$\left(\int_{X_1} \left(\int_{X_2} |f(x_1, x_2)| \, m_2(dx_2) \right)^q m_1(dx_1) \right)^{\frac{1}{q}} \leq \int_{X_2} \left(\int_{X_1} |f(x_1, x_2)|^q \, m_1(dx_1) \right)^{\frac{1}{q}} m_2(dx_2). \tag{A.17}$$

Next we show a tensor-type inequality for non-negative definite symmetric bilinear forms.

Proposition A.16. *Let V be a finite-dimensional vector space over $\mathbb{R}$, let $E \colon V \times V \to \mathbb{R}$ be a non-negative definite symmetric bilinear form, let $n_1, n_2 \in \mathbb{N}$ and let $A = (A_{lk})_{1 \leq l \leq n_2, 1 \leq k \leq n_1}$ be a real matrix. If $(u_1, \dots, u_{n_1}) \in V^{n_1}$, $q_1 \in (0, \infty)$, $q_2 \in (0, \infty]$ and $q_1 \leq q_2$, then*

$$\left\| \left(E\left(\sum_{k=1}^{n_1} A_{lk} u_k \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \|A\|_{\ell^{q_1}_{n_1} \to \ell^{q_2}_{n_2}} \left\| \left(E(u_k)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}, \tag{A.18}$$

where we set $E(u) := E(u, u)$ for $u \in V$ and $\|A\|_{\ell^{q_1}_{n_1} \to \ell^{q_2}_{n_2}} := \sup_{x \in \mathbb{R}^{n_1}, \|x\|_{\ell^{q_1}} \leq 1} \|Ax\|_{\ell^{q_2}}$.

Proof. The desired inequality follows from a Beckner-like result in [DF, 7.9.] (see also [Bec75, Lemma 2]). We present a complete proof for convenience. Let γ_n be the Gaussian measure on $\mathbb{R}^n$, i.e., $\gamma_n(dx) \coloneqq (2\pi)^{-n/2}\exp\bigl(-\|x\|^2/2\bigr)\,dx$, for each $n \in \mathbb{N}$ and set $n \coloneqq \dim(V/E^{-1}(0)) \in \mathbb{N}\cup\{0\}$. If $n = 0$, i.e., $E(u) = 0$ for any $u \in V$, then (A.18) is clear. Hence we assume that $n \geq 1$ in the rest of the proof. Let $\pi_j \colon \mathbb{R}^n \to \mathbb{R}$ be the projection map to the j-th coordinate for each $j \in \{1,\dots,n\}$. Then we have from [DF, Proposition in 8.7.] that for any $(\alpha_j)_{j=1}^n \in \mathbb{R}^n$,

$$\|\pi_1\|_{L^{q_1}(\mathbb{R},\gamma_1)}^{-1}\left(\int_{\mathbb{R}^n}\left|\sum_{j=1}^n \alpha_j\pi_j(x)\right|^{q_1} d\gamma_n(dx)\right)^{1/q_1} = \bigl\|(\alpha_j)_{j=1}^n\bigr\|_{\ell^2}. \tag{A.19}$$

Indeed, (A.19) is obviously true in the case of $(\alpha_j)_j = (\delta_{1j})_j$ and this together with the invariance of γ_n under ℓ_n^2-isometries implies the desired equality (A.19).

Let us fix a basis $\{e_j\}_{j=1}^n \subseteq V$ of V satisfying $E(e_j, e_{j'}) = \delta_{jj'}$ for each $j, j' \in \{1,\dots,n\}$, which exists by the Gram–Schmidt orthonormalization. Now we define $\iota \colon V \to L^{q_1}(\mathbb{R}^n,\gamma_n)$ by

$$\iota(u) \coloneqq \|\pi_1\|_{L^{q_1}(\mathbb{R},\gamma_1)}^{-1}\sum_{j=1}^n E(u,e_j)^{1/2}\pi_j, \quad u \in V. \tag{A.20}$$

Then $\|\iota(u)\|_{L^{q_1}(\mathbb{R}^n,\gamma_n)} = \bigl(\sum_{j=1}^n E(u,e_j)\bigr)^{1/2} = E(u,u)^{1/2}$ by (A.19). If $q_2 < \infty$, then

$$\begin{aligned}
\left\|\left(E\left(\sum_{k=1}^{n_1} A_{lk}u_k\right)^{1/2}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}} &= \left(\sum_{l=1}^{n_2}\left(\int_{\mathbb{R}^n}\left|\sum_{k=1}^{n_1} A_{lk}\iota(u_k)\right|^{q_1} d\gamma_n\right)^{q_2/q_1}\right)^{\frac{q_1}{q_2}\cdot\frac{1}{q_1}}\\
&\overset{(*)}{\leq} \left(\int_{\mathbb{R}^n}\left(\sum_{l=1}^{n_2}\left|\sum_{k=1}^{n_1} A_{lk}\iota(u_k)\right|^{q_2}\right)^{q_1/q_2} d\gamma_n\right)^{1/q_1}\\
&\leq \|A\|_{\ell_{n_1}^{q_1}\to\ell_{n_2}^{q_2}}\left(\int_{\mathbb{R}^n}\sum_{k=1}^{n_1}|\iota(u_k)|^{q_1}\,d\gamma_n\right)^{1/q_1}\\
&= \|A\|_{\ell_{n_1}^{q_1}\to\ell_{n_2}^{q_2}}\left(\sum_{k=1}^{n_1} E(u_k)^{q_1/2}\right)^{1/q_1},
\end{aligned}$$

where we used (A.17) with $q = q_1/q_2$ in $(*)$. Since the case of $q_2 = \infty$ is similar, so we obtain (A.18). □

Let us recall the definition of p-energy forms introduced by Kuwae in [Kuw24]

Definition A.17 ([Kuw24, Definition 1.4])**.** Let μ be a minimal energy-dominant measure of $(\mathcal{E},\mathcal{F})$, $p \in (1,\infty)$ and $\mathscr{D} \subseteq \{u \in \mathcal{F}\cap L^p(X,m) \mid \Gamma_\mu(u)^{\frac{1}{2}} \in L^p(X,\mu)\}$ a linear subspace. Assume that $(\mathcal{E},\mathcal{F})$ is strongly local and that

$$\begin{array}{l}\lim_{n\to\infty}\int_X \Gamma_\mu(u_n)^{\frac{p}{2}}\,d\mu = 0 \text{ for any } \{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{D} \text{ satisfying}\\ \lim_{n\wedge k\to\infty}\int_X \Gamma_\mu(u_n-u_k)^{\frac{p}{2}}\,d\mu = 0 \text{ and } \lim_{n\to\infty}\|u_n\|_{L^p(X,m)} = 0.\end{array} \tag{A.21}$$

We define the norm $\|\cdot\|_{H^{1,p}}$ on $\mathscr{D}$ by $\|u\|_{H^{1,p}} := \bigl(\|u\|_{L^p(X,m)}^p + \int_X \Gamma_\mu(u)^{\frac{p}{2}}\,d\mu\bigr)^{1/p}$, and let $(H^{1,p}(X), \|\cdot\|_{H^{1,p}})$ denote the completion of $(\mathscr{D}, \|\cdot\|_{H^{1,p}})$, so that we may and do consider $H^{1,p}(X)$ as a linear subspace of $L^p(X,m)$ since the canonical bounded linear map from $H^{1,p}(X)$ to $L^p(X,m)$ extending $\mathrm{id}_{\mathscr{D}}$ is injective by (A.21). Then we can uniquely extend Γ_μ to $H^{1,p}(X)$ by defining $\Gamma_\mu(u)^{\frac{1}{2}} \in L^p(X,\mu)$ for $u \in H^{1,p}(X)$ as the $L^p(X,\mu)$-limit of $\Gamma_\mu(u_n)^{\frac{1}{2}}$, where $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{D}$ satisfies $\lim_{n\wedge k\to\infty}\int_X \Gamma_\mu(u_n - u_k)^{\frac{p}{2}}\,d\mu = 0$ and $\lim_{n\to\infty}\|u - u_n\|_{L^p(X,m)} = 0$.

