

Generalized Kac's moment formula for positive continuous additive functionals of Markov processes

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Abstract

We establish a formula for moments of certain random variables involving positive continuous additive functionals (PCAFs) of standard processes which have absolutely continuous transition functions and are in duality with standard processes with absolutely continuous transition functions, generalizing the classical Kac's moment formula. In particular, all our results are applicable to the more familiar case of symmetric Hunt processes which are associated with regular Dirichlet forms and have absolutely continuous transition functions.

Keywords: Standard process with a dual, symmetric Hunt process, positive continuous additive functional (PCAF) in the strict sense, smooth measure in the strict sense, Revuz correspondence, generalized Kac's moment formula.

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1 Introduction

The aim of this paper is to extend Kac's moment formula [7] (see (1.1) below) to general positive continuous additive functionals (PCAFs) of Markov processes and to certain more general random variables involving PCAFs. Let us first recall Kac's moment formula for the Brownian motion. Let $X = (X_t)_{t \geq 0}$ be the one-dimensional Brownian motion, $f: \mathbb{R} \rightarrow [0, \infty)$ bounded and Borel measurable, and $(p_t(x, y))_{t > 0, x, y \in \mathbb{R}}$ the transition density of the Brownian motion, i.e., $p_t(x, y) = (2\pi)^{-1/2} \exp(-(x - y)^2/2t)$. Consider the additive functional $A = (A_t)_{t \geq 0}$ of X given by setting $A_t := \int_0^t f(X_s) ds$. Then, Kac's moment formula reads as follows: for any $t \in (0, \infty)$, $x \in \mathbb{R}$, and positive integer k ,

$$\begin{aligned} E_x[A_t^k] = k! & \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t dt_k \int_{\mathbb{R}} f(y_1) dy_1 \int_{\mathbb{R}} f(y_2) dy_2 \\ & \cdots \int_{\mathbb{R}} f(y_k) dy_k p_{t_1}(x, y_1) p_{t_2 - t_1}(y_1, y_2) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k). \end{aligned} \quad (1.1)$$

Kac's moment formula plays fundamental roles in studying occupation times and Feynman–Kac semigroups (see, e.g., [3, 15]). To obtain (1.1), one does not need to use any specific properties of Brownian motion, except for the Markov property. Therefore, it is not difficult to extend (1.1) to general Markov processes with transition densities. On the other hand, the condition that the additive functional A is of the form $A_t = \int_0^t f(X_s) ds$ is crucial for the applicability of Fubini's theorem, and it is not clear how (1.1) can be extended to more general additive functionals.

In this paper, we first state and prove our results in the framework of a symmetric Hunt process X whose Dirichlet form is regular and whose transition function is absolutely continuous, and for positive continuous additive functionals (PCAFs) in the strict sense of X . (The precise definitions of these objects are recalled in Section 2 below.) In fact, the choice of this setting

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is solely for the sake of the better familiarity to the reader, and our results hold, without any changes in the proofs, in the more general framework, as detailed in Section 4 below, of a standard process with absolutely continuous transition function and in duality with another such standard process.

Since some preparation is required to state the main theorems of this paper (Theorems 2.7 and 2.9), here we explain briefly what our version of Kac's moment formula (1.1) looks like in the present general setting. Fix a locally compact separable metrizable topological space S , a Radon measure m on S with full support, and an m -symmetric Hunt process $X = (X_t)_{t \geq 0}$ on S whose Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular, and assume that X admits a jointly Borel measurable transition density $(p_t(x, y))_{t > 0, x, y \in S}$ with respect to m . (As mentioned in the previous paragraph, our results hold as long as m is a σ -finite Borel measure on S and X is a standard process on S with a jointly Borel measurable transition density $(p_t(x, y))_{t > 0, x, y \in S}$ with respect to m such that $(t, x, y) \mapsto p_t(y, x)$ is a transition density of another standard process on S with respect to m .) For each $\alpha \in (0, \infty)$, we define the α -potential density $(r_\alpha(x, y))_{x, y \in S}$ of X by setting $r_\alpha(x, y) := \int_0^\infty e^{-\alpha t} p_t(x, y) dt$. Let $A = (A_t)_{t \geq 0}$ be a PCAF in the strict sense of X . By the Revuz correspondence (see Lemma 2.4), there exists a unique σ -finite Borel measure μ on S , called the *Revuz measure* of A , satisfying

$$E_x \left[\int_0^\infty e^{-\alpha t} f(X_s) dA_s \right] = \int_S r_\alpha(x, y) f(y) \mu(dy) \quad (1.2)$$

for any $\alpha \in (0, \infty)$, $x \in S$, and Borel measurable function $f: S \rightarrow [0, \infty]$. In this setting, our second main theorem (Theorem 2.9) implies that, for any $t \in (0, \infty)$, $x \in S$, and positive integer k ,

$$E_x[A_t^k] = k! \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t dt_k \int_S \mu(dy_1) \int_S \mu(dy_2) \cdots \int_S \mu(dy_k) p_{t_1}(x, y_1) p_{t_2-t_1}(y_1, y_2) \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) \quad (1.3)$$

(see Corollary 2.10 below). This is a generalization of (1.1). Indeed, if we consider a PCAF $A = (A_t)_{t \geq 0}$ of X given by $A_t := \int_0^t f(X_s) ds$ for a bounded Borel measurable function $f: S \rightarrow [0, \infty)$, then the measure μ satisfying (1.2) is $\mu(dx) = f(x) m(dx)$. Thus, (1.1) is recovered from (1.3).

The special case of (1.3) where X is the α -dimensional Bessel process on $[0, \infty)$ for $\alpha \in (0, 2)$ and A is its local time at 0 was obtained by Molchanov and Ostrovskii in [10, p. 129], by applying (1.1) to absolutely continuous PCAFs $A^{(n)}$ of X converging to A and then by letting $n \rightarrow \infty$ on the basis of explicit quantitative estimates on X . Unlike this argument in [10], our proof of the main results of this paper including (1.3) is based only on the strong Markov property of X and a change-of-variable formula [1, Chapter V, Lemma 2.2] for Lebesgue–Stieltjes integrals and does not require any quantitative estimates on X or A .

There is also a work in a similar research direction by Fitzsimmons and Pitman [4], where they considered more general Markov processes and their moment formula is described in terms of certain potential operators associated with PCAFs rather than transition densities and corresponding Revuz measures as in (1.3). A striking difference between their results and ours is that they considered moments of additive functionals at Markov killing times of X , whereas we consider moments of them at deterministic times. Here, a Markov killing time of X refers to a random time with the property that the process X killed at the time is again Markov, and a finite deterministic time is usually not a Markov killing time of X . For further discussions, see Remarks 2.8 and 2.11.

We expect that our generalized moment formula will be useful in the study of various objects involving PCAFs such as Feynman–Kac semigroups, just as the classical Kac's moment formula is. In fact, in [11], on the basis of our results, the convergence of PCAFs is shown to be implied by the convergence of the potentials of the corresponding Revuz measures.

The remainder of this article is organized as follows. In Section 2, we introduce the framework of an m -symmetric Hunt process whose Dirichlet form is regular and whose transition function is absolutely continuous, state the main theorems of this article, Theorems 2.7 and 2.9, and give some corollaries of them. Then, in Section 3, we prove Theorems 2.7 and 2.9. Finally, in Section 4, we present the framework of a standard process with absolutely continuous transition function and in duality with another such standard process, and give the background necessary for the proofs of our results to extend immediately to this more general setting.

2 Setting and main results

This section is divided into two subsections. In Subsection 2.1, we set out the framework for the discussions of Sections 2 and 3 and introduce PCAFs and smooth measures in the strict sense. Then, in Subsection 2.2, we present our main theorems, Theorems 2.7 and 2.9, and two corollaries of them (Corollaries 2.10 and 2.12). Throughout this paper, we fix a locally compact separable metrizable topological space S , and write $S_\Delta = S \cup \{\Delta\}$ for the one-point compactification of S . (N.B. If S is compact, then we add Δ to S as an isolated point.) Any $[-\infty, \infty]$ -valued function f defined on S is regarded as a function on S_Δ by setting $f(\Delta) := 0$, and we define $\|f\|_\infty := \sup\{|f(x)| \mid x \in S\}$ for each such f . The symbol \mathbb{N} denotes the set of positive integers. For $a, b \in [-\infty, \infty]$, or $[-\infty, \infty]$ -valued functions a, b on a common set, we set $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. For a non-empty set E and $E_0 \subseteq E$, we write $1_{E_0} = 1_{E_0}^E : E \rightarrow \mathbb{R}$ for the function defined by setting $1_{E_0}(x) := 1$ for $x \in E_0$ and $1_{E_0}(x) := 0$ for $x \in E \setminus E_0$. Given a topological space E , we write $\mathcal{B}(E)$ for the Borel σ -algebra of E .

2.1 Setting, PCAFs and smooth measures in the strict sense

In this subsection, we clarify the setting for our discussions and introduce the main objects: PCAFs and smooth measures in the strict sense. Our results are established within the theory of regular symmetric Dirichlet forms and symmetric Hunt processes. For details of this theory, the reader is referred to [2, 5].

We first fix the setting that is assumed throughout Sections 2 and 3. We let m be a Radon measure on S with full support, $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L^2(S, m)$ (see [5, Section 1.1] for the definition of the notion of regular symmetric Dirichlet form), and $X = (\Omega, \mathcal{M}, (X_t)_{t \in [0, \infty)}, (P_x)_{x \in S_\Delta}, (\theta_t)_{t \in [0, \infty)})$ be an m -symmetric Hunt process on S whose Dirichlet form is $(\mathcal{E}, \mathcal{F})$ in the sense of [5, Theorem 7.2.1]. Here, θ_t denotes the shift operator, i.e., a map $\theta_t : \Omega \rightarrow \Omega$ satisfying $X_s \circ \theta_t = X_{s+t}$ for any $s \in [0, \infty]$. We let ζ denote the life time of X , i.e., a $[0, \infty]$ -valued function on Ω satisfying $\{X_t = \Delta\} = \{\zeta \leq t\}$ for each $t \in [0, \infty]$, and write $\mathcal{F}_* = (\mathcal{F}_t)_{t \in [0, \infty]}$ for the minimum augmented admissible filtration of X in Ω (see [2, p. 397]). A Borel subset N of S is said to be *properly exceptional* for X if it satisfies $m(N) = 0$ and $P_x(\{X_t \mid t \in (0, \infty)\} \subseteq S_\Delta \setminus N) = 1$ for all $x \in S \setminus N$. (Note that by [5, Theorem A.2.3], which is applicable due to the present assumption that X is a Hunt process on S , for each Borel subset N of S and each $x \in S_\Delta$, $P_x(\{X_t \mid t \in (0, \infty)\} \subseteq S_\Delta \setminus N) = 1$ if and only if $P_x(\{X_t, X_{t-} \mid t \in (0, \infty)\} \subseteq S_\Delta \setminus N) = 1$, where $X_{t-}(\omega) := \lim_{s \uparrow t} X_s(\omega) \in S_\Delta$ for $(t, \omega) \in (0, \infty) \times \Omega$.) We assume that X satisfies the following *absolute continuity condition* (AC) (with respect to m).

(AC) For all $x \in S$ and $t \in (0, \infty)$, the Borel measure $P_x(X_t \in dy)$ on S is absolutely continuous with respect to $m(dy)$.

Then, by [16, Theorem 2], there exists a unique Borel measurable function $p : (0, \infty) \times S \times S \rightarrow [0, \infty]$ satisfying, for any $s, t \in (0, \infty)$ and $x, y \in S$, $P_x(X_t \in dz) = p_t(x, z) m(dz)$ (as Borel measures on S), $p_t(x, y) = p_t(y, x)$, and $p_{t+s}(x, y) = \int_S p_t(x, z) p_s(z, y) m(dz)$. The function p is called the *transition density* (or *heat kernel*) of X (with respect to m). For convenience, we

extend the domain of p to $(0, \infty] \times S \times S_\Delta$ by setting

$$p_t(x, \Delta) := 1 - \int_S p_t(x, y) m(dy) = P_x(X_t = \Delta) \quad \text{and} \quad p_\infty(x, y) := 1_{\{\Delta\}}(y) \quad (2.1)$$

for $t \in (0, \infty)$, $x \in S$, and $y \in S_\Delta$, and extend the measure m to a Borel measure m_Δ on S_Δ by setting $m_\Delta := m(\cdot \cap S) + \delta_{S_\Delta, \Delta}$, where $\delta_{S_\Delta, \Delta}$ denotes the Dirac measure on S_Δ putting mass 1 at Δ . It is then easy to check that, for any $t \in (0, \infty]$, $x \in S$, and Borel measurable function $f: S_\Delta \rightarrow [0, \infty]$ (that is not necessarily 0 at Δ),

$$E_x[f(X_t)] = \int_{S_\Delta} p_t(x, y) f(y) m_\Delta(dy). \quad (2.2)$$

For each $\alpha \in (0, \infty)$, the α -potential density $r_\alpha: S \times S \rightarrow [0, \infty]$ of X is defined by setting

$$r_\alpha(x, y) := \int_0^\infty e^{-\alpha t} p_t(x, y) dt, \quad (2.3)$$

so that by Fubini's theorem we have, for any $\alpha, \beta \in (0, \infty)$ and $x, y \in S$,

$$r_{\alpha \wedge \beta}(x, y) = r_{\alpha \vee \beta}(x, y) + |\alpha - \beta| \int_S r_\alpha(x, z) r_\beta(z, y) m(dz). \quad (2.4)$$

Given a Borel measure ν on S and $\alpha \in (0, \infty)$, we define $R_\alpha \nu: S \rightarrow [0, \infty]$ by setting

$$R_\alpha \nu(x) := \int_S r_\alpha(x, y) \nu(dy). \quad (2.5)$$

Now, we introduce PCAFs and smooth measures in the strict sense below.

Definition 2.1 (PCAF, [5, p. 222]). An \mathcal{F}_* -adapted $[0, \infty]$ -valued stochastic process $A = (A_t)_{t \geq 0}$ defined on Ω is called a *positive continuous additive functional (PCAF)* of X if there exist a set $\Lambda \in \mathcal{F}_\infty$, called a *defining set* of A , and a properly exceptional set $N \in \mathcal{B}(S)$ for X , called an *exceptional set* of A , satisfying the following.

- (i) It holds that $P_x(\Lambda) = 1$ for all $x \in S \setminus N$ and $\theta_t(\Lambda) \subseteq \Lambda$ for all $t \in [0, \infty)$.
- (ii) For every $\omega \in \Lambda$, $[0, \infty) \ni t \mapsto A_t(\omega)$ is a $[0, \infty]$ -valued continuous function with $A_0(\omega) = 0$ such that for any $s, t \in [0, \infty)$, $A_t(\omega) < \infty$ if $t < \zeta(\omega)$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ if $t \geq \zeta(\omega)$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t(\omega))$.