Remark A.18. The condition (A.21) always holds if $p \geq 2$ and $\mu(F_n) < \infty$ for any $n \in \mathbb{N}$ for some $\mathcal{E}$-nest $\{F_n\}_{n\in\mathbb{N}}$[16] as proved in [Kuw24, Proposition 1.1]; the latter condition[17] on μ is not assumed there, but is necessary for [Kuw24, Proof of Proposition 1.1] to make sense.

Now we can show the main result in this subsection.

Theorem A.19. *Let μ be a minimal energy-dominant measure of $(\mathcal{E},\mathcal{F})$, $p \in (1,\infty)$ and $\mathscr{D} \subseteq \{u \in \mathcal{F}\cap L^p(X,m) \mid \Gamma_\mu(u)^{\frac{1}{2}} \in L^p(X,\mu)\}$ a linear subspace. Assume that $(\mathcal{E},\mathcal{F})$ is strongly local and that (A.21) holds. In addition, we assume that*

$$\begin{aligned}&\widehat{T}(u) \in \mathscr{D} \text{ for any } u \in \mathscr{D}^n \text{ and any } \widehat{T} \in C^\infty(\mathbb{R}^n) \text{ satisfying}\\ &\sup_{x,y\in\mathbb{R}^n;x\neq y}\frac{\bigl|\widehat{T}(x)-\widehat{T}(y)\bigr|}{\|x-y\|} < \infty \text{ and } \widehat{T}(0) = 0.\end{aligned} \tag{A.22}$$

If $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.2) and $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in H^{1,p}(X)^{n_1}$, then $T(\boldsymbol{u}) \in H^{1,p}(X)^{n_2}$ and

$$\Bigl\|\bigl(\Gamma_\mu(T_l(\boldsymbol{u}))(x)^{1/2}\bigr)_{l=1}^{n_2}\Bigr\|_{\ell^{q_2}} \leq \Bigl\|\bigl(\Gamma_\mu(u_k)(x)^{1/2}\bigr)_{k=1}^{n_1}\Bigr\|_{\ell^{q_1}} \quad \text{for } \mu\text{-a.e. } x \in X. \tag{A.23}$$

In particular, $\bigl\{\Gamma_\mu(u)^{\frac{p}{2}}\,d\mu\bigr\}_{u\in H^{1,p}(X)}$ is a family of p-energy measures on $(X,\mathcal{B}(X))$ dominated by $\bigl(\int_X \Gamma_\mu(\,\cdot\,)^{\frac{p}{2}}\,d\mu, H^{1,p}(X)\bigr)$ and satisfies (GC)$_p$.

Proof. Let us consider the same mollifiers as in [Kuw24, the last paragraph in p. 10], i.e., define $j\colon \mathbb{R}^{n_1} \to \mathbb{R}$ by $j(x) := \exp\bigl(-\frac{1}{1-\|x\|^2}\bigr)$ for $\|x\| \leq 1$ and $j(x) := 0$ for $\|x\| > 1$, set $j_m(x) := m^{n_1} j(mx)$ for each $m \in \mathbb{N}$. We define $T_{l,n}(x) := \int_{\mathbb{R}^{n_1}}(j_n(x-y) - j_n(y))T_l(y)\,dy = \int_{\mathbb{R}^{n_1}} j_n(y)(T_{l,n}(x-y) - T_{l,n}(y))\,dy$ so that $T_{l,n} \in C^\infty(\mathbb{R}^{n_1})$, $T_{l,n}(0) = 0$ and $\lim_{n\to\infty} T_{l,n}(x) =$

[16] Namely, a non-decreasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed subsets of X such that $\bigcup_{n\in\mathbb{N}}\mathcal{F}_{F_n}$ is dense in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$, where $\mathcal{F}_{F_n} := \{u \in \mathcal{F} \mid u = 0\ m\text{-a.e. on } X \setminus F_n\}$; see, e.g., [CF, Definition 1.2.12-(i) and Theorem 1.3.14-(ii)].

[17] Note that a minimal energy-dominant measure μ of $(\mathcal{E},\mathcal{F})$ does not satisfy this condition in general. Indeed, consider the case where $X = \mathbb{R}$, m is the Lebesgue measure on $\mathbb{R}$ and $(\mathcal{E},\mathcal{F})$ is the Dirichlet form of the Brownian motion on $\mathbb{R}$, and let μ be a Borel measure on $\mathbb{R}$. Then it is easy to see from [Kig12, Theorem 9.9] that μ satisfies the condition in Remark A.18 if and only if μ is a Radon measure on $\mathbb{R}$. On the other hand, since $\mathcal{F} = W^{1,2}(\mathbb{R})$ and $d\mu_{\langle u\rangle} = |u'|^2\,dm$ for any $u \in W^{1,2}(\mathbb{R})$, it is clear that μ is a minimal energy-dominant measure of $(\mathcal{E},\mathcal{F})$ if and only if μ is σ-finite and satisfies $\mu \ll m$ and $m \ll \mu$. Of course, the latter class of μ contains plenty of measures which are not Radon measures on $\mathbb{R}$ and thereby are minimal energy-dominant measures of $(\mathcal{E},\mathcal{F})$ failing to satisfy the condition in Remark A.18.

$T_l(x)$ for any $x \in \mathbb{R}^{n_1}$. Then (2.2) with $T^{(n)} \coloneqq (T_{1,n}, \ldots, T_{n_2,n})$ in place of T holds; indeed, for any $x, y \in \mathbb{R}^{n_1}$,

$$\begin{aligned}\left\|T^{(n)}(x) - T^{(n)}(y)\right\|_{\ell^{q_2}} &= \left\| \left(\int_{\mathbb{R}^{n_1}} j_n(z)(T_l(x-z) - T_l(y-z))\,dz \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\ &\overset{(*)}{\leq} \int_{\mathbb{R}^{n_1}} j_n(z) \left\|T(x-z) - T(y-z)\right\|_{\ell^{q_2}}\,dz \\ &\overset{(2.2)}{\leq} \|x-y\|_{\ell^{q_1}} \int_{\mathbb{R}^{n_1}} j_n(z)\,dz = \|x-y\|_{\ell^{q_1}}, \end{aligned} \tag{A.24}$$

where we used (A.17) with $q = q_2$ in $(*)$. Moreover,

$$\left\| \left(\sum_{k=1}^{n_1} \partial_k T_{l,n}(x) y_k \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left\|T^{(n)}(x) - T^{(n)}(x+\varepsilon y)\right\|_{\ell^{q_2}} \overset{(A.24)}{\leq} \|y\|_{\ell^{q_1}}, \tag{A.25}$$

whence $\|(\partial_k T_{l,n}(x))\|_{\ell^{q_1}_{n_1} \to \ell^{q_2}_{n_2}} \leq 1$ for any $x \in \mathbb{R}^{n_1}$.