We set $A_\infty := \sup_{t \in [0, \infty) \cap \mathbb{Q}} A_t$ for each such A . We say that two PCAFs $A = (A_t)_{t \geq 0}$ and $B = (B_t)_{t \geq 0}$ of X are *equivalent* if $\int_S P_x(A_t \neq B_t) m(dx) = 0$ for all $t \in (0, \infty)$; see [2, Lemma A.3.2] in this connection.

Definition 2.2 (PCAF in the strict sense, [5, pp. 235–236]). A PCAF of X is called a *PCAF in the strict sense* of X if it admits a defining set Λ with $P_x(\Lambda) = 1$ for all $x \in S$. In other words, it is a PCAF of X such that the empty set \emptyset can be taken as an exceptional set of it. We say that two PCAFs $A = (A_t)_{t \geq 0}$ and $B = (B_t)_{t \geq 0}$ in the strict sense of X are *equivalent* if $\int_S P_x(A_t \neq B_t) m(dx) = 0$ for all $t \in (0, \infty)$, which, by [2, Proof of Lemma A.3.2] combined with (AC) and [2, Theorem A.2.17(iii)], holds if and only if there exists a set $\Lambda \in \mathcal{F}_\infty$ which is a defining set of both A and B and satisfies $P_x(\Lambda) = 1$ for all $x \in S$ and $A_t(\omega) = B_t(\omega)$ for all $(t, \omega) \in [0, \infty) \times \Lambda$. We write $\mathbf{A}_{c,1}^+$ for the collection of all PCAFs in the strict sense of X .

The next definition requires one more piece of notation. For each $E \subseteq S_\Delta$, we define the first exit time $\tau_E: \Omega \rightarrow [0, \infty]$ of X from E by setting $\tau_E(\omega) := \inf\{t \in [0, \infty) \mid X_t(\omega) \notin E\}$ ($\inf \emptyset := \infty$), so that τ_E is an \mathcal{F}_* -stopping time for any $E \in \mathcal{B}(S_\Delta)$ by [5, Theorem A.2.3] or [2, Theorem A.1.19].

Definition 2.3 (Smooth measure in the strict sense, [5, p. 238]). We define \mathcal{S}_0 to be the collection of finite Borel measures μ on S satisfying $\|R_1\mu\|_\infty < \infty$. A Borel measure μ on S is called a *smooth measure in the strict sense* if there exists a non-decreasing sequence $(S_n)_{n \in \mathbb{N}}$ of Borel subsets of S such that $1_{S_n} \cdot \mu \in \mathcal{S}_0$ for every $n \in \mathbb{N}$ and $P_x(\lim_{n \rightarrow \infty} \tau_{S_n} \geq \zeta) = 1$ for each $x \in S$; note that then $S = \bigcup_{n \in \mathbb{N}} S_n$ since, for any $x \in S$, $P_x(\lim_{n \rightarrow \infty} \tau_{S_n} > 0) \geq P_x(\lim_{n \rightarrow \infty} \tau_{S_n} \geq \zeta) = 1$ and hence $x \in \bigcup_{n \in \mathbb{N}} S_n$. We write \mathcal{S}_1 for the collection of smooth measures in the strict sense.

It is known that there is a one-to-one correspondence between the set of equivalence classes of PCAFs of X and a certain class of Borel measures on S called smooth measures. This correspondence is due to Revuz [13] and referred to as the *Revuz correspondence*, and the smooth measure μ_A corresponding under it to a given PCAF A of X is given by

$$\mu_A(E) := \sup_{t \in (0, \infty)} \frac{1}{t} \int_S E_x \left[\int_0^t 1_E(X_s) dA_s \right] m(dx) = \lim_{t \downarrow 0} \frac{1}{t} \int_S E_x \left[\int_0^t 1_E(X_s) dA_s \right] m(dx) \quad (2.6)$$

for every Borel subset E of S and called the *Revuz measure* of A ; for details see also [2, Subsection A.3.1 and Theorem 4.1.1] and [5, Theorems 5.1.3 and 5.1.4], which we follow for the terminology. By [5, Theorem 5.1.7], restricting the Revuz correspondence gives a one-to-one correspondence between the equivalence classes of PCAFs in the strict sense of X and the smooth measures in the strict sense, and we also have the following characterization of this correspondence.

Lemma 2.4 (Revuz correspondence). *Let $A = (A_t)_{t \geq 0} \in \mathbf{A}_{c,1}^+$, let μ be a σ -finite Borel measure on S , and consider the following condition (RC).*

(RC) *For any $\alpha \in (0, \infty)$, $x \in S$, and Borel measurable function $f: S \rightarrow [0, \infty]$,*

$$E_x \left[\int_0^\infty e^{-\alpha t} f(X_t) dA_t \right] = \int_S r_\alpha(x, y) f(y) \mu(dy). \quad (2.7)$$

Then, μ is the Revuz measure of A if and only if (RC) holds.

Proof. Let μ_A denote the Revuz measure of A . Note that by the symmetry of p_t and Fubini's theorem we have, for any $\alpha \in (0, \infty)$ and $y \in S$, $\alpha \int_S r_\alpha(x, y) m(dx) = \int_0^\infty e^{-s} P_y(s/\alpha < \zeta) ds$, which is non-decreasing in α and converges to $P_y(0 < \zeta) = 1$ as $\alpha \rightarrow \infty$ by the monotone convergence theorem. Therefore α times the $m(dx)$ -integrals on S of the left- and right-hand sides of (2.7) converge as $\alpha \rightarrow \infty$, respectively, to $\int_S f d\mu_A$ by [2, Theorem A.3.5(iv)] and to $\int_S f d\mu$ by Fubini's and the monotone convergence theorems, whence (RC) implies $\mu = \mu_A$. Conversely, if $\mu = \mu_A$, then (RC) holds by [13, Théorème V.5] (or alternatively one can verify (RC) by following [8, Proof of Proposition 2.32]). \square

Remark 2.5. Let $A = (A_t)_{t \geq 0}$ be a PCAF in the strict sense of X with defining set Λ . Then, noting that $\{\zeta = 0\} = \{X_0 = \Delta\}$ satisfies either $\{\zeta = 0\} \subseteq \Omega_0$ or $\{\zeta = 0\} \cap \Omega_0 = \emptyset$ for any $\Omega_0 \in \mathcal{F}_\infty$, we easily see that $\Lambda \in \mathcal{F}_0$, and thus that $(A_t 1_\Lambda)_{t \geq 0}$ is a PCAF in the strict sense of X equivalent to A with defining set $\Lambda \cup \{\zeta = 0\} \in \mathcal{F}_0$. In view of this observation and the Revuz correspondence described just before Lemma 2.4, we may and do assume without loss of generality that *every PCAF $A = (A_t)_{t \geq 0}$ in the strict sense of X with defining set Λ considered in the rest of this paper satisfies $A_t(\omega) = 0$ for any $(t, \omega) \in [0, \infty) \times (\Omega \setminus \Lambda)$ and $\{\zeta = 0\} \subseteq \Lambda$.*

The following fact will be used in the proof of Proposition 3.2 below.

Lemma 2.6 ([2, Exercise A.3.3 and Theorem A.3.5(iii)]). *Let $A = (A_t)_{t \geq 0}$ be a PCAF in the strict sense of X , let μ be the Revuz measure of A , and let $E \in \mathcal{B}(S)$. Then, the process $B = (B_t)_{t \geq 0}$ defined by setting $B_t := \int_0^t 1_E(X_s) dA_s$ is a PCAF in the strict sense of X with Revuz measure $1_E \cdot \mu$.*

2.2 Main results

In this subsection, we state our main theorems, Theorems 2.7 and 2.9, and provide two corollaries of them, Corollaries 2.10 and 2.12. The proofs of Theorems 2.7 and 2.9 are given later in Section 3. We continue to assume the setting specified in Subsection 2.1. For measurable spaces (Y, \mathcal{A}) and (Z, \mathcal{B}) , we write $\mathcal{A} \otimes \mathcal{B}$ for the product σ -algebra of \mathcal{A} and \mathcal{B} in $Y \times Z$.