We first prove (A.23) with $T^{(n)}$ in place of T under the assumption that $\boldsymbol{u} = (u_1, \ldots, u_{n_1}) \in \mathscr{D}^{n_1}$. Set $\widetilde{\boldsymbol{u}} = (\widetilde{u}_1, \ldots, \widetilde{u}_{n_1})$ where $\widetilde{u}_k$ is a $\mathcal{E}$-quasicontinuous m-version of u_k (see [FOT, p. 69 and Theorem 2.1.3]). We have $T_{l,n}(\boldsymbol{u}) \in \mathscr{D}$ by (A.22) and

$$\Gamma_\mu(T_{l,n}(\boldsymbol{u}))(x) = \sum_{i,j=1}^{n_1} \partial_i T_{l,n}(\widetilde{\boldsymbol{u}}(x)) \partial_j T_{l,n}(\widetilde{\boldsymbol{u}}(x)) \Gamma_\mu(u_i, u_j)(x) \quad \text{for } \mu\text{-a.e. } x \in X \tag{A.26}$$

by the chain rule in [Kuw24, (7) in p. 2]. Let $\{f_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{F}$ be an algebraic basis of $\mathcal{F}$ over $\mathbb{R}$. Then there exist $n \in \mathbb{N}$, $\{\alpha_{k,j}\}_{j=1}^n \subseteq \mathbb{R}$, $k \in \{1, \ldots, n_1\}$, and $\{g_j\}_{j=1}^n \subseteq \{f_\lambda\}_{\lambda \in \Lambda}$ such that $u_k = \sum_{j=1}^n \alpha_{k,j} g_j$ for each $k \in \{1, \ldots, n_1\}$. Let R be the finitely generated algebra over $\mathbb{Q}$ generated by $\{\alpha_{k,j}\}_{1 \leq j \leq n, 1 \leq k \leq n_1} \cup \{1\}$ so that $\mathbb{Q} \subseteq R$ and R is countable. We set

$$\mathcal{U} \coloneqq \left\{ \sum_{j=1}^n a_j g_j \;\middle|\; a_j \in R \text{ for each } j \in \{1, \ldots, n\} \right\}$$

so that $\{u_k\}_{k=1}^{n_1} \subseteq \mathcal{U}$ and $\mathcal{U}$ is countable. Since R is dense in $\mathbb{R}$, for any $x \in X$, $N \in \mathbb{N}$, $k \in \{1, \ldots, n_1\}$ and $l \in \{1, \ldots, n_2\}$, there exists $A^{x,N}_{lk,n} \in R$ such that $\left| \partial_k T_{l,n}(\widetilde{\boldsymbol{u}}(x)) - A^{x,N}_{lk,n} \right| \leq N^{-1}$. Note that $\Gamma_\mu(\,\cdot\,,\,\cdot\,)(x) \colon \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ is a non-negative definite symmetric bilinear form for μ-a.e. $x \in X$ since $\mathcal{U}$ is countable. By Proposition A.16, for μ-a.e. $x \in X$,

$$\begin{aligned} &\left\| \left(\left(\sum_{i,j=1}^{n_1} A^{x,N}_{li,n} A^{x,N}_{lj,n} \Gamma_\mu(u_i, u_j)(x) \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\ &= \left\| \left(\Gamma_\mu \left(\sum_{k=1}^{n_1} A^{x,N}_{lk,n} u_k \right)(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\ &\leq \left(1 + \left\| (\partial_k T_{l,n}(\widetilde{\boldsymbol{u}}(x)))_{l,k} - (A^{x,N}_{lk,n})_{l,k} \right\|_{\ell^{q_1}_{n_1} \to \ell^{q_2}_{n_2}} \right) \left\| \left(\Gamma_\mu(u_k)(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \end{aligned}$$

Letting $N \to \infty$ in the estimate above and recalling (A.26), we obtain

$$\left\| \left(\Gamma_\mu(T_{l,n}(\boldsymbol{u}))(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left(\Gamma_\mu(u_k)(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} \quad \text{for } \mu\text{-a.e. } x \in X, \tag{A.27}$$

under the assumption that $\boldsymbol{u} \in \mathscr{D}^{n_1}$.

Next let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in H^{1,p}(X)^{n_1}$ and fix $\{\boldsymbol{u}^{(n)} = (u_{1,n}, \dots, u_{n_1,n})\}_{n \in \mathbb{N}} \subseteq \mathscr{D}^{n_1}$ so that $\lim_{n\to\infty} \max_{k \in \{1,\dots,n_1\}} \|u_k - u_{k,n}\|_{H^{1,p}} = 0$. Then (A.27) together with the same argument as in (A.16) implies that

$$\left\| \left(\left(\int_X \Gamma_\mu(T_{l,n}(\boldsymbol{u}^{(n)}))^{\frac{p}{2}} \, d\mu \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left(\left(\int_X \Gamma_\mu(u_{k,n})^{\frac{p}{2}} \, d\mu \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}.$$

In particular, $\{T_{l,n}(\boldsymbol{u}^{(n)})\}_{n\in\mathbb{N}}$ is bounded in $H^{1,p}(X)$. Noting that $H^{1,p}(X)$ is reflexive ([Kuw24, Theorem 1.7]) and that $\lim_{n\to\infty} \int_X \Gamma_\mu(u_k - u_{k,n})^{\frac{p}{2}} \, d\mu = 0$, we find $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ with $\inf_{j\in\mathbb{N}}(n_{j+1} - n_j) \geq 1$ such that $T^{(n_j)}(\boldsymbol{u}^{(n_j)})$ converges weakly in $H^{1,p}(X)^{\oplus n_2}$ to some $v = (v_1, \dots, v_{n_2}) \in H^{1,p}(X)^{\oplus n_2}$ and $\max_{k\in\{1,\dots,n_1\}} \Gamma_\mu(u_k - u_{k,n_j})(x) \to 0$ for μ-a.e. $x \in X$ as $j \to \infty$.[18] Since $\lim_{n\to\infty} \left\| T_{l,n}(\boldsymbol{u}^{(n)}) - T_l(\boldsymbol{u}) \right\|_{L^p(X,m)} = 0$ by (A.24) and the dominated convergence theorem, we have $v_l = T_l(\boldsymbol{u})$. By Mazur's lemma (Lemma 3.14), there exist $\{N(i)\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ and $\{\alpha_j\} \subseteq [0,1]$ with $\inf_{i\in\mathbb{N}}(N(i) - i) \geq 1$ and $\sum_{j=i}^{N(i)} \alpha_{i,j} = 1$ such that $\widehat{v}_{l,i} \coloneqq \sum_{j=i}^{N(i)} \alpha_{i,j} T_{l,n_j}(\boldsymbol{u}^{(n_j)})$ converges strongly in $H^{1,p}(X)$ to $T_l(\boldsymbol{u})$ for any $l \in \{1, \dots, n_2\}$ as $i \to \infty$. Then we easily see that for μ-a.e. $x \in X$ and any $i \in \mathbb{N}$,

$$\begin{aligned}
\left\| \left(\Gamma_\mu(\widehat{v}_{l,i})(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} &\leq \left\| \left(\sum_{j=i}^{N(i)} \alpha_{i,j} \Gamma_\mu(T_{l,n_j}(\boldsymbol{u}^{(n_j)}))(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\
&\leq \sum_{j=i}^{N(i)} \alpha_{i,j} \left\| \left(\Gamma_\mu(T_{l,n_j}(\boldsymbol{u}^{(n_j)}))(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\
&\overset{\text{(A.27)}}{\leq} \sum_{j=i}^{N(i)} \alpha_{i,j} \left\| \left(\Gamma_\mu(u_{k,n_j})(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}},
\end{aligned} \tag{A.28}$$

where we used the triangle inequality for the norm of ℓ^{q_2} in the second inequality. Note that for μ-a.e. $x \in X$,

$$\lim_{i\to\infty} \sum_{j=i}^{N(i)} \alpha_{i,j} \left\| \left(\Gamma_\mu(u_{k,n_j})(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} = \left\| \left(\Gamma_\mu(u_k)(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}.$$