Theorem 2.7. *Let $A = (A_t)_{t \geq 0}$ be a PCAF in the strict sense of X and let μ be the Revuz measure of A . Let $F: (0, \infty) \times \Omega \rightarrow [0, \infty]$ be $\mathcal{B}((0, \infty)) \otimes \mathcal{F}_\infty$ -measurable, set $F_t(\omega) := F(t, \omega)$ for $(t, \omega) \in (0, \infty) \times \Omega$, and assume that the following conditions are satisfied:*

- (i) *there exists $\Lambda \in \mathcal{F}_\infty$ with $P_x(\Lambda) = 1$ for all $x \in S$ such that the function $(0, \infty) \times \Omega \ni (t, \omega) \mapsto 1_\Lambda(\omega)F_t \circ \theta_t(\omega) = 1_\Lambda(\omega)F_t(\theta_t \omega)$ is $\mathcal{B}((0, \infty)) \otimes \mathcal{F}_\infty$ -measurable;*
- (ii) *the function $(0, \infty) \times S \ni (t, x) \mapsto E_x[F_t]$ is Borel measurable.*

Then, for any $t \in (0, \infty]$ and $x \in S$,

$$E_x \left[\int_0^t F_s \circ \theta_s dA_s \right] = \int_0^t \int_S p_s(x, y) E_y[F_s] \mu(dy) ds. \quad (2.8)$$

Remark 2.8. As mentioned in Section 1, a formula similar to (2.8) is stated in [4, Equation (21)], which deals with expectations of Lebesgue–Stieltjes integrals with respect to PCAFs up to Markov killing times of X rather than up to deterministic times t as in (2.8). As stated in Remark 2.11 below, it is possible to extend our results to the case where t is replaced by the first exit time τ_D from a non-empty open subset D of S , one of the most typical examples of Markov killing times of X . We emphasize here that our proof of Theorem 2.7 is quite elementary and based only on the strong Markov property of X and a change-of-variable formula for Lebesgue–Stieltjes integrals, whereas it is indicated in [4] that the (omitted) proof of [4, Equation (21)] uses the Ray–Knight compactification and is therefore technically demanding.

By using Theorem 2.7, we further obtain the following generalizations of Kac’s moment formula (1.1) to various random variables involving general PCAFs in the strict sense of X .

Theorem 2.9. *Let $k \in \mathbb{N}$, and for each $i \in \{1, 2, \dots, k\}$, let $A^{(i)} = (A_t^{(i)})_{t \geq 0}$ be a PCAF in the strict sense of X and let μ_i be the Revuz measure of $A^{(i)}$. Then, for any $t \in (0, \infty]$, $x \in S$, and Borel measurable function $f: S_\Delta \rightarrow [0, \infty]$,*

$$\begin{aligned} & E_x \left[f(X_t) \prod_{i=1}^k A_t^{(i)} \right] \\ &= \sum_{\pi \in \mathfrak{S}_k} \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t dt_k \int_S \mu_{\pi_1}(dy_1) \int_S \mu_{\pi_2}(dy_2) \cdots \int_S \mu_{\pi_k}(dy_k) \int_{S_\Delta} m_\Delta(dz) \\ & \quad p_{t_1}(x, y_1) p_{t_2-t_1}(y_1, y_2) \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) p_{t-t_k}(y_k, z) f(z), \end{aligned} \quad (2.9)$$

where \mathfrak{S}_k denotes the set of all the bijections $\pi = (\pi_i)_{i=1}^k$ from $\{1, 2, \dots, k\}$ to itself.

Corollary 2.10. *Let $A = (A_t)_{t \geq 0}$ and $B = (B_t)_{t \geq 0}$ be PCAFs in the strict sense of X , and let μ and ν be the Revuz measures of A and B , respectively. Then, the following statements hold.*

- (i) *For any $t \in (0, \infty]$, $x \in S$, and Borel measurable function $f: S_\Delta \rightarrow [0, \infty]$,*

$$E_x[f(X_t)A_t] = \int_0^t \int_S p_s(x, y) \int_{S_\Delta} p_{t-s}(y, z) f(z) m_\Delta(dz) \mu(dy) ds. \quad (2.10)$$

(ii) For any $t \in (0, \infty]$, $x \in S$, and $k \in \mathbb{N}$,

$$E_x[A_t^k] = k! \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t dt_k \int_S \mu(dy_1) \int_S \mu(dy_2) \cdots \int_S \mu(dy_k) p_{t_1}(x, y_1) p_{t_2-t_1}(y_1, y_2) \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k). \quad (2.11)$$

(iii) For any $t \in (0, \infty]$ and $x \in S$,

$$E_x[A_t B_t] = \int_0^t dt_1 \int_{t_1}^t dt_2 \int_S \mu(dy_1) \int_S \nu(dy_2) p_{t_1}(x, y_1) p_{t_2-t_1}(y_1, y_2) + \int_0^t dt_1 \int_{t_1}^t dt_2 \int_S \nu(dy_1) \int_S \mu(dy_2) p_{t_1}(x, y_1) p_{t_2-t_1}(y_1, y_2). \quad (2.12)$$

Proof. These are immediate consequences of Theorem 2.9. \square

Remark 2.11. More generally, we can extend each of (2.8), (2.9), (2.10), (2.11) and (2.12) to a similar formula for the random variable given by replacing t with $t \wedge \tau_D$ (and requiring $f|_{S_\Delta \setminus D}$ to be constant) for any non-empty open subset D of S . Namely, fix any such D , let $D_\Delta = D \cup \{\Delta_D\}$ be the one-point compactification of D , set $m|_D := m|_{\mathcal{B}(D)}$, $P_{\Delta_D} := P_\Delta$, and for each $t \in [0, \infty]$ define $X_t^D: \Omega \rightarrow D_\Delta$ by setting $X_t^D(\omega) := X_t(\omega)$ for $\omega \in \{t < \tau_D\}$ and $X_t^D(\omega) := \Delta_D$ for $\omega \in \{t \geq \tau_D\}$. Then $X^D := (\Omega, \mathcal{F}_\infty, (X_t^D)_{t \in [0, \infty]}, (P_x)_{x \in D_\Delta})$, called the *part process* of X on D (killed upon exiting D), is an $m|_D$ -symmetric Hunt process on D by [5, Theorem A.2.10 and Lemma 4.1.3] (see also [2, Exercise 3.3.7(ii), (3.3.4) and Exercise 4.1.9(i)]), the Dirichlet form of X^D is regular by [5, Theorems 4.4.2 and 4.4.3], and X^D satisfies (AC) by (AC) of X and hence has a unique transition density p^D with respect to $m|_D$. We further set $p_t^D(x, \Delta_D) := 1 - \int_D p_t^D(x, y) m(dy) = P_x(t \geq \tau_D)$ and $p_\infty^D(x, y) := 1_{\{\Delta_D\}}(y)$ for $t \in (0, \infty)$, $x \in D$, and $y \in D_\Delta$, and extend $m|_D$ to a Borel measure $(m|_D)_\Delta$ on D_Δ by setting $(m|_D)_\Delta := m|_D(\cdot \cap D) + \delta_{D_\Delta, \Delta_D}$. Then we have the following.