Since $\lim_{i\to\infty} \int_X \Gamma_\mu(\widehat{v}_{l,i} - T_l(\boldsymbol{u}))^{\frac{p}{2}} \, d\mu = 0$, there exists $\{m_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ with $\inf_{i\in\mathbb{N}}(m_{i+1} - m_i) \geq 1$ such that $\lim_{i\to\infty} \Gamma_\mu(\widehat{v}_{l,m_i} - T_l(\boldsymbol{u}))(x) = 0$ for μ-a.e. $x \in X$ and any $l \in \{1, \dots, n_2\}$. In view of the triangle inequality for $\Gamma_\mu(\,\cdot\,)^{\frac{1}{2}}$ (see [Kuw24, (3) in p. 2]), we have

[18] The direct sum $H^{1,p}(X)^{\oplus n_2}$ is equipped with the norm $\|f\|_{H^{1,p}(X)^{\oplus n_2}} \coloneqq \sum_{l=1}^{n_2} \|f_j\|_{H^{1,p}(X)}$ for any $f = (f_1, \dots, f_{n_2}) \in H^{1,p}(X)^{\oplus n_2}$.

$\lim_{i\to\infty}\max_{l\in\{1,\dots,n_2\}}|\Gamma_\mu(\widehat{v}_{l,m_i})(x)-\Gamma_\mu(T_l(\boldsymbol{u}))(x)|=0$ for μ-a.e. $x\in X$. Hence we obtain (A.23) by (A.28). Once we get (A.23), we easily see by the same argument as in (A.16) that $\bigl\{\Gamma_\mu(u)^{\frac{p}{2}}\,d\mu\bigr\}_{u\in H^{1,p}(X)}$, which is obviously a family of p-energy measures on $(X,\mathcal{B}(X))$ dominated by $\bigl(\int_X\Gamma_\mu(\,\cdot\,)^{\frac{p}{2}}\,d\mu, H^{1,p}(X)\bigr)$, satisfies (GC)$_p$. □

B Some results for p-resistance forms on p.-c.f. self-similar structures

B.1 Existence of p-resistance forms with non-arithmetic weights

In this subsection, we discuss a gap between the frameworks in Subsection 8.2 and in Subsection 8.3 for p.-c.f. self-similar structures. As in Subsection 8.3, we fix $p\in(1,\infty)$ and a p.-c.f. self-similar structure $\mathcal{L}=(K,S,\{F_i\}_{i\in S})$ with $\#S\geq 2$ and K connected.

The following proposition about the "eigenvalue" $\lambda(\boldsymbol{\rho}_p)$ in Theorem 8.38 is a key result.

Proposition B.1. *Let $\boldsymbol{\rho}_p=(\rho_{p,i})_{i\in S}\in(0,\infty)^S$. Assume that $\boldsymbol{\rho}_p$ satisfies (**A**) (recall Remark 8.39).*

(a) *For any $a\in(0,\infty)$, $a\boldsymbol{\rho}_p:=(a\rho_{p,i})_{i\in S}$ satisfies (**A**) and $\lambda(a\boldsymbol{\rho}_p)=a\lambda(\boldsymbol{\rho}_p)$.*

(b) *Let $\widetilde{\boldsymbol{\rho}}_p=(\widetilde{\rho}_{p,i})_{i\in S}\in(0,\infty)^S$. If $\widetilde{\boldsymbol{\rho}}_p$ satisfies (**A**) and $\rho_{p,i}\leq\widetilde{\rho}_{p,i}$ for any $i\in S$, then $\lambda(\boldsymbol{\rho}_p)\leq\lambda(\widetilde{\boldsymbol{\rho}}_p)$.*

Proof. Throughout this proof, we fix a p-resistance form E_0 on V_0.

(a): Since $\mathcal{R}^n_{a\boldsymbol{\rho}_p}(E_0)=a\mathcal{R}^n_{\boldsymbol{\rho}_p}(E_0)$ for any $n\in\mathbb{N}\cup\{0\}$, we easily see that $a\boldsymbol{\rho}_p$ satisfies (**A**). Recall from Theorem 8.38-(a) that $\lambda(a\boldsymbol{\rho}_p)\in(0,\infty)$ is the unique number satisfying the following: there exists $C\in[1,\infty)$ such that

$$C^{-1}\lambda(a\boldsymbol{\rho}_p)^nE_0(u)\leq\mathcal{R}^n_{a\boldsymbol{\rho}_p}(E_0)(u)\leq C\lambda(a\boldsymbol{\rho}_p)^nE_0(u)\quad\text{for any } n\in\mathbb{N}\cup\{0\},\ u\in\mathbb{R}^{V_0}. \tag{B.1}$$

Therefore, $\lambda(a\boldsymbol{\rho}_p)=a\lambda(\boldsymbol{\rho}_p)$.

(b): Since $\mathcal{R}^n_{\boldsymbol{\rho}_p}(E_0)(u)\leq\mathcal{R}^n_{\widetilde{\boldsymbol{\rho}}_p}(E_0)(u)$ for any $u\in\mathbb{R}^{V_0}$, by (B.1), there exists $C\in[1,\infty)$ such that for any $n\in\mathbb{N}\cup\{0\}$ and any $u\in\mathbb{R}^{V_0}$,

$$C^{-1}\lambda(\boldsymbol{\rho}_p)^nE_0(u)\leq\mathcal{R}^n_{\boldsymbol{\rho}_p}(E_0)(u)\leq\mathcal{R}^n_{\widetilde{\boldsymbol{\rho}}_p}(E_0)(u)\leq C\lambda(\widetilde{\boldsymbol{\rho}}_p)^nE_0(u).$$

Since $n\in\mathbb{N}\cup\{0\}$ is arbitrary and $E_0(u)>0$ for $u\in\mathbb{R}^{V_0}\setminus\mathbb{R}\mathbb{1}_{V_0}$, we conclude that $\lambda(\boldsymbol{\rho}_p)\leq\lambda(\widetilde{\boldsymbol{\rho}}_p)$. □

Now we can show the existence of p-resistance forms with non-arithmetic weights on a class of strongly symmetric p.-c.f. self-similar sets as follows. (Recall the notation in Subsection 8.4.)

Proposition B.2. *Let $\mathcal{L}$ be a strongly symmetric p.-c.f. self-similar set. Assume that there exists $i \in S$ such that*

$$\bigcup_{g \in \mathcal{G}} \tau_g(i) \neq S. \tag{B.2}$$

Then there exists $\boldsymbol{\rho}_p = (\rho_{p,i})_{i \in S} \in (0,\infty)^S$ such that $\lambda(\boldsymbol{\rho}_p) = 1$, $\rho_{p,i} > 1$ for any $i \in S$, $\boldsymbol{\rho}_p$ satisfies (8.64) and

$$\frac{\log \rho_{p,i}}{\log \rho_{p,j}} \notin \mathbb{Q} \quad \text{for some } i, j \in S. \tag{B.3}$$

In particular, there exists a self-similar p-resistance form $(\mathcal{E}_p, \mathcal{F}_p)$ on $\mathcal{L}$ with weight $\boldsymbol{\rho}_p$.

Remark B.3. (1) Any weight $\boldsymbol{\rho}_p = (\rho_{p,i})_{i \in S}$ of a p-energy form constructed in Theorem 8.30 must satisfy $\rho_{p,i} = \sigma_p^{n_i}$ for some $n_i \in \mathbb{N}$, where $\sigma_p \in (0,\infty)$ is the p-scaling factor. Hence constructions of self-similar p-energy forms with weight ρ_p which satisfies (B.3) are not covered by Theorem 8.30 (or by [Kig23, Theorem 4.6]).

(2) The condition (B.2) is not very restrictive. See Figure B.2 for examples of self-similar sets satisfying this condition. In Figure B.1, we present examples of self-similar sets that do not satisfy (B.2).

Proof of Proposition B.2. Fix $i \in S$ and set $S_1 \coloneqq \bigcup_{g \in \mathcal{G}} \tau_g(i)$ and $S_2 \coloneqq S \setminus S_1$, which is non-empty by (B.2). For $t \in \mathbb{R}$, we define $\boldsymbol{\rho}_p(t) \coloneqq (\rho_{p,s}(t))_{s \in S}$ by

$$\rho_{p,s}(t) \coloneqq 1 + t\mathbb{1}_{S_2}(s) \quad \text{for } s \in S.$$

It is easy to see that $\rho_p(t)$ satisfies (8.64). Set $\lambda_p(t) \coloneqq \lambda(\boldsymbol{\rho}_p(t))$ for simplicity. By Proposition B.1, for any $t \in \mathbb{R}$, any $\delta \in (0,\infty)$ and any $s \in S$,

$$(1 - t - \delta)\lambda_p(0) \le \lambda_p(t - \delta) \le \lambda_p(t) \le \lambda_p(t + \delta) \le (1 + t + \delta)\lambda_p(0),$$

whence $\lambda_p(t)$ is continuous in t.

Fix $j \in S_2$ and define

$$r_{i,j}(t) \coloneqq \frac{\log\left(\rho_{p,i}(t)/\lambda_p(t)\right)}{\log\left(\rho_{p,j}(t)/\lambda_p(t)\right)} = \frac{-\log\left(\lambda_p(t)\right)}{\log(1+t) - \log\left(\lambda_p(t)\right)}, \quad t \in \mathbb{R}.$$

Since $r_{i,j}(0) = 1$ and $r_{i,j}(t)$ is continuous in t, there exists $t_* \in \mathbb{R} \setminus \{0\}$ such that $r_{i,j}(t_*) \notin \mathbb{Q}$. The existence of a self-similar p-resistance form on $\mathcal{L}$ with weight $\boldsymbol{\rho}_p$ follows from Theorems 8.50 and 8.51, so we complete the proof. □

B.2 Ahlfors regular conformal dimension of affine nested fractals

In this subsection, we prove that the Ahlfors regular conformal dimension of any strongly symmetric self-similar set equipped with the p-resistance metric for any $p \in (1,\infty)$ is equal to one (Theorem B.5). We also show that the Ahlfors regular conformal dimension

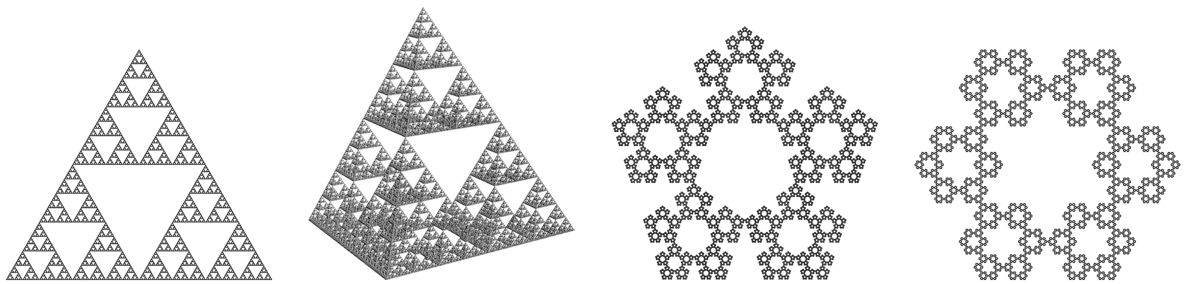

Figure B.1: Examples of affine nested fractals that do *NOT* satisfy (B.2). From the left, D-dimensional level-2 Sierpiński gasket ($D = 2, 3$), pentakun and hexagasket.

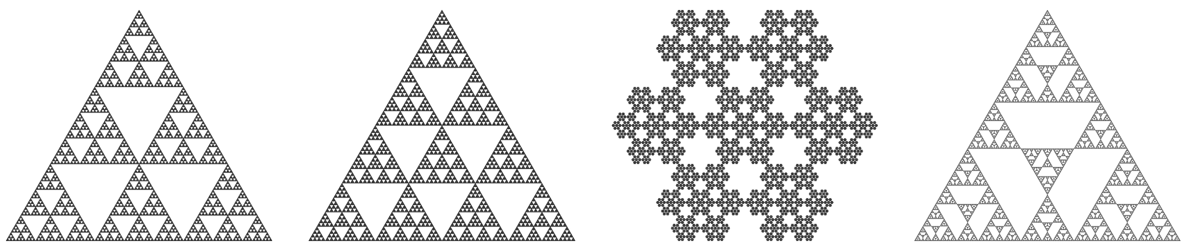

Figure B.2: Examples of affine nested fractals that satisfy (B.2). From the left, 2-dimensional level-l Sierpiński gasket ($l = 3, 4$), snowflake and a Sierpiński gasket-type fractal.

with respect to the Euclidean metric is also equal to one under some geometric condition (Theorem B.8).

Very similar results are already known in the literature. Indeed, Tyson and Wu [TW06, Theorems 1.3–1.5] showed that the (quasi)conformal dimensions (as defined in [TW06, p. 206]) of the D-dimensional level-2 Sierpiński gasket and of the N-polygasket with $N/4 \notin \mathbb{Z}$ are equal to one[19]. (The values of the conformal dimension and the Ahlfors regular conformal dimension coincide if the underlying metric space is compact, quasiself-similar [EB24, Definition 2.4], connected and locally connected [EB24, Theorem 1.6].) Also, Carrasco Piaggio [CP14, Theorem 1.2] provided a general criterion for a compact and metric doubling metric space to have Ahlfors regular conformal dimension one. This subsection is aimed at giving a new proof of a variant of these results in [TW06, CP14] based on the existence of self-similar p-resistance forms proved in Theorem 8.50.

Throughout this section, we assume that $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a strongly symmetric p.-c.f. self-similar set (recall Framework 8.46 and Definition 8.47). Let $c_i \in (0, 1)$ be the contraction ratio of F_i for each $i \in S$. Note that $(c_i)_{i\in S} \in (0, 1)^S$ must satisfy

$$c_i = c_{\tau_g(i)} \quad \text{for any } i \in S \text{ and any } g \in \mathcal{G}_{\mathrm{sym}}, \tag{B.4}$$

because of the symmetry of $\mathcal{L}$. For each $p \in (1, \infty)$, we also fix a self-similar p-resistance form $(\mathcal{E}^{\#}_p, \mathcal{F}^{\#}_p)$ on $\mathcal{L}$ with weight $(\rho_{\#,p})_{i\in S}$ for some $\rho_{\#,p} \in (1, \infty)$, i.e., a p-resistance form

[19] According to [TW06, the paragraph after Theorem 1.3], T. J. Laakso had shown before the work [TW06] that the conformal dimension of the 2-dimensional level-2 Sierpiński gasket (equipped with the Euclidean metric) is equal to one.