Theorems 2.7, 2.9 and Corollary 2.10, with S, S_Δ, p, m_Δ replaced by $D, D_\Delta, p^D, (m|_D)_\Delta$ and with t, X in the left-hand sides of (2.8), (2.9), (2.10), (2.11) and (2.12) replaced by $t \wedge \tau_D, X^D$, hold. (2.13)

Indeed, Theorem 2.9 and Corollary 2.10 (and Lemma 3.1 and Proposition 3.2 below) are shown to extend in this form to general D by applying them to X^D on the basis of the following fact implied by [2, Exercise 4.1.9 and Proposition 4.1.10].

If $A = \{A_t\}_{t \geq 0}$ is a PCAF in the strict sense of X with Revuz measure μ , then $\{A_{t \wedge \tau_D}\}_{t \geq 0}$ is a PCAF in the strict sense of X^D with Revuz measure $\mu|_D := \mu|_{\mathcal{B}(D)}$. (2.14)

(To be precise, $\{A_{t \wedge \tau_D}\}_{t \geq 0}$ might not be adapted to the minimum augmented admissible filtration $\mathcal{F}_*^D = \{\mathcal{F}_t^D\}_{t \in [0, \infty]}$ of X^D in Ω since \mathcal{F}_*^D involves smaller families of sets of probability zero than \mathcal{F}_* , but $\{A_{t \wedge \tau_D}\}_{t \geq 0}$ is still *equivalent* to an $(\mathcal{F}_*^D$ -adapted) PCAF A^D in the strict sense of X^D . Indeed, such A^D can be obtained by applying to A the limit construction of PCAFs of X presented in [1, Proofs of Chapter IV, Theorem 3.16 and Chapter V, Theorem 2.1] and restricting the approximating PCAFs to the stochastic interval $[0, \tau_D]$ in taking their limits to recover A ; in this connection see also [12, Remark 4.16].) Then the proof of Theorem 2.7 given in Section 3 below is easily seen to extend to general D by replacing $t, 1_S$ with $t \wedge \tau_D, 1_D$ in (3.3) and using the above extension to D of Proposition 3.2. (Note that this extension of Theorem 2.7 is *not* implied by a mere application of Theorem 2.7 to X^D , since $\mathcal{F}_\infty^D \subseteq \mathcal{F}_{\tau_D} \subseteq \mathcal{F}_\infty$ and the latter inclusion is usually strict.)

Following [9], we define $\mathcal{S}_{\text{EK}} := \{\mu \in \mathcal{S}_1 \mid \lim_{\alpha \rightarrow \infty} \|R_\alpha \mu\|_\infty < 1\}$, and call \mathcal{S}_{EK} the *extended Kato class*. This class of smooth measures plays an important role in the study of Feynman–Kac semigroups (see, e.g., [6, 17]). As a consequence of (2.11), we can prove that any PCAF in the strict sense of X with Revuz measure in \mathcal{S}_{EK} has uniformly bounded exponential moments at any finite time, as follows. Essentially the same result was proved in [6, 17], but we give a detailed proof here since our setting and class \mathcal{S}_{EK} of smooth measures are different from those in [6, 17].

Corollary 2.12. *Let $A = (A_t)_{t \geq 0}$ be a PCAF in the strict sense of X whose Revuz measure belongs to \mathcal{S}_{EK} . Then, there exist $c_1, c_2 \in (0, \infty)$ such that for any $t \in [0, \infty)$,*

$$\sup_{x \in S} E_x[e^{A_t}] \leq c_1 e^{c_2 t}.$$

Proof. Let μ be the Revuz measure of A . By $\mu \in \mathcal{S}_{\text{EK}}$, we can take $\alpha \in (0, \infty)$ satisfying $\|R_\alpha \mu\|_\infty < 1$. Choose $s_\alpha \in (0, \infty)$ so that $c := e^{s_\alpha} \|R_\alpha \mu\|_\infty < 1$, set $t_\alpha := s_\alpha / \alpha$, and let $x \in S$. Then,

$$\int_0^{t_\alpha} \int_S p_t(x, y) \mu(dy) dt \leq e^{\alpha t_\alpha} \int_S \int_0^{t_\alpha} e^{-\alpha t} p_t(x, y) dt \mu(dy) \leq e^{\alpha t_\alpha} \|R_\alpha \mu\|_\infty = c < 1, \quad (2.15)$$

which together with Corollary 2.10(ii) shows that $E_x[A_{t_\alpha}^k] \leq k!c^k$ for each $k \in \mathbb{N}$. This then yields

$$E_x[e^{A_{t_\alpha}}] = \sum_{k \geq 0} \frac{1}{k!} E_x[A_{t_\alpha}^k] \leq \frac{1}{1-c} =: c_1 \in [1, \infty). \quad (2.16)$$

Now for each $t \in [0, \infty)$, taking $n = n_t \in \mathbb{N}$ such that $(n-1)t_\alpha \leq t < nt_\alpha$, and noting that $E_\Delta[e^{A_{t_\alpha}}] = 1 \leq c_1$ by Remark 2.5, we see from the Markov property of X and (2.16) that

$$E_x[e^{A_t}] \leq E_x[e^{A_{nt_\alpha}}] = E_x[e^{A_{(n-1)t_\alpha}} E_{X_{(n-1)t_\alpha}}[e^{A_{t_\alpha}}]] \leq c_1 E_x[e^{A_{(n-1)t_\alpha}}] \leq c_1^n \leq c_1^{1+t/t_\alpha},$$

which completes the proof. \square

Remark 2.13. It is not always the case that a PCAF $A = (A_t)_{t \geq 0}$ in the strict sense of X with Revuz measure in \mathcal{S}_{EK} has uniformly bounded exponential moments at the life time ζ , i.e., $\sup_{x \in S} E_x[e^{A_\zeta}] < \infty$. If this is the case, then (X, A) is said to be *gaugeable*. The gaugeability is important in the study of Feynman–Kac semigroups and Schrödinger operators and there has been plenty of research on it. An analytic characterization of the gaugeability can be found in [9], for example.

3 Proofs of the main theorems

In this section, we prove our main theorems, Theorems 2.7 and 2.9. We continue to assume the setting specified in Subsection 2.1. The main ingredient of the proofs is the following fact.

Lemma 3.1. *Let $A = (A_t)_{t \geq 0}$ be a PCAF in the strict sense of X , let μ be the Revuz measure of A , and let $f: S \rightarrow [0, \infty]$ be Borel measurable. Then, for any $t \in (0, \infty]$ and $x \in S$,*

$$E_x \left[\int_0^t f(X_s) dA_s \right] = \int_0^t \int_S p_s(x, y) f(y) \mu(dy) ds. \quad (3.1)$$

Proof. This result is stated in [8, Proposition 2.32]. However, to ensure that it is implied solely by (RC) without relying on any additional assumptions, we provide a complete proof here. Recalling that $\mu \in \mathcal{S}_1$ as mentioned after (2.6), let $(S_n)_{n \in \mathbb{N}}$ be as in Definition 2.3. Let $x \in S$, $n \in \mathbb{N}$, and consider the Borel measures η_1, η_2 on $[0, \infty)$ given by $\eta_1(B) := E_x \left[\int_B ((f \wedge n) 1_{S_n})(X_s) dA_s \right]$

and $\eta_2(B) := \int_B \int_{S_n} p_s(x, y) (f \wedge n)(y) \mu(dy) ds$ for $B \in \mathcal{B}([0, \infty))$. It then follows from Fubini's theorem, (2.3), (RC) and $1_{S_n} \cdot \mu \in \mathcal{S}_{00}$ that the Laplace–Stieltjes transforms of η_1, η_2 coincide and are finite on $(0, \infty)$. Thus $\eta_1 = \eta_2$, namely (3.1) with $(f \wedge n)1_{S_n}$ in place of f holds for any $t \in (0, \infty]$, and letting $n \rightarrow \infty$ yields (3.1) by the monotone convergence theorem. \square

A simple application of the monotone class theorem allows us to extend Lemma 3.1 as follows.

Proposition 3.2. *Let $A = (A_t)_{t \geq 0}$ be a PCAF in the strict sense of X , let μ be the Revuz measure of A , and let $f: (0, \infty) \times S \rightarrow [0, \infty]$ be Borel measurable. Then, with the convention that $f(t, \Delta) := 0$ for any $t \in (0, \infty)$, it holds that for any $t \in (0, \infty]$ and $x \in S$,*

$$E_x \left[\int_0^t f(s, X_s) dA_s \right] = \int_0^t \int_S p_s(x, y) f(s, y) \mu(dy) ds. \quad (3.2)$$

Proof. By the monotone convergence theorem, it suffices to show (3.2) for $t \in (0, \infty)$. Let $t \in (0, \infty)$, $x \in S$, and assume that (3.2) holds when $\mu \in \mathcal{S}_{00}$. To see (3.2) in the general case, recalling that $\mu \in \mathcal{S}_1$ as mentioned after (2.6), let $(S_n)_{n \in \mathbb{N}}$ be as in Definition 2.3, so that $(S_n)_{n \in \mathbb{N}}$ is non-decreasing and $S = \bigcup_{n \in \mathbb{N}} S_n$. By Lemma 2.6, for each $n \in \mathbb{N}$, the process $A^{(n)} = (A_t^{(n)})_{t \geq 0}$ defined by setting $A_t^{(n)} := \int_0^t 1_{S_n}(X_s) dA_s$ is a PCAF in the strict sense of X with Revuz measure $1_{S_n} \cdot \mu_n$. Now we conclude (3.2) by using the monotone convergence theorem to let $n \rightarrow \infty$ in the equality

$$E_x \left[\int_0^t f(s, X_s) 1_{S_n}(X_s) dA_s \right] = E_x \left[\int_0^t f(s, X_s) dA_s^{(n)} \right] = \int_0^t \int_S p_s(x, y) f(s, y) 1_{S_n}(y) \mu(dy) ds$$

implied for any $n \in \mathbb{N}$ by the assumed validity of (3.2) for $A^{(n)}$ and $1_{S_n} \cdot \mu \in \mathcal{S}_{00}$.

It thus remains to prove (3.2) when $\mu \in \mathcal{S}_{00}$. Define \mathcal{H} to be the collection of $[0, \infty]$ -valued Borel measurable functions on $(0, \infty) \times S$ satisfying (3.2) for any $t \in (0, \infty)$ and $x \in S$. Then \mathcal{H} is closed under multiplication by any element of $[0, \infty]$ and under sum, and $1_{(0, \infty) \times S} \in \mathcal{H}$ by Lemma 3.1. Moreover, for any $t \in (0, \infty)$, $x \in S$, and $E \in \mathcal{B}(S)$, the same argument as (2.15) yields $\int_0^t \int_S p_s(x, y) 1_E(y) \mu(dy) ds \leq e^t \|R_1 \mu\|_\infty$, which is finite since $\mu \in \mathcal{S}_{00}$. It therefore follows from Lemma 3.1 that $1_{(s_1, s_2] \times E} \in \mathcal{H}$ for any $s_1, s_2 \in [0, \infty)$ with $s_1 < s_2$ and any $E \in \mathcal{B}(S)$, and hence from the monotone class theorem (see [1, Chapter 0, Proof of Theorem 2.3]) that \mathcal{H} is equal to the set of all $[0, \infty]$ -valued Borel measurable functions on $(0, \infty) \times S$, completing the proof. \square

Now, we are ready to prove Theorems 2.7 and 2.9.

Proof of Theorem 2.7. Set $F_\infty(\omega) := 0 =: F_0(\omega)$ for $\omega \in \Omega$ and, for each $t \in [0, \infty)$, define $\tau_t: \Omega \rightarrow [0, \infty]$ by setting $\tau_t(\omega) := \inf\{s \in [0, \infty) \mid A_s(\omega) > t\}$ ($\inf \emptyset := \infty$), so that τ_t is an \mathcal{F}_* -stopping time by [2, Proposition A.3.8(i)] and the function $[0, \infty) \ni s \mapsto \tau_s(\omega) \in [0, \infty]$ is non-decreasing and right-continuous for any $\omega \in \Omega$. In particular, by [5, Proof of Lemma A.2.4], as maps in $(s, \omega) \in [0, \infty) \times \Omega$, the functions $\tau_s(\omega)$ and $1_{[0, t \wedge \zeta(\omega))}(\tau_s(\omega))$ for $t \in (0, \infty]$ are $\mathcal{B}([0, \infty)) \otimes \mathcal{F}_\infty$ -measurable, the S_Δ -valued map $X_{\tau_s}(\omega) := X_{\tau_s(\omega)}(\omega)$ is $\mathcal{B}([0, \infty)) \otimes \mathcal{F}_\infty / \mathcal{B}(S_\Delta)$ -measurable, and hence the functions $1_\Lambda(\omega) F_{\tau_s} \circ \theta_{\tau_s}(\omega)$ and $E_{X_{\tau_s}(\omega)}[F_u]_{u=\tau_s(\omega)}$ are $\mathcal{B}([0, \infty)) \otimes \mathcal{F}_\infty$ -measurable, where Λ is as in condition (i) and $F_{\tau_s} \circ \theta_{\tau_s}(\omega) := F_{\tau_s(\omega)}(\theta_{\tau_s(\omega)} \omega)$. Letting $t \in (0, \infty]$ and $x \in S$, and noting $P_x(\Lambda) = 1$, the $\mathcal{B}([0, \infty)) \otimes \mathcal{F}_\infty$ -measurability of the functions in the previous sentence, and that $1_{[0, t \wedge \zeta)}(\tau_s)$ is \mathcal{F}_{τ_s} -measurable for each $s \in [0, \infty)$ (see, e.g., [1,

Chapter I, Proposition 6.8]), now we obtain

$$\begin{aligned}
E_x \left[\int_0^t F_s \circ \theta_s dA_s \right] &= E_x \left[\int_0^\infty 1_{[0, t \wedge \zeta)}(\tau_s) 1_\Lambda \cdot F_{\tau_s} \circ \theta_{\tau_s} ds \right] \quad (\text{by [1, Chapter V, Lemma 2.2]}) \\
&= \int_0^\infty E_x [1_{[0, t \wedge \zeta)}(\tau_s) 1_\Lambda \cdot F_{\tau_s} \circ \theta_{\tau_s}] ds \quad (\text{by Fubini's theorem}) \\
&= \int_0^\infty E_x [1_{[0, t \wedge \zeta)}(\tau_s) E_x [F_{\tau_s} \circ \theta_{\tau_s} | \mathcal{F}_{\tau_s}]] ds \\
&= \int_0^\infty E_x [1_{[0, t \wedge \zeta)}(\tau_s) E_{X_{\tau_s}} [F_u] |_{u=\tau_s}] ds \\
&= E_x \left[\int_0^\infty 1_{[0, t \wedge \zeta)}(\tau_s) E_{X_{\tau_s}} [F_u] |_{u=\tau_s} ds \right] \quad (\text{by Fubini's theorem}) \\
&= E_x \left[\int_0^t 1_S(X_s) E_{X_s} [F_s] dA_s \right] \quad (\text{by [1, Chapter V, Lemma 2.2]}); \quad (3.3)
\end{aligned}$$

here the fourth equality follows by applying [1, Chapter I, Exercise 8.16], which holds by the strong Markov property of X (see, e.g., [2, Theorem A.1.21]), to the \mathcal{F}_* -stopping time τ_s and the $\mathcal{F}_{\tau_s} \otimes \mathcal{F}_\infty$ -measurable function $\Omega \times \Omega \ni (\omega, \omega') \mapsto F_{\tau_s(\omega)}(\omega')$. Combining (3.3) with Proposition 3.2, we get (2.8). \square