$(\mathcal{E}_p^{\#}, \mathcal{F}_p^{\#})$ on K such that

$$\mathcal{F}_p^{\#} = \{u \in C(K) \mid u \circ F_i \in \mathcal{F}_p^{\#} \text{ for any } i \in S\},$$
$$\mathcal{E}_p^{\#}(u) = \rho_{\#,p} \sum_{i \in S} \mathcal{E}_p^{\#}(u \circ F_i) \quad \text{for any } u \in \mathcal{F}_p^{\#}.$$

By Theorem 8.50, such a self-similar p-resistance form on $\mathcal{L}$ exists and the number $\rho_{\#,p}$ is uniquely determined. Let $\widehat{R}_p^{\#}$ denote the p-resistance metric associated with $(\mathcal{E}_p^{\#}, \mathcal{F}_p^{\#})$.

The next proposition ensures that $\widehat{R}_p^{\#}$ is quasisymmetric to the q-resistance metric with respect to *any* self-similar q-resistance form arising from Theorem 8.50. (Recall Definition 8.5-(3).)

Proposition B.4. *Let $p, q \in (1,\infty)$ and assume that $\boldsymbol{\rho}_q = (\rho_{q,i})_{i \in S} \in (0,\infty)^S$ satisfies (8.64), $\rho_{q,i} > 1$ for any $i \in S$ and $\lambda(\boldsymbol{\rho}_q) = 1$, where $\lambda(\boldsymbol{\rho}_q) \in (0,\infty)$ is the unique number given in Theorem 8.50. Let $(\mathcal{E}_q, \mathcal{F}_q)$ be a self-similar q-resistance form on $\mathcal{L}$ with weight $\boldsymbol{\rho}_q$, which exists by Theorems 8.50, and let $\widehat{R}_q$ be the q-resistance metric associated with $(\mathcal{E}_q, \mathcal{F}_q)$. Then $\widehat{R}_{q,\mathcal{E}_q}$ is quasisymmetric to $\widehat{R}_p^{\#}$.*

Proof. We will use [Kig20, Corollary 3.6.7] to show the desired statement. We first show that there exist $\alpha_1, \alpha_2 \in (0,\infty)$ such that

$$\alpha_1 \rho_{q,w}^{-1/(p-1)} \leq \operatorname{diam}(K_w, \widehat{R}_q) \leq \alpha_2 \rho_{q,w}^{-1/(p-1)} \quad \text{for any } w \in W_*. \tag{B.5}$$

The upper estimate in (B.5) is immediate from (7.1). To prove the lower estimate in (B.5), note that we can easily find $m_0 \in \mathbb{N}$ such that for any $w \in W_*$ there exist $v^1, v^2 \in W_{|w|+m_0}$ with $v^i \leq w$, $i = 1, 2$, and $K_{v^1} \cap K_{v^2} = \emptyset$. (It is enough to choose m_0 satisfying $2(\max_{i \in S} c_i)^{m_0} < 1$.) Then, by the proof of Proposition 7.14-(a) and $\rho_{p,v^i} \leq \rho_{q,w}(\max_{i \in S} \rho_{q,i})^{m_0}$, there exists $\alpha_1 \in (0,\infty)$ that is independent of $w \in W_*$ such that

$$\inf_{(x,y) \in K_{v^1} \times K_{v^2}} \widehat{R}_q(x,y) \geq \alpha_1 \rho_{q,w}^{-1/(p-1)},$$

which implies the desired lower estimate in (B.5).

Next we note that $\mathcal{L}$ is a rationally ramified self-similar structure by [Kig09, Proposition 1.6.12]; moreover, by combining [Kig09, Proposition 1.6.12], $K_v \cap K_w = F_v(V_0) \cap F_w(V_0)$ for any $v, w \in W_*$ with $\Sigma_v \cap \Sigma_w = \emptyset$ (see [Kig01, Proposition 1.3.5-(2)]) and the fact that each element of V_0 is a fixed point of F_i for some $i \in S_{\mathrm{fix}} := \{i \in S \mid K_i \cap V_0 \neq \emptyset\}$, $\mathcal{L}$ is rationally ramified with a relation set

$$\mathcal{R} = \bigl\{\{(\{w(j)\}, \{v(j)\}, \varphi_j, x(j), y(j)) \mid w(j), v(j), x(j), y(j) \in W_* \setminus \{\emptyset\}\}\bigr\}_{j=1}^{k} \tag{B.6}$$

satisfying $w(j), v(j) \in S_{\mathrm{fix}}$. (See [Kig09, Sections 1.5 and 1.6 and Chapter 8] for details about rationally ramified self-similar structures.)

With these preparations, we will apply [Kig20, Corollary 3.6.7] to $\widehat{R}_{q,\mathcal{E}_q}$ and $\widehat{R}_p^{\#}$. By Proposition 7.14-(a) and (B.5), $\widehat{R}_{q,\mathcal{E}_q}$ is 1-*adapted* and *exponential* (see [Kig20, Definition

2.4.7 and 3.1.15-(2)] for these definitions; see also Remark in [Kig20, p. 108]). Similarly, $\widehat{R}_p^{\#}$ is also 1-adapted and exponential. Hence, by [Kig20, Corollary 3.6.7], $\widehat{R}_{q,\mathcal{E}_q}$ is quasisymmetric to $\widehat{R}_p^{\#}$ if and only if $\widehat{R}_{q,\mathcal{E}_q}$ is *gentle* with respect to $\widehat{R}_p^{\#}$ (see [Kig20, Definition 3.3.1] for the definition of the gentleness). Define $g_q(w) \coloneqq \rho_{q,w}^{-1/(q-1)}$ and $g_{\#,p}(w) \coloneqq \rho_{\#,p}^{-|w|}$ for $w \in W_*$. Since g_q and $g_{\#,p}$ satisfy the condition (R1) in [Kig09, Theorem 1.6.6] by (8.64) and (B.6), we obtain the desired gentleness by [Kig09, Theorem 1.6.6] and (B.5). This completes the proof. $\square$

Now we can determine the Ahlfors regular conformal dimension of $(K, \widehat{R}_p^{\#})$ by using the discrete characterization of the Ahlfors regular conformal dimension due to Keith and Kleiner (see [CP13, the paragraph before Corollary 1.4]).

Theorem B.5. $\dim_{\mathrm{ARC}}(K, \widehat{R}_p^{\#}) = 1$.

Proof. We will use a version of the characterization of $\dim_{\mathrm{ARC}}(K, \widehat{R}_p^{\#})$ in [Kig20, Theorem 4.6.9]. Note that $(K, \widehat{R}_p^{\#})$ satisfies (BF1) and (BF2) in [Kig20, Section 4.3] by Proposition 7.14-(a), (B.5), [Kig09, Proposition 1.6.12, Lemmas 1.3.6 and 1.3.12]. We define a graph $G_n = (V_n, E_n)$ and q-energy $\mathcal{E}_p^{G_n}$, $q \in (1, \infty)$, on G_n by

$$E_n \coloneqq \{(x, y) \mid x, y \in F_w(V_0) \text{ for some } w \in W_n\},$$

and

$$\mathcal{E}_q^{G_n}(f) \coloneqq \frac{1}{2} \sum_{(x,y) \in E_n} |f(x) - f(y)|^q, \quad f \in \mathbb{R}^{V_n}.$$

Note that $\{G_n\}_{n \geq 0}$ is a *proper system of horizontal networks* with indices $(1, 2(\#V_0 - 1)\#V_0, 1, 1)$ in the sense of [Kig20, Definition 4.6.5]. Therefore by [Kig20, Theorem 4.6.9], $\dim_{\mathrm{ARC}}(K, \widehat{R}_p^{\#}) = 1$ if and only if the following holds: for any $q \in (1, \infty)$,

$$\liminf_{k \to \infty} \sup_{w \in W_*} \inf\Bigl\{\mathcal{E}_q^{G_{|w|+k}}(f) \Bigm| f \in \mathbb{R}^{V_{|w|+k}}, f|_{F_w(V_k)} = 1, f|_{Z_{w,k}} = 0\Bigr\} = 0, \tag{B.7}$$

where $Z_{w,k} \coloneqq \{x \in V_{|w|+n} \mid x \in F_v(V_k) \text{ for some } v \in W_{|w|} \text{ with } K_v \cap K_w = \emptyset\}$. Since both $\mathcal{E}_q^{\#}|_{V_0}(\,\cdot\,)^{1/q}$ and $\mathcal{E}_q^{G_0}(\,\cdot\,)^{1/q}$ are norms on the finite-dimensional vector space $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$, there exists $C \geq 1$ such that $C^{-1}\mathcal{E}_q^{\#}|_{V_0}(u) \leq \mathcal{E}_q^{G_0}(u) \leq C\mathcal{E}_q^{\#}|_{V_0}(u)$ for any $u \in \mathbb{R}^{V_0}$. Hence, by Propositions 7.2-(2) and 7.4, we obtain $C^{-1}\mathcal{E}_q^{\#}|_{V_n}(u) \leq \rho_{\#,q}^n \mathcal{E}_q^{G_n}(u) \leq C\mathcal{E}_q^{\#}|_{V_n}(u)$ for any $n \in \mathbb{N} \cup \{0\}$ and any $u \in \mathbb{R}^{V_n}$. Recall that $\Gamma_1(w) = \{v \in W_{|w|} \mid K_v \cap K_w \neq \emptyset\}$ for $w \in W_*$ (Definition 8.3). Let $h_{q,w} \in \mathcal{F}_q^{\#}$ be the unique function satisfying $h_{q,w}|_{K_w} = 1$, $h_{q,w}|_{K_v} = 0$ for any $v \in W_{|w|} \setminus \Gamma_1(w)$ and

$$\mathcal{E}_q^{\#}(h_{q,w}) = \inf\Bigl\{\mathcal{E}_q^{\#}(u) \Bigm| u|_{K_w} = 1, u|_{K_v} = 0 \text{ for any } v \in W_{|w|} \setminus \Gamma_1(w)\Bigr\}.$$

Then we see from (7.20), (7.18) and (B.5) that

$$\sup_{w \in W_*} \inf\Bigl\{\mathcal{E}_q^{G_{|w|+k}}(f) \Bigm| f \in \mathbb{R}^{V_{|w|+k}}, f|_{F_w(V_k)} = 1, f|_{Z_{w,k}} = 0\Bigr\}$$

$$\leq C\rho_{\#,q}^{-(|w|+k)} \sup_{w \in W_*} \mathcal{E}_q^{\#}|_{V_{|w|+k}}(h_{q,w}|_{V_{|w|+k}}) \leq C\rho_{\#,q}^{-(|w|+k)} \sup_{w \in W_*} \mathcal{E}_q^{\#}(h_{q,w}) \lesssim \rho_{\#,q}^{-k}.$$

Since $\rho_{\#,q} \in (1, \infty)$ for any $q \in (0,1)$, we obtain (B.7). The proof is completed. $\square$

To discuss the Ahlfors regular conformal dimension of K with respect to the Euclidean metric, we need the following assumption.

Assumption B.6. We define $\Lambda_1^d \coloneqq \{\emptyset\}$,

$$\Lambda_s^d \coloneqq \{w \mid w = w_1 \dots w_n \in W_* \setminus \{\emptyset\}, \mathrm{diam}(K_{w_1 \dots w_{n-1}}, d) > s \geq \mathrm{diam}(K_w, d)\}$$

for each $s \in (0,1)$. For $s \in (0,1]$, $M \in \mathbb{N} \cup \{0\}$ and $x \in K$, define

$$\Lambda_{s,M}^d(x) \coloneqq \left\{ v \,\middle|\, \begin{array}{l} v \in \Lambda_s^d, \text{ there exists } w \in \Lambda_s^d \text{ with } x \in K_w \text{ and} \\ \{z(j)\}_{j=1}^k \subseteq \Lambda_s^d \text{ with } k \leq M+1,\ z(1) = w,\ z(k) = v \\ \text{such that } K_{z(j)} \cap K_{z(j+1)} \neq \emptyset \text{ for any } j \in \{1, \dots, k-1\} \end{array} \right\},$$

and $U_M^d(x,s) \coloneqq \bigcup_{w \in \Lambda_{s,M}^d(x)} K_w$. Then there exist $M_* \in \mathbb{N}$, $\alpha_0, \alpha_1 \in (0,\infty)$ such that

$$U_{M_*}^d(x, \alpha_0 s) \subseteq B_d(x,s) \subseteq U_{M_*}^d(x, \alpha_1 s) \quad \text{for any } (x,s) \in K \times (0,1].$$

(Equivalently, d is M_*-adapted; see [Kig20, Definition 2.4.1].)

Remark B.7. We do not know whether Assumption B.6 is true for any strongly symmetric self-similar set. Even for nested fractals, being 1-adapted with respect to the Euclidean metric is required as an additional assumption in [Kig23, Assumption 4.41].

Now we can show the main result in this section under Assumption B.6.

Theorem B.8. *Assume that Assumption B.6 holds. Then* $\dim_{\mathrm{ARC}}(K,d) = 1$.

Proof. Thanks to Theorem B.5, it suffices to prove that $\widehat{R}_p^{\#}$ is quasisymmetric to d. Obviously, d is exponential since $\mathrm{diam}(K_w, d) = c_w \,\mathrm{diam}(K, d)$. By (B.4), a similar argument as in the proof of Proposition B.4 implies that $\widehat{R}_p^{\#}$ is gentle with respect to d. Hence [Kig20, Corollary 3.6.7] together with Assumption B.6 implies that $\widehat{R}_p^{\#}$ is quasisymmetric to d. $\square$

B.3 An estimate on self-similar regular p-resistance forms on p.-c.f. self-similar structures

This subsection is devoted to proving the following theorem, which is a generalization of [Kig03, Theorem A.1].

Theorem B.9. *Let $p \in (1,\infty)$, let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a p.-c.f. self-similar structure with $\#S \geq 2$ and K connected, and let $(\mathcal{E}, \mathcal{F})$ be a self-similar p-resistance form on $\mathcal{L}$ with weight $\boldsymbol{\rho} = (\rho_i)_{i \in S} \in (1,\infty)^S$. Then there exists $c \in (0,1)$ such that for any $x, y \in K$ and any $w \in W_*$,*

$$c\rho_w^{-1} R_{\mathcal{E}}(x,y) \leq R_{\mathcal{E}}(F_w(x), F_w(y)) \leq \rho_w^{-1} R_{\mathcal{E}}(x,y). \tag{B.8}$$

Since the upper estimate in (B.8) is obtained in (7.1), what matters is the lower estimate in (B.8). To prove it, we need the following lemma.