Remark 3.3. The arguments in (3.3) can be interpreted in terms of the general theory of optional projections (see [14, pp. 390–391 and Section 22]) as identifying the optional projection of $(1_S(X_t)F_t \circ \theta_t)_{t \geq 0}$ as $(1_S(X_t)E_{X_t}[F_t])_{t \geq 0}$ and then applying [14, Equation (A5.18)] (for an arbitrarily chosen \mathcal{F}_∞ -measurable function $F_0: \Omega \rightarrow [0, \infty]$). Note, however, that this observation does not allow us to reduce the proof of (3.3) to a mere application of [14, Equation (A5.18)], because the former identification is a non-trivial consequence of a version [1, Chapter I, Exercise 8.16] of the strong Markov property of X .

Proof of Theorem 2.9. Let $f: S_\Delta \rightarrow [0, \infty]$ be Borel measurable. First, for any $t \in (0, \infty]$ and $x \in S$,

$$E_x \left[f(X_t) \prod_{i=1}^k A_t^{(i)} \right] = \sum_{\pi \in \mathfrak{S}_k} E_x \left[\int_0^t dA_{t_1}^{(\pi_1)} \int_{t_1}^t dA_{t_2}^{(\pi_2)} \cdots \int_{t_{k-1}}^t dA_{t_k}^{(\pi_k)} f(X_t) \right].$$

It thus suffices to show by induction on $k \in \mathbb{N}$ that, for any $A^{(1)}, \dots, A^{(k)}$ with their respective Revuz measures μ_1, \dots, μ_k as in the statement, $t \in (0, \infty]$, and $x \in S$,

$$\begin{aligned}
&E_x \left[\int_0^t dA_{t_1}^{(1)} \int_{t_1}^t dA_{t_2}^{(2)} \cdots \int_{t_{k-1}}^t dA_{t_k}^{(k)} f(X_t) \right] \\
&= \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t dt_k \int_S \mu_1(dy_1) \int_S \mu_2(dy_2) \cdots \int_S \mu_k(dy_k) \int_{S_\Delta} m_\Delta(dz) \\
&\quad p_{t_1}(x, y_1) p_{t_2-t_1}(y_1, y_2) \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) p_{t-t_k}(y_k, z) f(z). \quad (3.4)
\end{aligned}$$

This claim holds for $k = 1$; indeed, applying Theorem 2.7 with $F_s = f(X_{(t-s)\vee 0})$, we have

$$\begin{aligned}
E_x \left[\int_0^t dA_s^{(1)} f(X_t) \right] &= E_x \left[\int_0^t f(X_{t-s}) \circ \theta_s dA_s^{(1)} \right] \\
&= \int_0^t \int_S p_s(x, y) E_y [f(X_{t-s})] \mu_1(dy) ds \quad (\text{by Theorem 2.7}) \\
&= \int_0^t \int_S \int_{S_\Delta} p_s(x, y) p_{t-s}(y, z) f(z) m_\Delta(dz) \mu_1(dy) ds \quad (\text{by (2.2)}).
\end{aligned}$$

Next, let $k \geq 2$ and suppose that the above claim with $k-1$ in place of k holds. Let $t \in (0, \infty]$ and, for each $s \in (0, \infty)$, define $F_s^{(1)}, F_s^{(2)}, F_s: \Omega \rightarrow [0, \infty]$ by setting

$$F_s^{(1)} := \int_0^{(t-s)\vee 0} dA_{t_2}^{(2)} \int_{t_2}^{(t-s)\vee 0} dA_{t_3}^{(3)} \cdots \int_{t_{k-1}}^{(t-s)\vee 0} dA_{t_k}^{(k)}, \quad F_s^{(2)} := f(X_{(t-s)\vee 0}), \quad F_s := F_s^{(1)} F_s^{(2)},$$

so that, taking a defining set $\Lambda_i \in \mathcal{F}_\infty$ of $A^{(i)}$ for each $i \in \{2, \dots, k\}$ and setting $\Lambda := \bigcap_{i=2}^k \Lambda_i$, we have $P_x(\Lambda) = 1$ for all $x \in S$ and

$$1_\Lambda \cdot F_s^{(1)} \circ \theta_s = 1_\Lambda \int_{s \wedge t}^t dA_{t_2}^{(2)} \int_{t_2}^t dA_{t_3}^{(3)} \cdots \int_{t_{k-1}}^t dA_{t_k}^{(k)}. \quad (3.5)$$

In particular, $F_s^{(1)}(\omega)$ and $1_\Lambda(\omega) F_s^{(1)} \circ \theta_s(\omega)$ are right-continuous as $[0, \infty]$ -valued maps in $s \in (0, \infty)$ for each $\omega \in \Omega$ by the monotone convergence theorem, and hence are $\mathcal{B}((0, \infty)) \otimes \mathcal{F}_\infty$ -measurable as functions in $(s, \omega) \in (0, \infty) \times \Omega$ by [5, Proof of Lemma A.2.4]. Moreover, for any $s \in (0, \infty)$ and $x \in S$, $E_x[F_s] = 0$ if $s \geq t$, otherwise by the induction hypothesis $E_x[F_s]$ is equal to the right-hand side of (3.4) with $k, A^{(i)}, \mu_i, t$ replaced by $k-1, A^{(i+1)}, \mu_{i+1}, t-s$, and therefore $(0, \infty) \times S \ni (s, x) \mapsto E_x[F_s]$ is Borel measurable. Theorem 2.7 is thus applicable to $F_s = F_s^{(1)} F_s^{(2)}$ and, combined with (3.5), shows that for any $x \in S$,

$$\begin{aligned} E_x \left[\int_0^t dA_{t_1}^{(1)} \int_{t_1}^t dA_{t_2}^{(2)} \cdots \int_{t_{k-1}}^t dA_{t_k}^{(k)} f(X_t) \right] &= E_x \left[\int_0^t F_{t_1} \circ \theta_{t_1} dA_{t_1}^{(1)} \right] \\ &= \int_0^t \int_S p_{t_1}(x, y_1) E_{y_1}[F_{t_1}] \mu_1(dy_1) dt_1, \end{aligned}$$

which, together with the above-mentioned expression of $E_{y_1}[F_{t_1}]$ implied by the induction hypothesis, yields (3.4) and thereby completes the proof. \square

4 Extension to standard processes with duals

As already mentioned in the second paragraph of Section 1, our results in Subsection 2.2 hold, without any changes in the proofs, in the more general framework of a standard process with absolutely continuous transition function and in duality with another such standard process. In this last section, we set out this framework and give the background necessary for the discussions in Subsection 2.2 and Section 3 to extend immediately to this more general setting.

Recall the notation and conventions introduced at the beginning of Section 2. In particular, S is a fixed locally compact separable metrizable topological space, and its one-point compactification is denoted by $S_\Delta = S \cup \{\Delta\}$. Let $X = (\Omega, \mathcal{M}, (X_t)_{t \in [0, \infty)}, (P_x)_{x \in S_\Delta}, (\theta_t)_{t \in [0, \infty)})$ be a (Borel) standard process on S (see [5, the paragraph before Theorem A.2.1] or [2, Definition A.1.23(i)]), and write $\mathcal{F}_* = (\mathcal{F}_t)_{t \in [0, \infty)}$ for the minimum augmented admissible filtration of X in Ω (see [2, p. 397]). Let m be a σ -finite Borel measure on S . We assume that X satisfies the absolute continuity condition (AC) and, instead of being m -symmetric, the following duality assumption: there exists a standard process $\hat{X} = (\hat{\Omega}, \hat{\mathcal{M}}, (\hat{X}_t)_{t \in [0, \infty)}, (\hat{P}_x)_{x \in S_\Delta}, (\hat{\theta}_t)_{t \in [0, \infty)})$ on S satisfying (AC) such that, for any $t \in (0, \infty)$ and Borel measurable functions $f, g: S \rightarrow [0, \infty]$,

$$\int_S E_x[f(X_t)] g(x) m(dx) = \int_S f(x) \hat{E}_x[g(\hat{X}_t)] m(dx). \quad (4.1)$$

By (4.1), (AC) for X, \hat{X} , and [16, Theorem 2], there exists a unique Borel measurable function $p: (0, \infty) \times S \times S \rightarrow [0, \infty]$ satisfying, for any $s, t \in (0, \infty)$ and $x, y \in S$, $P_x(X_t \in dz) = p_t(x, z) m(dz)$, $\hat{P}_y(\hat{X}_t \in dz) = p_t(z, y) m(dz)$, and $p_{t+s}(x, y) = \int_S p_t(x, z) p_s(z, y) m(dz)$, and p is called the *transition density* (or *heat kernel*) of (X, \hat{X}) (with respect to m). Note that m is

then an excessive reference measure for X by (4.1), (AC) of X and [2, Theorem A.2.13(ii)] (see [2, p. 418] and [1, Chapter V, Definition 1.1] for the definitions of m being excessive and being a reference measure for X , respectively). We apply the definition of $N \in \mathcal{B}(S)$ being properly exceptional for X , (2.1), the definition of m_Δ , (2.3), (2.5), Definitions 2.1, 2.2, the definition of τ_E , and Definition 2.3 all as they are in Subsection 2.1, so that (2.2) and (2.4) hold.

In the present setting, the Revuz correspondence between PCAFs of X and a class of Borel measures on S is formulated as follows (see [2, Subsection A.3.1] and [13] for details). For each PCAF $A = (A_t)_{t \geq 0}$ of X , by [2, Lemma A.3.4(ii) and Theorem A.3.5(i)] one can define a Borel measure μ_A on S given by (2.6), and μ_A charges no set semipolar for X (i.e., $\mu_A(E) = 0$ for any $E \in \mathcal{B}(S)$ that is semipolar for X) by [2, Theorem A.3.5(ii)] (see [2, Definition A.2.6] for the definition of $E \subseteq S$ being semipolar for X). Furthermore by [13, Théorème VI.1], the Revuz measure of a PCAF in the strict sense of X is a smooth measure in the strict sense which, as already mentioned, charges no set semipolar for X , and conversely each such measure is the Revuz measure of a PCAF $A = (A_t)_{t \geq 0}$ in the strict sense of X , which is unique up to equivalence (recall Definition 2.2) by [1, Chapter V, Corollary 2.8], (2.6), [1, Chapter IV, Proposition 2.4(i)], (AC) and [2, Theorem A.2.17(iii)]. (To be precise, the existence of a defining set $\Lambda \in \mathcal{F}_\infty$ of such A is not treated in [13], but can be verified by examining [1, Proofs of Chapter IV, Theorem 3.16 and Chapter V, Theorem 2.1] carefully; in this connection see also [12, Remark 4.16].) Lemma 2.4 is proved in the same way as before, with the only change being that $\alpha \int_S r_\alpha(x, y) m(dx) = \int_0^\infty e^{-s} \hat{P}_y(s/\alpha < \hat{\zeta}) ds$ instead, where $\hat{\zeta}$ denotes the life time of \hat{X} . Remark 2.5 and Lemma 2.6 also remain applicable as they are.

Now, with the only exception of Remark 2.11, whose necessary changes are described in Remark 4.2 below, all the results and arguments in Subsection 2.2 and Section 3 remain valid even in the present wider framework, and we thus obtain the following theorem.

Theorem 4.1. *Assume the setting introduced so far in this section. Then Theorems 2.7 and 2.9 hold, and consequently so do Corollaries 2.10 and 2.12.*

Remark 4.2. Assume the setting introduced so far in this section. Let D be a non-empty open subset of S , let $D_\Delta = D \cup \{\Delta_D\}$ be the one-point compactification of D , set $m|_D := m|_{\mathcal{B}(D)}$, let $X^D = (\Omega, \mathcal{F}_\infty, (X_t^D)_{t \in [0, \infty]}, (P_x)_{x \in D_\Delta})$ be the part process of X on D as defined in Remark 2.11, and let $\hat{X}^D = (\hat{\Omega}, \hat{\mathcal{F}}_\infty, (\hat{X}_t^D)_{t \in [0, \infty]}, (\hat{P}_x)_{x \in D_\Delta})$ be the part process of \hat{X} on D , which is defined from \hat{X} in exactly the same way as X^D is from X . Then X^D, \hat{X}^D are standard processes on D by [5, Proof of Theorem A.2.10] (see also [2, Exercises 3.3.7(ii) and 4.1.9(i)]) and satisfy (AC) with respect to $m|_D$ by (AC) of X, \hat{X} , and we easily see from [5, Proofs of Lemmas 4.1.2 and 4.1.3], with the uses of the m -symmetry of X and [5, Theorem A.2.4] replaced by those of (4.1) and [1, Chapter I, Theorem 10.16], respectively, that X^D, \hat{X}^D satisfy (4.1) with respect to $m|_D$. Hence there exists a unique transition density p^D of (X^D, \hat{X}^D) with respect to $m|_D$, and then, extending p^D to $(0, \infty) \times D \times D_\Delta$ and $m|_D$ to a Borel measure $(m|_D)_\Delta$ on D_Δ as in Remark 2.11, we have the following.

$$\text{The statement (2.13) holds in the present setting.} \tag{4.2}$$

Indeed, the arguments in Remark 2.11 after (2.13) remain applicable and show (4.2) (and the analogous extensions of Lemma 3.1 and Proposition 3.2 to general D), with the only change being that (2.14) is now implied by [2, Exercise 4.1.9] and [13, Théorème V.5] instead.

To be precise, this deduction of (2.14) from [13, Théorème V.5] requires a justification because, for each $\alpha \in (0, \infty)$, the function $V_B^\alpha(x, y)$ in [13, Théorème V.5] with $B := S \setminus D$ might not be defined, or coincide with $r_\alpha^D(x, y) := \int_0^\infty e^{-\alpha t} p_t^D(x, y) dt$, for any $x, y \in D$. Indeed, we easily see from Fubini's theorem and the strong Markov property of \hat{X} (see, e.g., [2, Theorem A.1.21]) that $V_B^\alpha(x, y) = r_\alpha^D(x, y)$ for m -a.e. $x \in D$ for each $y \in D$, and then for any Borel

measurable function $f: D \rightarrow [0, \infty]$ we have

$$E_x \left[\int_0^{\tau_D} e^{-\alpha t} f(X_t) dA_t \right] = \int_D V_B^\alpha(x, y) f(y) \mu(dy) = \int_D r_\alpha^D(x, y) f(y) \mu(dy)$$

for m -a.e. $x \in D$ by [13, Théorème V.5] and Fubini's theorem and thus for any $x \in D$ by [1, Chapter IV, Proposition 2.4(i)], (AC) and [2, Theorem A.2.17(iii)], proving (2.14) by virtue of Lemma 2.4.

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