Lemma B.10. *Assume the same conditions as in Theorem B.9. Let* $x, y \in K$ *and* $w \in W_*$. *Set* $\Lambda \coloneqq \{\tau = \tau_1 \ldots \tau_n \in W_* \mid (\rho_{\tau_1 \cdots \tau_{n-1}})^{-1} > \rho_w \geq \rho_\tau^{-1}\}$, $U \coloneqq V_0 \cup \{x, y\}$, $V_\Lambda \coloneqq \bigcup_{w \in \Lambda} F_w(V_0)$ *and* $V \coloneqq V_\Lambda \cup \{F_w(x), F_w(y)\}$. *Then* Λ *is a partition of* Σ *and*

$$\mathcal{E}|_V(u) = \rho_w \mathcal{E}|_U(u \circ F_w) + \sum_{\tau \in \Lambda \setminus \{w\}} \rho_\tau \mathcal{E}|_{V_0}(u \circ F_\tau) \quad \textit{for any } u \in \mathcal{F}|_V. \tag{B.9}$$

Proof. The proof is very similar to Proposition 7.4. It is clear that Λ is a partition of Σ. Note that, by Proposition 7.2-(2), $R_{\mathcal{E}}^{1/p}$ is compatible with the original topology of K and thereby $\operatorname{diam}(K, R_{\mathcal{E}}^{1/p}) < \infty$. For any $u \in \mathcal{F}|_V$,

$$\begin{aligned}
&\mathcal{E}|_V(u)\\
&= \min\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F}, v|_V = u\bigr\}\\
&\overset{(5.7)}{=} \min\left\{\rho_w \mathcal{E}(v \circ F_w) + \sum_{\tau \in \Lambda \setminus \{w\}} \rho_\tau \mathcal{E}(v \circ F_\tau) \;\middle|\; v \in \mathcal{F},\, v|_V = u\right\}\\
&\geq \min\left\{\rho_w \mathcal{E}(v \circ F_w) \;\middle|\; v \in \mathcal{F}, v|_V = u\right\} + \min\left\{\sum_{\tau \in \Lambda \setminus \{w\}} \rho_\tau \mathcal{E}(v \circ F_\tau) \;\middle|\; v \in \mathcal{F}, v|_V = u\right\}\\
&\geq \rho_w \min\{\mathcal{E}(v) \mid v \in \mathcal{F}, v|_U = u \circ F_w\} + \sum_{\tau \in \Lambda \setminus \{w\}} \rho_\tau \min\{\mathcal{E}(v) \mid v \in \mathcal{F}, v|_{V_0} = u \circ F_\tau\}\\
&= \rho_w \mathcal{E}|_U(u \circ F_w) + \sum_{\tau \in \Lambda \setminus \{w\}} \rho_\tau \mathcal{E}|_{V_0}(u \circ F_\tau).
\end{aligned}$$

To prove the converse, let $v \in C(K)$ satisfy $v \circ F_w = h_U^{\mathcal{E}}[u \circ F_w]$ and, for $\tau \in \Lambda \setminus \{w\}$, $v \circ F_\tau = h_{V_0}^{\mathcal{E}}[u \circ F_\tau]$. Such v is well-defined since $K_w \cap K_\tau = F_w(V_0) \cap F_\tau(V_0)$. Also, we have $v|_V = u$ and $v \in \mathcal{F}$ by (5.5). Moreover,

$$\mathcal{E}|_V(u) \leq \mathcal{E}(v) \overset{(5.7)}{=} \sum_{\tau \in \Lambda} \rho_\tau \mathcal{E}(v \circ F_\tau) = \rho_w \mathcal{E}|_U(u \circ F_w) + \sum_{\tau \in \Lambda \setminus \{w\}} \rho_\tau \mathcal{E}|_{V_0}(u \circ F_\tau).$$

This completes the proof. □

Proof of Theorem B.9. Let Λ, U, V_Λ, V be the same as in Lemma B.10. Set $\Gamma_1(w; \Lambda) \coloneqq \{\tau \in \Lambda \mid w \neq \tau, K_w \cap K_\tau \neq \emptyset\}$ for simplicity. Then $\#\Gamma_1(w; \Lambda) \leq \#(\mathcal{C}_{\mathcal{L}})\#(V_0)$ by [Kig01, Lemma 4.2.3]. Let $\psi_{xy} \in \mathcal{F}$ satisfy $\psi_{xy}(x) = 1$, $\psi_{xy}(y) = 0$ and $\mathcal{E}(\psi_{xy}) = R_{\mathcal{E}}(x, y)^{-1}$. Let $u_* \in \mathcal{F}$ satisfy $u_*(x) = 1$, $u_*(y) = 0$, $u|_{V \setminus F_w(U)} \in \mathbb{R}\mathbb{1}_{V \setminus F_w(U)}$ and

$$\mathcal{E}(u_*) = \inf\{\mathcal{E}(v) \mid v \in \mathcal{F}, (v \circ F_w)|_U = \psi_{xy}, v|_{V \setminus F_w(U)} \in \mathbb{R}\mathbb{1}_{V \setminus F_w(U)}\}.$$

Such u_* is uniquely exists by a standard argument in the variational analysis. Also, by Proposition 2.3-(b), we easily see that $0 \leq u_* \leq 1$. Since $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$ is a finite dimensional vector space, there exists a constant $C \in (0, \infty)$ such that

$$\mathcal{E}|_{V_0}(u)^{1/p} \leq C \max_{z, z' \in V_0} |u(z) - u(z')| \quad \text{for any } u \in \mathbb{R}^{V_0}. \tag{B.10}$$

Then, by using Lemma B.10, we see that

$$
\begin{aligned}
R_{\mathcal{E}}(F_w(x), F_w(y))^{-1} \leq \mathcal{E}(u_*) &= \mathcal{E}|_V(u_*) \\
&= \rho_w \mathcal{E}|_U(u_* \circ F_w) + \sum_{\tau \in \Lambda \setminus \{w\}} \rho_\tau \mathcal{E}|_{V_0}(u_* \circ F_\tau) \\
&= \rho_w \mathcal{E}|_U(u_* \circ F_w) + \sum_{\tau \in \Gamma_1(w;\Lambda)} \rho_\tau \mathcal{E}|_{V_0}(u_* \circ F_\tau) \\
&\overset{\text{(B.10)}}{\leq} \frac{\rho_w}{R_{\mathcal{E}}(x,y)} + C^p \sum_{\tau \in \Gamma_1(w;\Lambda)} \rho_\tau \\
&\leq \rho_w \left(\frac{1}{R_{\mathcal{E}}(x,y)} + C^p \Bigl(\max_{i \in S} \rho_i \Bigr) (\#\Gamma_1(w;\Lambda)) \right) \\
&= \rho_w \left(\frac{1}{R_{\mathcal{E}}(x,y)} + C' \frac{R_{\mathcal{E}}(x,y)}{R_{\mathcal{E}}(x,y)} \right) \\
&\leq \rho_w \left(1 + C' \sup_{z,z' \in K} R_{\mathcal{E}}(z,z') \right) R_{\mathcal{E}}(x,y)^{-1},
\end{aligned}
$$

which shows the desired lower estimate in (B.8). □

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